Commutative Algebra and its Applications

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Preface

The University S. M. Ben Abdellah, Fez, Morocco organized the Fifth International Fez Conference on Commutative Algebra and Applications during the period of June 23–28, 2008. The purpose of the conference was to present recent progress and new trends in commutative algebra and applications. Several talks were centered on topics influenced by Professor Alain Bouvier to pay tribute to the role he played in initiating diverse contributions to the field of commutative algebra.

This volume comprises the proceedings of the conference which consist mainly of research papers and draw on the authors' contributions to the conference. The book features 29 articles providing an up-to-date account of current research in commutative algebra and covering the following topics: amalgamated algebras, chain conditions and spectral topology, class groups and t-class semigroups, factorization and divisibility, Gorenstein homological algebra, homological aspects of commutative rings, idealization, integer-valued polynomial rings and generalizations, Matlis domains, numerical semigroups, Prüfer-like conditions in monoids and rings, pullbacks and trivial extensions, Schubert varieties, semidualizing modules, spectra and dimension theory.

The aim of this book is to be a reference and a learning tool for different audiences including graduate and postgraduate students, commutative algebraists, and also researchers in other fields of mathematics (e.g., algebraic number theory and algebraic geometry).

The conference was sponsored by S. M. Ben Abdellah University, Faculty of Sciences and Technology (FST) Fez-Saiss, Faculty of Sciences Dhar El-Mehraz, Laboratoire de Modélisation et Calcul Scientifique at FST Fez-Saiss, UFR "Algèbre Commutative et Aspects Homologiques" at FST Fez-Saiss, Ministère de l'Enseignement Supérieur et de la Recherche Scientifique, Centre National de la Recherche Scientifique et Technique, International Mathematical Union, and Les Eaux Minérales D'Oulmès.

We wish to express our gratitude to the local organizing committee, particularly Prof. N. Mahdou (chair) and Prof. R. Ameziane Hassani (treasurer). Also we would like to thank Prof. M. Zouak and Prof. A. Iraqi, Deans of FST Fez-Saiss and FS Dhar El-Mehraz, respectively. Special thanks are due to Mrs. Z. Bouziane, Director of Hotel Aghlias, and her staff for the hospitality and conference arrangements. We greatly appreciate the efforts of the contributors and referees and thank the editorial staff at Walter de Gruyter, especially Dr. R. Plato and S. Albroscheit for their patience and assistance with this volume.

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Splitting sets and weakly Matlis domains

D. D. Anderson and Muhammad Zafrullah

Abstract. An integral domain D is *weakly Matlis* if the intersection $D = \cap\{D_P \mid P \in t\text{-Max}(D)\}$ is independent of finite character. We investigate the question of when $D[X]$ or D_S is weakly Matlis.

Keywords. Weakly Matlis domain, splitting set, t-splitting set.

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Call an integral domain D a *weakly Matlis domain* if D is of finite t-character and no two distinct maximal t-ideals of D contain a nonzero prime ideal. Recently Gabelli, Houston and Picozza [13] have studied polynomial rings over weakly Matlis domains and have shown that in some cases a polynomial ring over a weakly Matlis domain need not be weakly Matlis. The purpose of this paper is to indicate the use of splitting sets and t-splitting sets in the study of polynomial rings over weakly Matlis domains. We show for instance that if $K \subseteq L$ is an extension of fields and X an indeterminate over L, then the polynomial ring over $K + XL[X]$ is a weakly Matlis domain.

Let D be an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A saturated multiplicative set S of D is said to be a *splitting set* if for all $d \in D\backslash\{0\}$ we can write $d = st$ where $s \in S$ and $tD \cap kD = tkD$ for all $k \in S$. (When $t, k \in D\backslash\{0\}$ are such that $tD \cap kD = tkD$ we say that t and k are *v-coprime*, because in this case $(t,k)_v = D$.) Splitting sets and their properties important for ideal theory were studied in [1]. Splitting sets have proved to be useful in many situations (see [20]). A saturated multiplicative set S is said to be a *t-splitting set* if for each $d \in D\backslash\{0\}$ we can write $(d) = (AB)_t$ where A and B are integral ideals such that $A_t \cap S \neq \phi$ and $(B,s)_t = D$ for all $s \in S$. Here the subscript v (resp., t) indicates the *v-operation* (resp., *t-operation*) defined on $F(D)$ by $A \mapsto A_v = (A^{-1})^{-1}$ (resp., $A_t = \cup\{F_v \mid F$ a finitely generated nonzero subideal of $A\}$). We shall freely use known facts about the v- and t-operations. A reader in need of a quick review on this topic may consult Sections 32 and 34 of Gilmer's book [14]. Let us note for now that a proper integral ideal maximal with respect to being a t-ideal is a prime ideal called a *maximal t-ideal*. We note that if S is a splitting set or t-splitting set, then any prime t-ideal P intersecting S *intersects S in detail*, i.e., every nonzero prime ideal contained in P also intersects S (see [5, Proposition 2.8] and [2, Lemma 4.2]). Thus a splitting or t-splitting set induces a bifurcation of t-Max(D), the set of maximal t-ideals, into those that intersect S (in detail) and those that are disjoint from S. The aim of this article is to show how the splitting sets and t-splitting sets can be used to prove useful results and provide interesting examples. Our focus will be on proving results about and providing examples of weakly Matlis domains which, as defined in [4], are domains D such that every nonzero nonunit is contained in at most a finite number of maximal t-ideals and no two distinct maximal t-ideals contain a common nonzero prime ideal. Indeed, as any nonzero prime ideal contains a minimal prime of a principal ideal which is necessarily a t-ideal, one can require that in a weakly Matlis domain no two maximal t-ideals contain a prime t-ideal.

A domain that satisfies ACC on integral divisorial ideals is called a *Mori domain*. In [13] Houston, Gabelli and Picozza give an example of a semiquasilocal one dimensional Mori domain (and hence a weakly Matlis domain) D such that the polynomial ring $D[X]$ is not a weakly Matlis domain. They also show that if D is a *t-local domain* (i.e., a quasilocal domain with maximal ideal a t-ideal) or a *UMT domain* (i.e., uppers to zero are maximal t-ideals, or, equivalently, t-invertible), then D is a weakly Matlis domain if and only if $D[X]$ is. One aim of this paper is to give a class of examples of one dimensional Mori domains D such that the polynomial ring $D[X]$ is a weakly Matlis domain. We do this by proving Theorem 1. We also provide a family of examples of non-UMT weakly Matlis domains such that polynomial rings over them are again weakly Matlis.

Theorem 1. *Let $K \subseteq L$ be an extension of fields and let T be an indeterminate over L. The domain $D = K + TL[T]$ is a one dimensional Mori domain such that the polynomial ring $D[X]$ is a weakly Matlis domain.*

To facilitate the proof of this theorem we shall need a sequence of lemmas, which will find other uses as well.

Lemma 2. *Let S be a splitting set of D. If B is an integral t-ideal of D, then BD_S is an integral t-ideal. In fact, for a nonzero ideal A of D, $A_tD_S = (AD_S)_t$. If E is an integral t-ideal of D_S, then $E \cap D$ is a t-ideal of D. Consequently a maximal t-ideal of D that is disjoint from S extends to a maximal t-ideal of D_S, and every maximal t-ideal of D_S contracts to a maximal t-ideal of D. Hence $t\text{-Max}(D_S) = \{PD_S \mid P \in t\text{-Max}(D)$ and $P \cap S = \phi\}$.*

Proof. Only the "consequently" part is new. (See [1, Section 3] for the other parts of the proof.) Thus suppose that P is a maximal t-ideal of D such that $P \cap S = \phi$, so PD_S is a proper t-ideal. Suppose that PD_S is not a maximal t-ideal. Let Q be a maximal t-ideal of D_S that properly contains PD_S. Thus $Q \cap D \supsetneq P$, but by the earlier part of the lemma, $Q \cap D$ is a t-ideal which contradicts the maximality of P. Further, if Q is a maximal t-ideal of D_S, then $Q \cap D = \mathcal{P}$ is a prime t-ideal of D. If $\mathcal{P}$ is not a maximal t-ideal, then $\mathcal{P}$ is properly contained in a maximal t-ideal M. There are two cases: $M \cap S = \phi$ and $M \cap S \neq \phi$. In the first case MD_S is a maximal t-ideal properly containing Q which contradicts the maximality of Q. In the second case, let $s \in M \cap S$ and let $p \in \mathcal{P}\backslash\{0\}$. Then since S is a splitting set, $p = s_1 t$ where $s_1 \in S$ and t is v-coprime to every element of S. Since $\mathcal{P} \cap S = \phi$, $t \in \mathcal{P}$. But then $t, s \in M$; so $M \supseteq (t, s)_v = D$, contradicting the assumption that M is a proper t-ideal. □

Note that the proof of Lemma 2 shows that the set $t\text{-Max}(D)$ is bifurcated by the splitting set S into two sets: those disjoint from S and those that intersect S in detail.

Lemma 2a. *Let S be a splitting set of a domain D with the following properties:*

(1) *Every member of S belongs to only a finite number of maximal t-ideals of D,*

(2) *Every prime t-ideal intersecting S is contained in a unique maximal t-ideal of D.*

Then D_S is a weakly Matlis domain if and only if D is.

Proof. Let D be a weakly Matlis domain and consider D_S. Take a nonzero nonunit $x \in D_S$. Then since S is a splitting set we conclude that $xD_S \cap D = dD$ a principal ideal [1]. Now $dD_S = xD_S$. Since every maximal t-ideal of D_S is extended from a maximal t-ideal of D disjoint from S, and because D, being weakly Matlis, is of finite t-character, we conclude that d and hence x belongs to only a finite number of maximal t-ideals of D_S. Since x is arbitrary, D_S is of finite t-character. Next suppose that there is a prime ideal P of D_S such that P is contained in two distinct maximal t-ideals M_1 and M_2 of D_S. Then $P \cap D \subseteq M_1 \cap D$, $M_2 \cap D$, two distinct maximal t-ideals of D, contradicting the fact that D is a weakly Matlis domain.

For the converse, suppose that D_S is a weakly Matlis domain and let d be a nonzero nonunit of D. Then $d = sr$ where $s \in S$ and r is v-coprime to every member of S. Now s and r being v-coprime do not share any maximal t-ideals. Next the number of maximal t-ideals containing s is finite because of (1) and the number of maximal t-ideals containing r is finite because D_S is weakly Matlis (and by Lemma 2). Next let P be a prime t-ideal of D such that P is contained in two distinct maximal t-ideals M and N. Then $P \cap S = \phi$ by (2). But then both M and N are disjoint from S (for if they intersect S, they intersect S in detail) and so the prime t-ideal PD_S is contained in the two maximal t-ideals MD_S and ND_S contradicting the assumption that D_S is a weakly Matlis domain. □

Proposition 2b. *Let X be an indeterminate over D and let D be of finite t-character. Then D and $D[X]$ are weakly Matlis if and only if for every pair of distinct maximal t-ideals P and Q of D, $P[X] \cap Q[X]$ does not contain a nonzero prime ideal.*

Proof. We know that D is of finite t-character if and only if $D[X]$ is [17, Proposition 4.2] (while this is stated for D being integrally closed, their proof does not use this hypothesis). Also, every maximal t-ideal M of $D[X]$ with $M \cap D \neq (0)$ is of the form $P[X]$ where P is a maximal t-ideal of D [16, Proposition 1.1]. Suppose that for every pair of distinct maximal t-ideals P and Q of D, $P[X] \cap Q[X]$ does not contain a nonzero prime ideal. Let us show that $D[X]$ is weakly Matlis. For this we must show that no two maximal t-ideals of $D[X]$ contain a nonzero prime ideal. Now the complement of the set of maximal t-ideals used in the condition is the set of maximal t-ideals that are uppers to zero and these are height-one prime ideals. But if at least one of a pair of maximal t-ideals is of height one, then obviously the condition is satisfied. So no pair of maximal t-ideals contains a nonzero prime ideal. The condition clearly indicates that D is also weakly Matlis. Conversely if both D and $D[X]$ are weakly Matlis, then the condition is obviously satisfied. □

There are examples of weakly Matlis domains D, such as weakly Krull domains, with the property that for every multiplicative subset S the ring of fractions D_S is weakly Matlis. Recall that an integral domain D is a *weakly Krull domain* if $D = \cap D_P$ is a locally finite intersection of localizations at height-one prime ideals of D. A *Krull domain* is a weakly Krull domain such that the localization at every height one prime ideal is a discrete rank one valuation domain. Now recall from [3] that D is a *weakly factorial domain* (i.e., every nonzero nonunit is a product of primary elements) if and only if every saturated multiplicative subset of D is a splitting set, if and only if D is a weakly Krull domain with zero t-class group. The t-class group is precisely the divisor class group for a Krull domain and a Krull domain with a zero divisor class group is a UFD, and there exist Krull domains that are not UFD's. So, in general, a saturated multiplicative set in a weakly Krull domain is not a splitting set. In other words there do exist weakly Matlis domains D such that D_S is a weakly Matlis domain while S is not a splitting set. There are other examples of weakly Matlis domains D that have nonsplitting saturated multiplicative sets S such that D_S is weakly Matlis. Now the question is: Is there a weakly Matlis domain D such that D_S is not weakly Matlis? The example below answers this question.

Example 2c. Let X and Y be indeterminates over the field of rational numbers $\mathbb{Q}$ and let $T = \mathbb{Q}[[X, Y]]$. The ring $R = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[[X, Y]]$, p a nonzero prime of $\mathbb{Z}$, is an integral domain of the general $D + M$ type [7] and obviously a quasilocal ring with the maximal ideal a principal ideal. Let K be the quotient field of R and let T be an indeterminate over K. Then the ring $S = R + TK[[T]] = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[[X, Y]] + TK[[T]]$ is a quasilocal ring with the maximal ideal a principal ideal. Since a principal ideal is a divisorial ideal and hence a t-ideal, we conclude that S is a t-local ring and hence is a weakly Matlis domain. But if $N = \{p^n\}_{n=0}^{\infty}$, then $S_N = \mathbb{Q}[[X, Y]] + TK[[T]]$ is a GCD domain such that every nonzero prime ideal of S_N contains $TK[T]$; so S_N cannot be a weakly Matlis domain.

A splitting set S of D is called *lcm splitting* if every element of S has an lcm with every member of $D\backslash\{0\}$. A splitting set generated by prime elements is obviously an lcm splitting set. This gives us a Nagata type theorem for weakly Matlis domains.

Lemma 3. *Let S be an lcm splitting set of D generated by primes. Then D_S is a weakly Matlis domain if and only if D is.*

Proof. Note that S satisfies (1) and (2) of Lemma 2a since a nonzero principal prime ideal is a maximal t-ideal. □

Proposition 4. *Let D be an integral domain that contains a splitting set S generated by primes such that $D_S[X]$ is a weakly Matlis domain. Then $D[X]$ is a weakly Matlis domain.*

Proof. The proof follows from the fact that if S is an lcm splitting set in D then S is an lcm splitting set in $D[X]$ [5, Theorem 2.2] and of course that $D_S[X] = D[X]_S$ which in turn makes the proof an application of Lemma 3. □

Proof of Theorem 1. The proof depends upon the fact that $TL[T]$ is a maximal ideal of $D = K + TL[T]$ of height one and every element of $K + TL[T]\backslash TL[T]$ is an associate of an element of the form $1 + Tf(T)$ which being common to both $L[T]$ and $K + TL[T]$ is a product of primes which are of height one and maximal [12]. From this it follows that $\mathcal{S} = K + TL[T] \backslash TL[T]$ is an lcm splitting set generated by primes. Also, because $D_{\mathcal{S}}$ is a one dimensional local domain, $D_{\mathcal{S}}[X] = D[X]_{\mathcal{S}}$ is a weakly Matlis domain, [6, page 389]. Now Proposition 4 applies. □

It is shown in [13] that if D is a UMT domain or a t-local domain, then D is a weakly Matlis domain if and only if the polynomial ring $D[X]$ is. Now a PVMD being an integrally closed UMT domain we conclude that for a PVMD and hence for a GCD domain D being weakly Matlis is equivalent to $D[X]$ being weakly Matlis. We shall use t-splitting sets to bring to light the behind the scenes goings on, in this matter, later. For now we shall show, that even a weakly Matlis domain that is neither t-local nor UMT can have a weakly Matlis polynomial ring. The example has already appeared in Section 2 of [19]. So we shall briefly describe this example and let the reader check the details.

Example 5. Let V be a valuation domain of rank > 1 and let Q be a nonzero non-maximal prime ideal of V. The domain $R = V + TV_Q[T]$ is a non-UMT weakly Krull domain such that $R[X]$ is a weakly Matlis domain.

Before we start to illustrate this example let us recall that an element x in $D \backslash \{0\}$ is called *primal* if for all $r, s \in D \backslash \{0\}$, $x|rs$ in D implies that $x = uv$ where $u|r$ and $v|s$. An integral domain D is a *Schreier domain* if D is integrally closed and every nonzero element of D is primal. Schreier domains were introduced by P.M. Cohn in [10] where it was shown that a GCD domain is Schreier and that every irreducible element in a Schreier domain is a prime. It was noted in [11, page 424] that if D is a GCD domain, S a multiplicative set in D and X is an indeterminate over D_S, then $D + XD_S[X]$ is a Schreier domain.

Illustration: That R is a Schreier domain that is not a GCD domain (and hence not a UMT domain) can be checked from [19, Section 2]. Following [19] let us call $f \in R \backslash \{0\}$ *discrete* if $f(0)$ is a unit in V. Now according to Lemma 2.2 of [19] every nonzero nonunit f of R can be written uniquely up to associates as $f = gd$ where d is a discrete element and g is not divisible by a nonunit discrete element of R. Indeed it is also shown in [19] after Lemma 2.2 that every discrete element is a product of finitely many height-one principal primes. So the set $S = \{d \in R \mid d \text{ is discrete}\}$ is an lcm splitting set generated by primes. Next, as shown in Lemma 2.4 of [19] $M = R \backslash S$ is a prime t-ideal of R such that MR_M is a prime t-ideal. So, $R_M = R_S$ is t-local and according to [13] $R_S[X] = R[X]_S$ is a weakly Matlis domain. Now Lemma 3 facilitates the conclusion that $R[X]$ is a weakly Matlis domain.

Let us do some analysis here. Our main tool in Lemma 3 is the fact that we can split every nonzero nonunit x of D as a product $x = st$ where s is a finite product of height-one principal primes (coming from an lcm splitting set S) and hence is contained in a finite number of maximal t-ideals and t is not divisible by any such primes, i.e., t is coprime to every member of S. So, if we can show that each t for a general x belongs to at most a finite number of maximal t-ideals such that no two of those maximal t-ideals contain a nonzero prime ideal we have accomplished our task.

Following Theorem 4.9 of [2] we can prove the following lemma similar to Lemma 2.

Lemma 6. *Let D be an integral domain and S a t-splitting set of D. If B is an (integral) t-ideal of D, then BD_S is an (integral) t-ideal of D_S. In fact, for a nonzero ideal A of D, $A_t D_S = (AD_S)_t$. If E is a t-ideal of D_S, then $E \cap D$ is a t-ideal of D. Consequently, if P is a maximal t-ideal of D with $P \cap S = \phi$, then PD_S is a maximal t-ideal of D_S. Hence $t\text{-Max}(D_S) = \{PD_S \mid P \in t\text{-Max}(D)$ and $P \cap S = \phi\}$.*

Indeed the "consequently" part of Lemma 6 can be handled in precisely the same manner as we did in the proof of Lemma 2. For the other parts of the proof the reader may consult [2, Theorem 4.9].

Let S be a t-splitting set of D and let $\tau = \{A_1 A_2 \cdots A_n \mid A_i = d_i D_S \cap D\}$ be the multiplicative set generated by ideals that are contractions of dD_S to D for each nonzero $d \in D$. Call τ a *t-complement* of S. Also, let D_τ be the *τ-transform*, i.e., $D_\tau = \{x \in K \mid xA \subseteq D$ for some $A \in \tau\}$. It is easy to show that $D = D_S \cap D_\tau$ and as shown in Theorem 4.3 of [2], $D_S = \cap D_P$ where P ranges over the maximal t-ideals P of D with $P \cap S = \phi$ and $D_\tau = \cap D_Q$ where Q ranges over the maximal t-ideals Q of D with Q intersecting S in detail.

This discussion leads to the following result.

Lemma 7. *Let S be a t-splitting set of D, $F = \{P \in t\text{-Max}(D) \mid P \cap S = \phi\}$ and $G = \{Q \in t\text{-Max}(D) \mid Q \cap S \neq \phi\}$. Suppose that D_S is a ring of finite t-character and every nonzero nonunit of D belongs to at most a finite number of members of G. Then D is a ring of finite t-character. If in addition D_S is a weakly Matlis domain and no two members of G contain a nonzero prime ideal, then D is a weakly Matlis domain. Moreover, if S is a t-splitting set and D is a ring of finite t-character (resp., weakly Matlis domain), then D_S is a ring of finite t-character (resp., weakly Matlis).*

Let D be an integral domain, X an indeterminate over D, and $S = \{f \in D[X] \mid (A_f)_v = D\}$. It is easy to check that the set S is multiplicative and saturated. Our next result gives an alternate proof to parts of Lemma 2.1 and Proposition 2.2 of [13]; also see Corollary 3.5 of that paper.

Proposition 8. *Let D be an integral domain, X be an indeterminate over D, and $S = \{f \in D[X] \mid (A_f)_v = D\}$. If $D[X]_S$ is a ring of finite t-character (resp., weakly Matlis domain), then $D[X]$ is a ring of finite t-character (resp., weakly Matlis domain) and so is D. Moreover, if $D[X]$ is a ring of finite t-character (resp., weakly Matlis domain), then so is $D[X]_S$.*

Proof. It has been shown in [8, Proposition 3.7] that S is a t-splitting set. So all we have to do for the first part is to check that the requirements of Lemma 7 are met. For this set $F = \{P \in t\text{-Max}(D[X]) \mid P \cap S = \phi\}$ and $G = \{Q \in t\text{-Max}(D[X]) \mid Q \cap S \neq \phi\}$. Now every nonzero nonunit of $D[X]$ belongs to at most a finite number of members of G, because every nonzero nonunit of $D[X]$ belongs to at most a finite number of uppers to zero. The other requirement is met automatically because the members of G are all height-one primes. □

The bifurcation induced by the t-splitting set $S = \{f \in D[X] \mid (A_f)_v = D\}$ does indeed shed useful light on the construction $D[X]_{N_v}$ where $N_v = S = \{f \in D[X] \mid (A_f)_v = D\}$ by B. G. Kang [18]. For details the reader may consult [8] and [9].

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Divisibility properties in ultrapowers of commutative rings

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Abstract. All rings considered are commutative with identity. We study the preservation of certain properties in the passage from a ring R to the ultrapower R^* relative to a free ultrafilter on the set of all positive integers. Our main result is that if R is a locally pseudo-valuation domain (LPVD) of finite character (for instance, a semi-quasilocal LPVD), then R^* is also an LPVD. In the same vein, it is shown that the classes of pseudo-valuation domains and pseudo-valuation rings are each stable under the passage from R to R^*. An example is given of a divided domain R such that the domain R^* is not divided. A divisibility condition is found which characterizes the divided (respectively, quasilocal) rings R such that R^* is a divided (respectively, treed) ring.

Keywords. Ultrapower, linearly ordered, divided prime, divided ring, pseudo-valuation domain, ϕ-ring.

AMS classification. Primary 13A05, 13G05; secondary 13H10.

1 Introduction

All rings considered in this note are commutative with identity. Our interest here is in the preservation of certain properties in the passage from a ring R to the ultrapower R^* relative to a free ultrafilter $\mathcal{U}$ on a denumerable index set I. For convenience, we identify I with the set $\mathbb{N}$ of all positive integers. (The interested reader is invited to check that our methods extend to the case in which $\mathcal{U}$ is any countably incomplete ultrafilter on an infinite index set I; and that many of our results carry over to ultraproducts.) By definition, R^* is the factor ring of $\prod_I R$ modulo the ideal $\{(a_i)_{i\in I} \mid Z(a_i) \in \mathcal{U}\}$, where $Z(a_i)$ denotes the set of coordinates i where $a_i = 0$. By an abuse of notation, we will also denote the elements of R^* by $(a_i)_{i\in I}$. It will be clear from the context whether we are working in the product or the ultrapower.

For some time, there has been considerable interest in the transfer of ring-theoretic properties between R and R^*; see, for instance, [19], [20], [21], as well as [16, pp. 179–180] for a brief introduction to ultrafilters and ultraproducts. Among the assembled lore is the fact that if R is an integral domain with quotient field K, then R^* is an integral domain with quotient field K^*. More significantly, if a ring R is semi-quasilocal with exactly n maximal ideals, then R^* is also semi-quasilocal with exactly n maximal ideals [20]. In particular, if (R, M) is a quasilocal ring, then R^* is also quasilocal, with unique maximal ideal M^*. Much of our motivation comes from the result [15] that if R is a Prüfer domain, then R^* is also a Prüfer domain. Hence, if R is a valuation

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domain, then so is R^*. In the same spirit, our main result, Corollary 3.3, establishes that if R is a locally pseudo-valuation domain (LPVD, in the sense of [12]) of finite character (for instance, a semi-quasilocal LPVD), then R^* is also an LPVD. Section 3 is devoted to a proof of Corollary 3.3, together with the supporting technical results on ultrafilters. Section 2 deals with easier material, primarily for certain quasilocal rings, suggested by the Prüfer ↔ LPVD interplay.

The "interplay" that was just mentioned refers to the fact that LPVDs are, perhaps, the best behaved members in the class of locally divided integral domains (in the sense of [10]). This class is particularly interesting for several reasons: it contains all Prüfer domains as well as many integral domains that are not necessarily integrally closed; and the "locally divided integral domain" concept figures in several characterizations of Prüfer domains. Generalizations to the context of rings possibly with nontrivial zero-divisors have led to concepts such as divided rings [3], locally divided rings [7], and pseudo-valuation rings ([14], [6]). Among the results in Section 2, we establish in Corollary 2.8 and Proposition 2.9 that the classes of pseudo-valuation domains and pseudo-valuation rings are each stable under the passage from R to R^*. However, Example 2.4 shows that the class of divided domains does not exhibit similar stability. Section 2 also contains sharper facts, such as Example 2.5, involving the concept of a treed ring. (Recall that a ring R is called *treed* in case no maximal ideal of R can contain incomparable prime ideals of R; thus, a ring R is quasilocal and treed if and only if $\operatorname{Spec}(R)$, the set of all prime ideals of R, is linearly ordered with respect to inclusion.) The study of the various classes of divided rings in Section 2 is aided by a divisibility condition established in Proposition 2.3 as a characterization of rings R such that R^* is divided. This result is paired naturally with our first result, Proposition 2.1 which, in the context of ultrapowers, permits a permutation in the quantifications in a characterization of rings R such that $\operatorname{Spec}(R)$ is linearly ordered [2, Theorem 0].

Our reasoning with ultrapowers often depends on a number of facts that are used without further mention. For instance, consider elements $x = (x_i), y = (y_i)$ of the ultrapower R^*. Then $x^k = 0$ (for some $k \in \mathbb{N}$) if and only if $\{i \in I \mid x_i^k = 0\} \in \mathcal{U}$. Similarly, $x|y$ if and only if $\{i \in I \mid x_i|y_i\} \in \mathcal{U}$. Also, note that if P is an ideal of a ring R, then $P^* := \{(a_i) \mid \{i \in I \mid a_i \in P\} \in \mathcal{U}\}$ is an ideal of R^*; and $P^* \in \operatorname{Spec}(R^*)$ if and only if $P \in \operatorname{Spec}(R)$. Viewing R as canonically embedded in R^* via the diagonal map, we see easily that $P^* \cap R = P$. Furthermore, there are canonical isomorphisms $(R/P)^* \cong R^*/P^*$ and (if P is a prime ideal) $(R_P)^* \cong R^*_{P^*}$.

In addition to the notation and conventions mentioned above, we use dim(ension) to refer to Krull dimension; and, for a ring R, we use $\operatorname{Max}(R)$ to denote the set of maximal ideals of R, $\operatorname{Min}(R)$ to denote the set of minimal prime ideals of R, $\operatorname{Nil}(R)$ to denote the set of nilpotent elements of R, and $\operatorname{Rad}(J)$ to denote the nilradical of an ideal J of R. Any unexplained material is standard, as in [13].

2 Comparability properties in Spec(R^*)

Recall from [2, Theorem 0] that for any ring R, $\operatorname{Spec}(R)$ is linearly ordered (with respect to inclusion) if and only if, for any $a, b \in R$, there exists $n = n(a, b) \in \mathbb{N}$

such that either $a|b^n$ or $b|a^n$. This criterion is sharpened in the following result for ultrapowers. One consequence of Proposition 2.1 is the fact that if R is a ring such that $\operatorname{Spec}(R^*)$ is linearly ordered, then $\operatorname{Spec}(R)$ is also linearly ordered; a more elementary proof of this fact follows since $P^* \cap R = P$ for each $P \in \operatorname{Spec}(R)$.

Proposition 2.1. *Let R be a ring. Then* $\operatorname{Spec}(R^*)$ *is linearly ordered if and only if there exists $n \in \mathbb{N}$ such that for all $a, b \in R$, either $a|b^n$ or $b|a^n$.*

Proof. We first prove the contrapositive of the "only if" assertion. Suppose, then, that there does not exist $n \in \mathbb{N}$ such that for all $a, b \in R$, one has that either $a|b^n$ or $b|a^n$. Thus, for each $n \in \mathbb{N}$, there exist elements $a_n, b_n \in R$ such that $a_n \nmid b_n^n$ and $b_n \nmid a_n^n$. Define $\alpha, \beta \in R^*$ by $\alpha := (a_n)_{n\in\mathbb{N}}$ and $\beta := (b_n)_{n\in\mathbb{N}}$. We claim that for all $n \in \mathbb{N}$, $\alpha \nmid \beta^n$ and $\beta \nmid \alpha^n$. Given the claim, one sees via the criterion in [2, Theorem 0] that, as desired, $\operatorname{Spec}(R^*)$ is not linearly ordered.

Suppose that the above claim fails. Then, without loss of generality, $\alpha|\beta^k$ for some $k \in \mathbb{N}$. It follows, by a fact recalled in the Introduction, that $\{n \in \mathbb{N} \mid a_n|b_n^k\} \in \mathcal{U}$. However, for all $n \geq k$, we know that $a_n \nmid b_n^k$. Hence, $\{n \in \mathbb{N} \mid a_n|b_n^k\} \subseteq \{1, 2, \ldots, k-1\}$. This is a contradiction, since a finite set cannot be a member of a free ultrafilter. This establishes the claim and completes the proof of the "only if" assertion.

We next turn to the "if" assertion. Suppose, then, that there exists $k \in \mathbb{N}$ such that for $a, b \in R$, either $a|b^k$ or $b|a^k$. By applying the above criterion from [2, Theorem 0], our task is translated to showing that if $\alpha := (a_n)_{n\in\mathbb{N}}$ and $\beta := (b_n)_{n\in\mathbb{N}}$, then there exists $n \in \mathbb{N}$ such that either $\alpha|\beta^n$ or $\beta|\alpha^n$. It suffices to show that either $\alpha|\beta^k$ or $\beta|\alpha^k$. Putting $V := \{i \in \mathbb{N} \mid a_i|b_i^k\}$ and $W := \{i \in \mathbb{N} \mid b_i|a_i^k\}$, we see from a fact recalled in the Introduction that an equivalent task is to show that either $V \in \mathcal{U}$ or $W \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it is enough to show that $V \cup W = \mathbb{N}$. This equality is, however, ensured by the hypothesis of the "if" assertion, to complete the proof. □

Recall that a local (Noetherian) integral domain (R, M) is called *analytically unramified* (resp., *analytically irreducible*) if its completion with respect to the filtration given by the powers M^n is a reduced ring (resp., an integral domain). Our next result, which re-encounters the criterion from Proposition 2.1, is also interesting in that neither its hypothesis nor its conclusion mentions an ultrapower.

Corollary 2.2. *Let R be an analytically unramified one-dimensional local integral domain. Then R is analytically irreducible if and only if there exists $n \in \mathbb{N}$ such that for all $a, b \in R$, either $a|b^n$ or $b|a^n$.*

Proof. The integral closure of R (in its quotient field) is a finitely generated R-module [22]. It therefore follows from well-known results (cf. [18, (43.20), (32.2)], [8, Proposition III.5.2]) that R is analytically irreducible if and only if the integral closure of R is local. By [21, Theorem 6.3], the latter condition is equivalent to requiring that $\operatorname{Spec}(R^*)$ is linearly ordered. Accordingly, an application of Proposition 2.1 completes the proof. □

A ring R is called *divided* if, for each $P \in \operatorname{Spec}(R)$ and $a \in R$, either $a \in P$ or $P \subseteq Ra$. Recall from [3, Proposition 2] that a ring R is divided if and only if, for any elements $a, b \in R$, either $a|b$ or there exists $n = n(a,b) \in \mathbb{N}$ such that $b|a^n$. It therefore follows that one consequence of Proposition 2.3 is the fact that if R is a ring such that R^* is divided, then R is also divided. The proof of Proposition 2.3 is similar to the proof of Proposition 2.1 and is hence omitted.

Proposition 2.3. *Let R be a divided ring. Then R^* is divided if and only if there exists $n \in \mathbb{N}$ such that for all $a, b \in R$, either $a|b$ or $b|a^n$.*

We next construct an example to show that a divided ring R need not satisfy the divisibility condition in the statement of Proposition 2.3.

Example 2.4. There exists a divided ring R such that R^* is not divided. Our construction uses an infinite strictly ascending chain of fields $\mathbb{Q} \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$, with $F := \cup F_n$. Fix a prime integer p and let

$$R := \mathbb{Z}_{(p)} + F_1 X + F_2 X^2 + \cdots + F_n X^n + \cdots \subseteq F[[X]].$$

To show that R is divided, it will be convenient to first show that the ring

$$T := \mathbb{Q} + F_1 X + F_2 X^2 + \cdots + F_n X^n + \cdots \subseteq F[[X]]$$

is divided. For this, it suffices, by [3, Proposition 2], to show that if $a = \sum_{i=n}^{\infty} a_i X^i$ and $b = \sum_{j=m}^{\infty} b_j X^j$ are nonzero nonunits of T, with $1 \leq n \leq m$, then there exists a positive integer ν such that $a^\nu / b \in T$. Choose ν so that $\nu n > 2m$. Then $p := \nu n - m > m$ and we easily see by the usual process of long division that when a^ν $(= (a_n)^\nu X^{n\nu} +$ higher degree terms) is divided by b, the quotient in $F[[X]]$ actually lies in T. In other words, b divides a^ν in T, and so T is a divided domain. Next, consider the maximal ideal $M := F_1 X + F_2 X^2 + \cdots + F_n X^n + \cdots$ of T. Observe that $T/M \cong \mathbb{Q}$. Then, since R is the pullback $T \times_{T/M} \mathbb{Z}_{(p)}$ with both T and $\mathbb{Z}_{(p)}$ being divided domains, it follows from [11, Corollary 2.6] that R is a divided domain, and hence a divided ring, as asserted.

Moreover, we claim that R^* is not divided. To prove this claim, pick $d_n \in F_n \setminus F_{n-1}$ for each $n \in \mathbb{N}$. Set $a_n := X$ and $b_n := d_n X^n$. Then for each n, $a_n \nmid b_n$ and $b_n \nmid a_n^n$. Define $\alpha, \beta \in R^*$ via $\alpha := (a_n)_{n \in \mathbb{N}}$ and $\beta := (b_n)_{n \in \mathbb{N}}$. Then $\alpha \nmid \beta$ and $\beta \nmid \alpha^n$ for all $n \in \mathbb{N}$. So, by [3, Proposition 2], R^* is not divided, thus proving the above claim.

Let R be the divided ring in the above example. Although R^* is not divided, note that R^* *is* treed (in contrast to the situation in Example 2.5 below). To see this, observe that for all $a, b \in R$, either $a|b^2$ or $b|a^2$, and apply Proposition 2.1.

Next, we present a family of examples of divided rings whose ultrapowers are not treed.

Example 2.5. There exists a divided ring R such that R^* is not treed. Indeed, consider any analytically unramified one-dimensional local integral domain R. Trivially, R is a divided ring. Moreover, as recalled in the Introduction, R^* inherits the "quasilocal" condition from R. Therefore, by also arranging that R is not analytically irreducible (for instance, taking R as in the examples in [9, pp. 54–55]), we see from Corollary 2.2 and Proposition 2.1 that R^* is not treed.

Our next result shows that the property involving a uniform bound that was mentioned in the hypothesis of Proposition 2.3 ascends and descends in the context of a certain pullback, namely, the "$\mathrm{Spec}(R) = \mathrm{Spec}(T)$" context of [1]. First, for $n \in \mathbb{N}$, it is convenient to say that a ring R has *property* $*_n$ if, for all elements $a, b \in R$, either $a|b$ or $b|a^n$. It is clear that if a ring R satisfies $*_n$ for some $n \in \mathbb{N}$, then R satisfies $*_m$ for all $m \geq n$.

Proposition 2.6. *Let $R \subset T$ be quasilocal rings with common maximal ideal M. Then:*

(a) *If R satisfies $*_n$, then T satisfies $*_n$.*

(b) *If T satisfies $*_1$, then R satisfies $*_2$.*

(c) *If T satisfies $*_n$ for some $n \geq 2$, then R satisfies $*_n$.*

Proof. (a) The assertion is clear because R and T have the same set of nonunits.

(b) Suppose that T satisfies $*_1$, and let $a, b \in R$. We need to show that either $a|b$ in R or $b|a^2$ in R. Without loss of generality, neither a nor b is a unit; thus, $a, b \in M$. Suppose that $a \nmid b$ in R. We show that $b|a^2$ in R. There are two cases.

In the first case, $a|b$ in T. Then $b = ax$ for some $x \in T \setminus R$. Since $M \subset R$, we conclude that x is a unit of T. Then $a = x^{-1}b$ (in T), and so $a^2 = (ax^{-1})b \in Rb$, as $ax^{-1} \in M \subset R$. In particular, $b|a^2$ in R.

In the remaining case, $a \nmid b$ in T. Then, by hypothesis, $b|a$ in T. Write $by = a$, with $y \in T$. It follows that $a^2 = b(ay)$ and so, since $ay \in M \subset R$, we have $b|a^2$ in R, as desired.

(c) Suppose that T satisfies $*_n$ for some $n \geq 2$. As in the proof of (b), we must show that if $a, b \in M$, then either $a|b$ in R or $b|a^n$ in R. Suppose that $a \nmid b$ in R. We show that $b|a^n$ in R. If $a|b$ in T, we can argue as in the proof of (b) to show that $b|a^2$ in R, whence $b|a^n$ in R. Thus, without loss of generality, $a \nmid b$ in T. Then, by hypothesis, $b|a^n$ in T. Write $a^n = bx$, with $x \in T$. If $x \in R$, we are done, and so we may assume that x is a unit of T. Since $n \geq 2$, we have $b = x^{-1}a^n = (x^{-1}a^{n-1})a$, with $(x^{-1}a^{n-1}) \in M \subset R$. Thus, in the case to which we have reduced, it follows that $a|b$ (in T), a contradiction. The proof is complete. □

Proposition 2.7. *Let T be an overring of an integral domain R. If $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ (as sets), then $\mathrm{Spec}(R^*) = \mathrm{Spec}(T^*)$.*

Proof. Without loss of generality $R \neq T$. Then, by [1, Lemma 3.2], R and T are quasilocal, with the same maximal ideal, say M. By a fact recalled in the Introduction, it follows that R^* and T^* are each quasilocal with unique maximal ideal M^*. Therefore, by [1, Proposition 3.8], $\mathrm{Spec}(R^*) = \mathrm{Spec}(T^*)$. □

Recall from [14] that a quasilocal domain (R, M) is called a *pseudo-valuation domain* (PVD) if R has a (uniquely determined) valuation overring V such that V has maximal ideal M; equivalently, such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ (as sets).

Corollary 2.8. *If R is a PVD, then R^* is a PVD.*

Proof. Since the ultrapower of a valuation domain is also a valuation domain, the result follows by combining Proposition 2.7 and the second of the above characterizations of PVDs. □

The preceding result generalizes to arbitrary (commutative) rings. In the process of proving this (see Proposition 2.9 below), we make contact with the following interesting class of divided rings. Recall from [6] that a ring R is called a *pseudo-valuation ring* (PVR) if Pa and Rb are comparable (with respect to inclusion) for all $P \in \operatorname{Spec}(R)$ and $a, b \in R$. An integral domain is a PVR if and only if it is a PVD. Any PVR is a divided, hence quasilocal, ring. It was shown in [6] that a quasilocal ring (R, M) is a PVR if and only if, for all elements $a, b \in R$, either $a|b$ or $b|am$ for each $m \in M$.

Proposition 2.9. *If R is a PVR, then R^* is a PVR.*

Proof. By the above remarks, R is quasilocal, say with maximal ideal M. Therefore, R^* is quasilocal, with maximal ideal M^*. Consider arbitrary elements $\alpha = (a_n)_{n\in\mathbb{N}}, \beta = (b_n)_{n\in\mathbb{N}} \in R^*$ and $\mu = (m_n)_{n\in\mathbb{N}} \in M^*$. Without loss of generality, the coordinates m_n may be chosen so that $m_n \in M$ for each $n \in \mathbb{N}$. Put $V := \{i \in \mathbb{N} \mid a_i|b_i\}$ and $W := \{i \in \mathbb{N} \mid b_i|a_i m_i\}$. Since R is a PVR, the last-mentioned characterization of PVRs yields that $V \cup W = \mathbb{N}$. As $\mathcal{U}$ is an ultrafilter, it follows that either $V \in \mathcal{U}$ or $W \in \mathcal{U}$. In the first (resp., second) case, $\alpha|\beta$ (resp., $\beta|\alpha\mu$). Thus, either $\alpha|\beta$ or $\beta|\alpha\mu$ for each $\mu \in M^*$. In other words, R^* is a PVR. □

Recently, much attention has been paid to a certain class $\mathcal{C}$ of divided rings that contains the class of PVRs. We next recall the definition of $\mathcal{C}$ and show that, unlike the classes of Prüfer domains, valuation domains, PVDs and PVRs, $\mathcal{C}$ is *not* stable under the passage from R to R^*.

Recall that a prime ideal P of a ring R is said to be *divided (in R)* if P is comparable (with respect to inclusion) to Rb for each $b \in R$. A ring R is called a Φ-*pseudo-valuation ring* (Φ-PVR) if $\operatorname{Nil}(R)$ is a divided prime ideal of R and, for all $a, b \in R \setminus \operatorname{Nil}(R)$, either $a|b$ or $b|an$ for all nonunits n of R. The following particularly useful characterization of the Φ-PVR concept appears in [5]. A ring R is a Φ-PVR if and only if $\operatorname{Nil}(R)$ is a divided prime ideal of R and $R/\operatorname{Nil}(R)$ is a PVD.

Example 2.10. There exists a Φ-PVR R such that R^* is not a Φ-PVR. It can be further arranged that $\operatorname{Nil}(R)$ is a prime ideal of R but $\operatorname{Nil}(R^*)$ is not a prime ideal of R^*. For a construction of such, begin with any field K and set

$$R := K[X_1, X_2, \ldots, X_n, \ldots]/(\{X_n^n, X_i X_j \mid n, i, j \in \mathbb{N}, i \neq j\}).$$

Notice that the images of the X_i's are nilpotent and generate the unique maximal ideal, say M, of R. In particular, $\operatorname{Nil}(R) = M \in \operatorname{Spec}(R)$. It is then trivial via the above criterion from [5] that R is a Φ-PVR. To show that R^* is not a Φ-PVR, we produce elements $\alpha \in \operatorname{Nil}(R^*)$, $\beta \in R^* \setminus \operatorname{Nil}(R^*)$ such that $\beta \nmid \alpha$. Observe that the elements $\alpha := (X_2, X_3, X_2, X_3, \ldots)$ and $\beta := (X_1, X_2, X_3, \ldots, X_n, \ldots)$ have the asserted properties. Thus, $\operatorname{Nil}(R^*)$ is not a divided prime ideal of R^*, and so R^* is not a Φ-PVR.

It remains to verify that $\mathrm{Nil}(R^*)$ is not a prime ideal of R^*. Consider the element $\gamma := (X_2, X_3, \ldots, X_n, \ldots) \in R^*$. Evidently, $\beta\gamma = 0$. We noted above that β is not nilpotent; and in the same way, one checks that γ is not nilpotent. The verification is complete.

In view of the unexpected behavior of $\mathrm{Nil}(R^*)$ in the preceding example, we devote the final two results of this section to additional scrutiny of related behavior. We begin by analyzing the behavior of "Rad" in the passage from R to R^*.

Remark 2.11. Let J be an ideal of a ring R. Then $\mathrm{Rad}(J^*) \subseteq \mathrm{Rad}(J)^*$, with equality if and only if there exists $n \in \mathbb{N}$ such that $a^n \in J$ for all $a \in \mathrm{Rad}(J)$. The proof is similar to that of Proposition 2.1; see [19, Proposition 2.28].

Example 2.12. There exists a PVR, R, such that $\mathrm{Nil}(R)$ is a prime ideal of R and $\mathrm{Nil}(R^*)$ is a prime ideal of R^*, but $\mathrm{Nil}(R^*) \neq \mathrm{Nil}(R)^*$. (By taking $J := 0$ in the preceding remark, one trivially has that $\mathrm{Nil}(S^*) \subseteq \mathrm{Nil}(S)^*$ for any ring S.) For a construction, consider any rank one non-discrete valuation domain (D, M). Choose any nonzero element $d \in M$; then $\mathrm{Rad}(Dd) = M$. Then, of course, [6, Corollary 3] ensures that $R := D/Dd$ is a PVR, and so $\mathrm{Nil}(R)$ is a prime (in fact, the unique maximal) ideal of R. Moreover, by Proposition 2.9, R^* is a PVR, and so $\mathrm{Nil}(R^*)$ is a prime ideal of R^*.

It remains to verify that $\mathrm{Nil}(R^*) \neq \mathrm{Nil}(R)^*$. Deny. Then, by the criterion in Remark 2.11, there exists $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in \mathrm{Rad}(0) = \mathrm{Nil}(R)$. Therefore, for all $m \in M$, we have $m^n \in Dd$. Letting v denote any (real-valued) valuation associated to D, we infer the existence of $d_1 \in D$ such that

$$nv(m) = v(m^n) = v(d_1 d) = v(d_1) + v(d) \geq v(d),$$

whence $v(m) \geq \frac{v(d)}{n}$, an absurdity since the non-discreteness of D guarantees the existence of elements $m \in M$ with arbitrarily small (positive) v-value. This (desired) contradiction completes the verification.

3 Ultrapowers of LPVDs

For the remainder of the paper, R will denote an integral domain. Recall from [12] that R is called a *locally pseudo-valuation domain* (LPVD) if R_P is a PVD for all $P \in \mathrm{Spec}(R)$ (equivalently, for all $P \in \mathrm{Max}(R)$). We will show that with an extra assumption on R, the property of being an LPVD is inherited by R^*. First, we need to describe the maximal ideals of R^*. In [23], a description of some of the maximal ideals of the product $T := \prod_I R$ is given (an equivalent formulation is given in [20]). Furthermore, it is shown in [23] that with an additional hypothesis, these are all the maximal ideals of T. We next present that description here and then pass to considerations involving the factor ring R^*.

Let $\mathcal{J}$ denote the set of all functions from the index set I to the set of finite subsets of $\mathrm{Max}(R)$. For $\sigma, \rho \in \mathcal{J}$, we say $\rho \leq \sigma$ (ρ is called a *subfunction of* σ) if $\rho(i) \subseteq \sigma(i)$ for all $i \in I$. For $\rho \le \sigma \in \mathcal{I}$, we define $\sigma \setminus \rho$ to be the function given by $\sigma(i) \setminus \rho(i)$ for each $i \in I$. Also, we define the functions $\sigma \vee \rho$ (resp., $\sigma \wedge \rho$) via $(\sigma \vee \rho)(i) := \sigma(i) \cup \rho(i)$ (resp., $(\sigma \wedge \rho)(i) := \sigma(i) \cap \rho(i)$) for each $i \in I$. Finally, the *blank function* Φ is defined by $\Phi(i) := \varnothing$ for each $i \in I$. Now, consider a fixed element $\sigma \in \mathcal{I}$. The set of subfunctions of σ forms a Boolean algebra with σ as 1, Φ as 0, and the complement of $\rho \le \sigma$ is $\rho' = \sigma \setminus \rho$. Therefore, it makes sense to talk about ultrafilters on the set of subfunctions of σ. In particular, by an *ultrafilter on* σ, we mean a collection of functions $\mathcal{F} \subseteq \{\rho \mid \rho \le \sigma\}$ such that:

- $\sigma \in \mathcal{F}$ and $\Phi \notin \mathcal{F}$;
- If $\rho \in \mathcal{F}$ and $\rho \le \tau$, then $\tau \in \mathcal{F}$;
- If $\rho, \tau \in \mathcal{F}$, then $\rho \wedge \tau \in \mathcal{F}$;
- If ρ is an element of the Boolean algebra of subfunctions of σ, then either $\rho \in \mathcal{F}$ or $\rho' \in \mathcal{F}$.

If $\mathcal{F}$ is an ultrafilter on σ, then by a standard argument, one can show that if $\rho \vee \tau \in \mathcal{F}$, then either $\rho \in \mathcal{F}$ or $\tau \in \mathcal{F}$.

For $a = (a_i)_{i \in I} \in T$, we can obtain an interesting subfunction of σ by defining $\sigma_a(i) := \{P \in \sigma(i) \mid a_i \in P\}$. Now, consider any ultrafilter $\mathcal{F}$ on σ. Set $(\mathcal{F}) := \{a \in T \mid \sigma_a \in \mathcal{F}\}$. As shown in [17] or [23], $(\mathcal{F})$ is a maximal ideal of T. Moreover, if each nonzero element of R is contained in only finitely many maximal ideals of R (such rings R are said to have *finite character*), then these $(\mathcal{F})$'s are all the maximal ideals of T [23, Theorem 1.2].

It is well known that for a (commutative) ring S, each $P \in \operatorname{Spec}(S)$ induces an ultrafilter $\mathcal{U}_P$ on the Boolean algebra of idempotents of S via: $e \in \mathcal{U}_P$ if and only if $1 - e \in P$. Furthermore, it is easy to see that if $P \subseteq Q$ are elements of $\operatorname{Spec}(S)$, then $\mathcal{U}_P \subseteq \mathcal{U}_Q$; hence, since $\mathcal{U}_P$ and $\mathcal{U}_Q$ are maximal filters, we have $\mathcal{U}_P = \mathcal{U}_Q$. Now if (as above) $T := \prod_I R$, where R is an integral domain, then the Boolean algebra of idempotents of T is isomorphic to the Boolean algebra of subsets of I. Thus, each prime ideal P of T determines an ultrafilter $\mathcal{U}_P$ on (the Boolean algebra of subsets of) I. In particular, $\mathcal{U}_P := \{A \subseteq I \mid 1 - e_A \in P\}$, where e_A denotes the characteristic function on A.

Conversely, given an ultrafilter $\mathcal{U}$ on (the Boolean algebra of subsets of) I, one can construct an ideal $P_{\mathcal{U}} \subset T$ by setting $P_{\mathcal{U}} := (\{1 - e_A \mid A \in \mathcal{U}\})$. Observe that if $(a_i) \in T$, then $(a_i)(1 - e_A) = (a_i)$, where $A := Z(a_i)$. Thus $P_{\mathcal{U}}$ is the ideal of relations that defines the ultrapower R^* and so $T/P_{\mathcal{U}} = R^*$. Therefore, if R is an integral domain, $P_{\mathcal{U}}$ is a prime ideal of T. Furthermore, one sees that $\mathcal{U}_{P_{\mathcal{U}}} = \mathcal{U}$ and $P_{\mathcal{U}_P} \subseteq P$. Hence the assignment $\mathcal{U} \mapsto P_{\mathcal{U}}$ defines a bijection between the set of ultrafilters on I and $\operatorname{Min}(T)$.

In addition, we claim that each $Q \in \operatorname{Spec}(T)$ contains a unique minimal prime ideal P. Suppose the claim is false, and take $Q \in \operatorname{Spec}(T)$ such that Q contains two distinct minimal prime ideals P_1 and P_2. Since each minimal prime ideal of T is generated by idempotents, there exists an idempotent $e \in P_1 \setminus P_2$. Therefore $1 - e \in P_2$, whence $1 = e + (1 - e) \in Q$, a contradiction, thus proving the claim.

Furthermore, suppose $Q \in \operatorname{Spec}(T)$, and let P denote the unique minimal prime ideal of T that is contained in Q. Then, since the only idempotents in a local ring are 0 and 1, it follows that $PT_Q = 0$. Hence, there are canonical isomorphisms $T_Q \cong T_Q/PT_Q \cong (T/P)_{Q/P}$.

Now, we return to ultrapowers of an integral domain R with respect to an ultrafilter $\mathcal{U}$ on I. As noted above, $R^* = T/P_{\mathcal{U}}$. Therefore, to examine the localization of R^* at an arbitrary maximal ideal, it suffices, by the preceding remarks, to consider the localization of T at a suitable maximal ideal. First, we need the following technical lemma.

Lemma 3.1. *Let $\mathcal{F}$ be an ultrafilter on $\sigma \in \mathcal{I}$ and let $\tau \in \mathcal{F}$. Let $M := (\mathcal{F})$ be the maximal ideal determined by $\mathcal{F}$ and put $\mathcal{U} := \mathcal{U}_M$. Then there exists $W \in \mathcal{U}$ such that $\tau(i) \neq \varnothing$ for all $i \in W$.*

Proof. Partition the set I into $V := \{i \in I \mid \tau(i) = \varnothing\}$ and $W := I \setminus V$. Observe that $\sigma_{e_V} \geq \tau \in \mathcal{F}$, whence $\sigma_{e_V} \in \mathcal{F}$. Thus, $e_V \in M = (\mathcal{F})$. Since $\mathcal{U} = \mathcal{U}_M$, it follows that $W \in \mathcal{U}$, and the result is proved. □

Before moving on to our result for LPVDs, we give some general definitions. Let R be a ring with $a, b \in R$ and let S be a multiplicative subset of R. We say that *a divides b with respect to S* if the image of a divides the image of b in the ring R_S. This is equivalent to saying that there exists $r \in R$ and $s \in S$ such that $ar = bs$. Note that if X is any finite subset of $\operatorname{Max}(R)$, then a divides b with respect to all $R \setminus P$ for all $P \in X$ if and only if a divides b with respect to $R \setminus \cup_{P \in X} P$.

Theorem 3.2. *Let R be an LPVD and let $(\mathcal{F})$ be the maximal ideal of T determined by an ultrafilter $\mathcal{F}$ on some σ. Then $T_{(\mathcal{F})}$ is a PVD.*

Proof. Let $M = (\mathcal{F})$. To show that T_M is a PVD, we must show that given any two elements $\alpha, \beta \in M$, either α divides β with respect to $T \setminus M$ or β divides αm with respect to $T \setminus M$ for all $m \in M$ (cf. [6]).

Let $\alpha = (a_i)_{i \in I}$ and $\beta = (b_i)_{i \in I}$ be elements of M. Thus the element $\sigma_\alpha \wedge \sigma_\beta$ is in $\mathcal{F}$. Define $\tau \leq \sigma_\alpha \wedge \sigma_\beta$ by $\tau(i) := \{P \in (\sigma_\alpha \wedge \sigma_\beta)(i) \mid a_i$ divides b_i with respect to $R \setminus P\}$. Set $\rho := (\sigma_\alpha \wedge \sigma_\beta) \setminus \tau$.

Since $\mathcal{F}$ is an ultrafilter, either $\tau \in \mathcal{F}$ or $\rho \in \mathcal{F}$. First, assume the former. In this case, we claim that α divides β with respect to $T \setminus M$. To see this, note that by Lemma 3.1, there exists $W \in \mathcal{U}$ such that for all $i \in W$, $\tau(i) \neq \varnothing$. Furthermore, from the definition of τ, it follows that for each $i \in W$, there exists $r_i \in R$ and $s_i \in R \setminus \cup_{P \in \tau(i)} P$ such that $a_i r_i = b_i s_i$. For all $i \in I \setminus W$, let $r_i := s_i := 1$. Use this data to define two elements of R^*, namely, $r := (r_i)$ and $s := (s_i)$. It follows that $\alpha r = \beta s$. However, it is also clear from the definition that $s \notin M$, thus proving the claim.

In the remaining case, we can assume that $\rho \in \mathcal{F}$. Let $m := (m_i)_{i \in I} \in M$. It suffices to show that β divides αm with respect to $T \setminus M$. Since $\sigma_m \in \mathcal{F}$, we have $\sigma_m \wedge \rho \in \mathcal{F}$. Therefore, it again follows from Lemma 3.1 that there exists $W \in \mathcal{U}$ such that for all $i \in W$, $(\sigma_m \wedge \rho)(i) \neq \varnothing$. Also from the definition of ρ, a_i does not divide b_i with respect to $R \setminus P$ for any $P \in (\sigma_m \wedge \rho)(i)$. Therefore, since R is an LPVD, we have that for each $i \in W$, there exist $s_i \in R \setminus \cup_{P \in (\sigma_m \wedge \tau)(i)} P$ and $r_i \in R$ such that $m_i a_i s_i = b_i r_i$. For all other i, set $r_i := s_i := 1$. Thus, $s := (s_i) \in T \setminus M$, $r := (r_i) \in T$, and $m\alpha s = \beta r$. The last equation is the statement that β divides αm with respect to $T \setminus M$, as desired. □

Corollary 3.3. *Let R be an LPVD with finite character. Then any ultrapower R^* is also an LPVD.*

Proof. Since R has finite character, it follows from [23, Theorem 1.2] that each maximal ideal of R^* is the image of a maximal ideal of T of the form $(\mathcal{F})$. Thus, the result follows directly from Theorem 3.2. □

We close with two observations. First, the "locally divided" analogue of the preceding result is false. Indeed, Examples 2.4 and 2.5 each show that if R is a quasilocal locally divided integral domain (necessarily of finite character), in other words a divided domain, then R^* need not be locally divided. Second, we do not know whether Corollary 3.3 remains valid if one deletes the "finite character" hypothesis.

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On rings with divided nil ideal: a survey

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Abstract. Let R be a commutative ring with $1 \neq 0$ and $\mathrm{Nil}(R)$ be its set of nilpotent elements. Recall that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In many articles, the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$ (Observe that if R is an integral domain, then $R \in \mathcal{H}$.) If $R \in \mathcal{H}$, then R is called a ϕ-ring. Recently, David Anderson and the author generalized the concept of Prüfer domains, Bezout domains, Dedekind domains, and Krull domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the author generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. In this paper, we state many of the main results on ϕ-rings.

Keywords. Prüfer ring, ϕ-Prüfer ring, Dedekind ring, ϕ-Dedekind ring, Krull ring, ϕ-Krull ring, Mori ring, ϕ-Mori ring, divided ring.

AMS classification. 13F05, 13A15.

1 Introduction

Let R be a commutative ring with $1 \neq 0$ and $\mathrm{Nil}(R)$ be its set of nilpotent elements. Recall from [26] and [7] that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [6], [8], [9], [10], and [11], the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. (Observe that if R is an integral domain, then $R \in \mathcal{H}$.) If $R \in \mathcal{H}$, then R is called a ϕ-ring. Recently, David Anderson and the author, [3] and [4], generalized the concept of Prüfer, Bezout domains, Dedekind domains, and Krull domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the author, [17], generalized the concept of Mori domain to the context of rings that are in the class $\mathcal{H}$. Yet, another paper by Dobbs and the author [14] investigated going-down ϕ-rings. In this paper, we state many of the main results on ϕ-rings.

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, and $Z(R)$ denotes the set of zerodivisors of R. We start by recalling some background material. A non-zerodivisor of a ring R is called a *regular element* and an ideal of R is said to be *regular* if it contains a regular element. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then $\mathrm{Nil}(R) \subset I$. In particular, this holds if I is a regular ideal of a ring $R \in \mathcal{H}$.

Recall from [6] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \in R \setminus Z(R)$ is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $\mathrm{Ker}(\phi) \subseteq \mathrm{Nil}(R)$, $\mathrm{Nil}(T(R)) = \mathrm{Nil}(R)$, $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \phi(\mathrm{Nil}(R)) = \mathrm{Nil}(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{\mathrm{Nil}(R)}$ is quasilocal with maximal ideal $\mathrm{Nil}(\phi(R))$, and $R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\mathrm{Nil}(\phi(R))$.

Recall that an ideal I of a ring R is called a *divisorial ideal* of R if $(I^{-1})^{-1} = I$, where $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. If a ring R satisfies the ascending chain condition (a.c.c.) on divisorial regular ideals of R, then R is called a *Mori ring* in the sense of [46]. An integral domain R is called a *Dedekind domain* if every nonzero ideal of R is invertible, i.e., if I is a nonzero ideal of R, then $II^{-1} = R$. If every finitely generated nonzero ideal I of an integral domain R is invertible, then R is said to be a *Prüfer domain*. If every finitely generated regular ideal of a ring R is invertible, then R is said to be a *Prüfer ring*. If R is an integral domain and $x^{-1} \in R$ for each $x \in T(R) \setminus R$, then R is called a *valuation domain*. Also, recall from [29] that an integral domain R is called a *Krull domain* if $R = \cap V_i$, where each V_i is a discrete valuation overring of R, and every nonzero element of R is a unit in all but finitely many V_i. Many characterizations and properties of Dedekind and Krull domains are given in [29], [30], and [40]. Recall from [32] that an integral domain R with quotient field K is called a *pseudo-valuation domain (PVD)* in case each prime ideal of R is *strongly prime* in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Every valuation domain is a pseudo-valuation domain. In [13], Anderson, Dobbs and the author generalized the concept of pseudo-valuation rings to the context of arbitrary rings. Recall from [13] that a prime ideal P of R is said to be *strongly prime* if either $aP \subset bR$ or $bR \subset aP$ for all $a, b \in R$. A ring R is said to be a *pseudo-valuation ring (PVR)* if every prime ideal of R is a strongly prime ideal of R.

Throughout the paper, we will use the technique of idealization of a module to construct examples. Recall that for an R-module B, the idealization of B over R is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for the "idealized ring" is $R(+)B$. See [38] for basic properties of these rings.

2 ϕ-pseudo-valuation rings and ϕ-chained rings

In [6], the author generalized the concept of pseudo-valuation domains to the context of rings that are in $\mathcal{H}$. Recall from [6] that a ring $R \in \mathcal{H}$ is said to be a *ϕ-pseudo-valuation ring (ϕ-PVR)* if every nonnil prime ideal of R is a *ϕ-strongly prime ideal* of $\phi(R)$, in the sense that $xy \in \phi(P)$, $x \in R_{\mathrm{Nil}(R)}$, $y \in R_{\mathrm{Nil}(R)}$ (observe that $R_{\mathrm{Nil}(R)} = T(\phi(R))$) implies that either $x \in \phi(P)$ or $y \in \phi(P)$. We state some of the main results on ϕ-pseudo-valuation rings.

Theorem 2.1 ([8, Proposition 2.1]). *Let D be a PVD and suppose that P, Q are prime ideal of D such that P is properly contained in Q. Let $d \geq 1$ and choose $x \in D$ such that* $\mathrm{Rad}(xD) = P$. *Then $J = x^{d+1}D_Q$ is an ideal of D and hence D/J is a PVR* *with the following properties:*

(i) $\operatorname{Nil}(R) = P/J$ *and* $x^d \notin J$*;*

(ii) $Z(R) = Q/J$.

Theorem 2.2 ([8, Corollary 2.7]). *Let* $d \geq 2$, D, P, Q, x, J, *and* R *be as in Theorem 2.1. Set* $B = R_{\operatorname{Nil}(R)}$. *Then the idealization ring* $R(+)B$ *is a* ϕ*-PVR that is not a PVR.*

Theorem 2.3 ([10, Proposition 2.9], also see [23, Theorem 3.1]). *Let* $R \in \mathcal{H}$. *Then* R *is a* ϕ*-PVR if and only if* $R/\operatorname{Nil}(R)$ *is a PVD.*

Recall from [9] that a ring $R \in \mathcal{H}$ is said to be a *ϕ-chained ring (ϕ-CR)* if for each $x \in R_{\operatorname{Nil}(R)} \setminus \phi(R)$ we have $x^{-1} \in \phi(R)$. A ring A is said to be a *chained ring* if for every $a, b \in A$, either $a \mid b$ (in A) or $b \mid a$ (in A).

Theorem 2.4 ([9, Corollary 2.7]). *Let* $d \geq 2$, D *be a valuation domain,* P, Q, x, J, R *be as in Theorem 2.1. Then* $R = D/J$ *is a chained ring. Furthermore, if* $B = R_{\operatorname{Nil}(R)}$, *then the idealization ring* $R(+)B$ *is a* ϕ*-CR that is not a chained ring.*

Theorem 2.5 ([9, Proposition 3.3]). *Let* $R \in \mathcal{H}$ *be a quasi-local ring with maximal ideal* M *such that* M *contains a regular element of* R. *Then* R *is a* ϕ*-PVR if and only if* $(M : M) = \{x \in T(R) \mid xM \subset M\}$ *is a* ϕ*-CR with maximal ideal* M.

Theorem 2.6 ([3, Theorem 2.7]). *Let* $R \in \mathcal{H}$. *Then* R *is a* ϕ*-CR if and only if* $R/\operatorname{Nil}(R)$ *is a valuation domain.*

Recall that B is said to be an *overring* of a ring A if B is a ring between A and $T(A)$.

Theorem 2.7 ([10, Corrollary 3.17]). *Let* $R \in \mathcal{H}$ *be a* ϕ*-PVR with maximal ideal* M. *The following statements are equivalent:*

(i) *Every overring of* R *is a* ϕ*-PVR;*

(ii) $R[u]$ *is a* ϕ*-PVR for each* $u \in (M : M) \setminus R$*;*

(iii) $R[u]$ *is quasi-local for each* $u \in (M : M) \setminus R$*;*

(iv) *If* B *is an overring of* R *and* $B \subset (M : M)$, *then* B *is a* ϕ*-PVR with maximal ideal* M*;*

(v) *If* B *is an overring of* R *and* $B \subset (M : M)$, *then* B *is quasi-local;*

(vi) *Every overring of* R *is quasi-local;*

(vii) *Every* ϕ*-CR between* R *and* $T(R)$ *other than* $(M : M)$ *is of the form* R_P *for some non-maximal prime ideal* P *of* R*;*

(viii) $R' = (M : M)$ *(where* R' *is the integral closure of* R *inside* $T(R)$*).*

3 Nonnil Noetherian rings (ϕ-Noetherian rings)

Recall that an ideal I of a ring R is said to be a nonnil ideal if $I \not\subseteq \mathrm{Nil}(R)$. Let $R \in \mathcal{H}$. Recall from [11] that R is said to be a *nonnil-Noetherian ring* or just a *ϕ-Noetherian ring* as in [16] if each nonnil ideal of R is finitely generated. We have the following results.

Theorem 3.1 ([11, Corollary 2.3]). *Let $R \in \mathcal{H}$. If every nonnil prime ideal of R is finitely generated, then R is a ϕ-Noetherian ring.*

Theorem 3.2 ([11, Theorem 2.4]). *Let $R \in \mathcal{H}$. The following statements are equivalent:*

(i) *R is a ϕ-Noetherian ring;*

(ii) *$R/\mathrm{Nil}(R)$ is a Noetherian domain;*

(iii) *$\phi(R)/\mathrm{Nil}(\phi(R))$ is a Noetherian domain;*

(iv) *$\phi(R)$ is a ϕ-Noetherian ring.*

Theorem 3.3 ([11, Theorem 2.6]). *Let $R \in \mathcal{H}$. Suppose that each nonnil prime ideal of R has a power that is finitely generated. Then R is a ϕ-Noetherian ring.*

Theorem 3.4 ([11, Theorem 2.7]). *Let $R \in \mathcal{H}$. Suppose that R is a ϕ-Noetherian ring. Then any localization of R is a ϕ-Noetherian ring, and any localization of $\phi(R)$ is a ϕ-Noetherian ring.*

Theorem 3.5 ([11, Theorem 2.9]). *Let $R \in \mathcal{H}$. Suppose that R satisfies the ascending chain condition on the nonnil finitely generated ideals. Then R is a ϕ-Noetherian ring.*

Theorem 3.6 ([11, Theorem 3.4]). *Let R be a Noetherian domain with quotient field K such that* $\dim(R) = 1$ *and R has infinitely many maximal ideals. Then $D = R(+)K \in \mathcal{H}$ is a ϕ-Noetherian ring with Krull dimension one which is not a Noetherian ring. In particular, $\mathbb{Z}(+)\mathbb{Q}$ is a ϕ-Noetherian ring with Krull dimension one which is not a Noetherian ring (where $\mathbb{Z}$ is the set of all integer numbers with quotient field $\mathbb{Q}$).*

Theorem 3.7 ([11, Theorem 3.5]). *Let R be a Noetherian domain with quotient field K and Krull dimension $n \geq 2$. Then $D = R(+)K \in \mathcal{H}$ is a ϕ-Noetherian ring with Krull dimension n which is not a Noetherian ring. In particular, if K is the quotient field of $R = \mathbb{Z}[x_1, \ldots, x_{n-1}]$, then $R(+)K$ is a ϕ-Noetherian ring with Krull dimension n which is not a Noetherian ring.*

In the following result, we show that a ϕ-Noetherian ring is related to a pullback of a Noetherian domain.

Theorem 3.8 ([16, Theorem 2.2]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Noetherian ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & S = A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring containing A with maximal ideal M, $S = A/M$ is a Noetherian subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Theorem 3.9 ([16, Proposition 2.4]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let $I \neq R$ be an ideal of R. If $I \subset \mathrm{Nil}(R)$, then R/I is a ϕ-Noetherian ring. If $I \not\subset \mathrm{Nil}(R)$, then $\mathrm{Nil}(R) \subset I$ and R/I is a Noetherian ring. Moreover, if $\mathrm{Nil}(R) \subset I$, then R/I is both Noetherian and ϕ-Noetherian if and only if I is either a prime ideal or a primary ideal whose radical is a maximal ideal.*

Theorem 3.10 ([16, Corollary 2.5]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Then a homomorphic image of R is either a ϕ-Noetherian ring or a Noetherian ring.*

Our next result shows that a ϕ-Noetherian ring satisfies the conclusion of the Principal Ideal Theorem (and the Generalized Principal Ideal Theorem).

Theorem 3.11 ([16, Theorem 2.7]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a prime ideal. If P is minimal over an ideal generated by n or fewer elements, then the height of P is less than or equal to n. In particular, each prime minimal over a nonnil element of R has height one.*

Other statements about primes of Noetherian rings that can be easily adapted to statements about primes of ϕ-Noetherian rings include the following.

Theorem 3.12 ([16, Proposition 2.8] and [40, Theorem 145]). *Let $R \in \mathcal{H}$ satisfy the ascending chain condition on radical ideals. If R has an infinite number of prime ideals of height one, then their intersection is $\mathrm{Nil}(R)$.*

Theorem 3.13 ([16, Proposition 2.9]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and P be a nonnil prime ideal of R of height n. Then there exist nonnil elements $a_1, \dots, a_n$ in R such that P is minimal over the ideal $(a_1, \dots, a_n)$ of R, and for any i $(1 \leq i \leq n)$, every (nonnil) prime ideal of R minimal over $(a_1, \dots, a_i)$ has height i.*

Theorem 3.14 ([16, Proposition 2.10]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let I be an ideal of R generated by n elements with $I \neq R$. If P is a prime ideal containing I with P/I of height k, then the height of P is less than or equal to $n + k$.*

Theorem 3.15 ([16, Proposition 3.1]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a height n prime of R. If Q is a prime of $R[x]$ that contracts to P but properly contains $PR[x]$, then $PR[x]$ has height n and Q has height $n + 1$.*

Similar height restrictions exist for the primes of $R[x_1, \ldots, x_m]$.

Theorem 3.16 ([16, Proposition 3.2]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a height n prime of R. If Q is a prime of $R[x_1, \ldots, x_m]$ that contracts to P but properly contains $PR[x_1, \ldots, x_m]$, then $PR[x_1, \ldots, x_m]$ has height n and Q has height at most $n + m$. Moreover the prime $PR[x_1, \ldots, x_m] + (x_1, \ldots, x_m)R[x_1, \ldots, x_m]$ has height $n + m$.*

Theorem 3.17 ([16, Corollary 3.3]). *If R is a finite dimensional ϕ-Noetherian ring of dimension n, then $\dim(R[x_1, \ldots, x_m]) = n + m$ for each integer $m > 0$.*

In our next result, we show that each ideal of $R[x]$ that contracts to a nonnil ideal of R is finitely generated.

Theorem 3.18 ([16, Proposition 3.4]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. If I is an ideal of $R[x_1, \ldots, x_n]$ for which $I \cap R$ is not contained in $\mathrm{Nil}(R)$, then I is a finitely generated ideal of $R[x_1, \ldots, x_n]$.*

Since three distinct comparable primes of $R[x]$ cannot contract to the same prime of R, a consequence of Theorem 3.18 is that the search for primes of $R[x]$ that are not finitely generated can be restricted to those of height one. A similar statement can be made for primes of $R[x_1, \ldots, x_n]$.

Theorem 3.19 ([16, Corollary 3.5]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a prime of $R[x_1, \ldots, x_n]$. If P has height greater than n, then P is finitely generated.*

The ring in our next example shows that the converse of Theorem 3.18 does not hold even for prime ideals.

Example 3.20 ([16, Example 3.6]). Let $R = D(+)L$ be the idealization of $L = K((y))/D$ over $D = K[[y]]$. Then R is a quasilocal ϕ-Noetherian ring with nilradical $\mathrm{Nil}(R)$ isomorphic to L. Consider the polynomial $g(x) = 1 - yx$. Since the coefficients of g generate D as an ideal and g is irreducible, $P = gD[x]$ is a height-one principal prime of $D[x]$ with $P \cap D = (0)$. Each nonzero element of L can be written in the form d/y^n where n is a positive integer, y denotes the image of y in L and $d = d_0 + d_1 y + \cdots + d_{n-1} y^{n-1}$ with $d_0 \neq 0$. Given such an element, let $f(x) = 1 + yx + \cdots + y^{n-1}x^{n-1} \in L[x]$. Then $g(x)(df(x)/y^n) = d/y^n$ since $dy^n/y^n = 0$ in L. It follows that $g(x)R[x]$ is a height-one principal prime of $R[x]$ that contracts to $\mathrm{Nil}(R)$.

4 ϕ-Prüfer rings and ϕ-Bezout rings

We say that a nonnil ideal I of R is *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$. Recall from [3] that R is called a *ϕ-Prüfer ring* if every finitely generated nonnil ideal of R is ϕ-invertible.

Theorem 4.1 ([3, Corollary 2.10]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Prüfer ring;*

(ii) *$\phi(R)$ is a Prüfer ring;*

(iii) *$\phi(R)/\operatorname{Nil}(\phi(R))$ is a Prüfer domain;*

(iv) *R_P is a ϕ-CR for each prime ideal P of R;*

(v) *$R_P/\operatorname{Nil}(R_P)$ is a valuation domain for each prime ideal P of R;*

(vi) *$R_M/\operatorname{Nil}(R_M)$ is a valuation domain for each maximal ideal M of R;*

(vii) *R_M is a ϕ-CR for each maximal ideal M of R.*

Theorem 4.2 ([3, Theorem 2.11]). *Let $R \in \mathcal{H}$ be a ϕ-Prüfer ring and let S be a ϕ-chained overring of R. Then $S = R_P$ for some prime ideal P of R containing $Z(R)$.*

The following is an example of a ring $R \in \mathcal{H}$ such that R is a Prüfer ring, but R is not a ϕ-Prüfer ring.

Example 4.3 ([3, Example 2.15]). Let $n \geq 1$ and let D be a non-integrally closed domain with quotient field K and Krull dimension n. Set $R = D(+)(K/D)$. Then $R \in \mathcal{H}$ and R is a Prüfer ring with Krull dimension n which is not a ϕ-Prüfer ring.

Theorem 4.4 ([3, Theorem 2.17]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if every overring of $\phi(R)$ is integrally closed.*

Example 4.5 ([3, Example 2.18]). Let $n \geq 1$ and let D be a Prüfer domain with quotient field K and Krull dimension n. Set $R = D(+)K$. Then $R \in \mathcal{H}$ is a (non-domain) ϕ-Prüfer ring with Krull dimension n.

Recall from [21] that a ring R is said to be a *pre-Prüfer ring* if R/I is a Prüfer ring for every nonzero proper ideal I of R.

Theorem 4.6 ([3, Theorem 2.19]). *Let $R \in \mathcal{H}$ such that* $\operatorname{Nil}(R) \neq \{0\}$. *Then R is a pre-Prüfer ring if and only if R is a ϕ-Prüfer ring.*

The following example shows that the hypothesis $\operatorname{Nil}(R) \neq \{0\}$ in Theorem 4.6 is crucial.

Example 4.7 ([3, Example 2.20] and [42, Example 2.9]). Let D be a Prüfer domain with quotient field F. For indeterminates X, Y, let $K = F(Y)$ and let V be the valuation domain $K + XK[[X]]$. Then V is one-dimensional with maximal ideal $M = XK[[X]]$. Set $R = D + M$. Then $\operatorname{Nil}(R) = \{0\}$, and R is a pre-Prüfer ring (domain) which is not a Prüfer ring (domain). Hence R is not a ϕ-Prüfer ring.

Recall from [3] that a ring $R \in \mathcal{H}$ is said to be a *ϕ-Bezout ring* if $\phi(I)$ is a principal ideal of $\phi(R)$ for every finitely generated nonnil ideal I of R. A ϕ-Bezout ring is a ϕ-Prüfer ring, but of course the converse is not true. A ring R is said to be a *Bezout ring* if every finitely generated regular ideal of R is principal.

Theorem 4.8 ([3, Corollary 3.5]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Bezout ring;*

(ii) *$R/\operatorname{Nil}(R)$ is a Bezout domain;*

(iii) *$\phi(R)/\operatorname{Nil}(\phi(R))$ is a Bezout domain;*

(iv) *$\phi(R)$ is a Bezout ring;*

(v) *Every finitely generated nonnil ideal of R is principal.*

Theorem 4.9 ([3, Theorem 3.9]). *Let $R \in \mathcal{H}$ be quasi-local. Then R is a ϕ-CR if and only if R is a ϕ-Bezout ring.*

Example 4.10 ([3, Example 3.8]). Let $n \geq 1$ and let D be a Bezout domain with quotient field K and Krull dimension n. Set $R = D(+)K$. Then $R \in \mathcal{H}$ is a (non-domain) ϕ-Bezout ring with Krull dimension n.

5 ϕ-Dedekind rings

Let $R \in \mathcal{H}$. We say that a nonnil ideal I of R is *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every nonnil ideal of R is ϕ-invertible, then we say that R is a *ϕ-Dedekind ring*.

Theorem 5.1 ([4, Theorem 2.6]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a Dedekind subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Example 5.2 ([4, Example 2.7]). Let D be a Dedekind domain with quotient field K, and let L be an extension ring of K. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-Dedekind ring which is not a Dedekind domain.

We say that a ring $R \in \mathcal{H}$ is *ϕ-(completely) integrally closed* if $\phi(R)$ is (completely) integrally closed in $T(\phi(R)) = R_{\operatorname{Nil}(R)}$. The following characterization of ϕ-Dedekind rings resembles that of Dedekind domains as in [40, Theorem 96].

Theorem 5.3 ([4, Theorem 2.10]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is ϕ-Dedekind;*

(ii) *R is nonnil-Noetherian (ϕ-Noetherian), ϕ-integrally closed, and of dimension ≤ 1;*

(iii) *R is nonnil-Noetherian and R_M is a discrete ϕ-chained ring for each maximal ideal M of R.*

A ring R is said to be a *Dedekind ring* if every nonzero ideal of R is invertible.

Theorem 5.4 ([4, Theorem 2.12]). *Let $R \in \mathcal{H}$ be a ϕ-Dedekind ring. Then R is a Dedekind ring.*

The following is an example of a ring $R \in \mathcal{H}$ which is a Dedekind ring but not a ϕ-Dedekind ring.

Example 5.5 ([4, Example 2.13]). Let D be a non-Dedekind domain with (proper) quotient field K. Set $R = D(+)K/D$. Then $R \in \mathcal{H}$ and $R = T(R)$. Hence R is a Dedekind ring. Since $R/\operatorname{Nil}(R)$ is ring-isomorphic to D, R is not a ϕ-Dedekind ring by [4, Theorem 2.5].

It is well known that an integral domain R is a Dedekind domain iff every nonzero proper ideal of R is (uniquely) a product of prime ideals of R. We have the following result.

Theorem 5.6 ([4, Theorem 2.15]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if every nonnil proper ideal of R is (uniquely) a product of nonnil prime ideals of R.*

Theorem 5.7 ([4, Theorem 2.16]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Dedekind ring;*

(ii) *Each nonnil proper principal ideal aR can be written in the form $aR = Q_1 \cdots Q_n$, where each Q_i is a power of a nonnil prime ideal of R and the Q_i's are pairwise comaximal;*

(iii) *Each nonnil proper ideal I of R can be written in the form $I = Q_1 \cdots Q_n$, where each Q_i is a power of a nonnil prime ideal of R and the Q_i's are pairwise comaximal.*

Theorem 5.8 ([4, Theorem 2.20]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Dedekind ring;*

(ii) *Each nonnil prime ideal of R is ϕ-invertible;*

(iii) *R is a nonnil-Noetherian ring and each nonnil maximal ideal of R is ϕ-invertible.*

Theorem 5.9 ([4, Theorem 2.23]). *Let $R \in \mathcal{H}$ be a ϕ-Dedekind ring. Then every overring of R is a ϕ-Dedekind ring.*

6 Factoring nonnil ideals into prime and invertible ideals

In this section, we give a generalization of the concept of factorization of ideals of an integral domain into a finite product of invertible and prime ideals which was extensively studied by Olberding [48] to the context of rings that are in the class $\mathcal{H}$. Observe that if R is an integral domain, then $R \in \mathcal{H}$. An ideal I of a ring R is said to be a *nonnil ideal* if $I \nsubseteq \mathrm{Nil}(R)$. Let $R \in \mathcal{H}$. Then R is said to be a *ϕ-ZPUI ring* if each nonnil ideal I of $\phi(R)$ can be written as $I = JP_1 \cdots P_n$, where J is an invertible ideal of $\phi(R)$ and $P_1, \ldots, P_n$ are prime ideals of $\phi(R)$. If every nonnil ideal I of R can be written as $I = JP_1 \cdots P_n$, where J is an invertible ideal of R and $P_1, \ldots, P_n$ are prime ideals of R, then R is said to be a *nonnil-ZPUI ring*. Commutative ϕ-ZPUI rings that are in $\mathcal{H}$ are characterized in [12, Theorem 2.9]. Examples of ϕ-ZPUI rings that are not ZPUI rings are constructed in [12, Theorem 2.13]. It is shown in [12, Theorem 2.14] that a ϕ-ZPUI ring is the pullback of a ZPUI domain. It is shown in [12, Theorem 3.1] that a nonnil-ZPUI ring is a ϕ-ZPUI ring. Examples of ϕ-ZPUI rings that are not nonnil-ZPUI rings are constructed in [12, Theorem 3.2]. We call a ring $R \in \mathcal{H}$ a *nonnil-strongly discrete ring* if R has no nonnil prime ideal P such that $P^2 = P$. A ring $R \in \mathcal{H}$ is said to be *nonnil-h-local* if each nonnil ideal of R is contained in at most finitely many maximal ideals of R and each nonnil prime ideal P of R is contained in a unique maximal ideal of R.

Since the class of integral domains is a subset of $\mathcal{H}$, the following result is a generalization of [48, Theorem 2.3].

Theorem 6.1 ([12, Theorem 2.9]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-ZPUI ring;*

(ii) *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R;*

(iii) *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$;*

(iv) *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring.*

In the following result, we show that a nonnil-ZPUI ring is a ϕ-ZPUI ring.

Theorem 6.2 ([12, Theorem 3.1]). *Let $R \in \mathcal{H}$ be a nonnil-ZPUI ring. Then R is a ϕ-ZPUI ring, and hence all the following statements hold:*

(i) *$R/\mathrm{Nil}(R)$ is a ZPUI domain.*

(ii) *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R.*

(iii) *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.*

(iv) *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring.*

(v) *R is a nonnil-strongly discrete nonnil-h-local Prüfer ring.*

Examples of ϕ-ZPUI rings that are not nonnil-ZPUI rings are constructed in the following result.

Theorem 6.3 ([12, Theorem 3.2]). *Let A be a ZPUI domain that is not a Dedekind domain with Krull dimension $n \geq 1$ and quotient field K. Then $R = A(+)K/A \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n which is not a nonnil-ZPUI ring.*

Olberding in [48, Corollary 2.4] showed that for each $n \geq 1$, there exists a ZPUI domain with Krull dimension n. A Dedekind domain is a trivial example of a ZPUI domain. We have the following result.

Theorem 6.4 ([12, Theorem 2.13]). *Let A be a ZPUI domain (i.e. A is a strongly discrete h-local Prüfer domain by* [48, Theorem2.3]*) with Krull dimension $n \geq 1$ and quotient field F, and let K be an extension ring of F (i.e. K is a ring and $F \subseteq K$). Then $R = A(+)K \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n that is not a ZPUI ring.*

In the following result, we show that a ϕ-ZPUI ring is the pullback of a ZPUI domain. A good paper for pullbacks is the article by Fontana [27].

Theorem 6.5 ([12, Theorem 2.14]). *Let $R \in \mathcal{H}$. Then R is a ϕ-ZPUI ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a ZPUI ring that is a subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

7 ϕ-Krull rings

We say that a ring $R \in \mathcal{H}$ is a *discrete ϕ-chained ring* if R is a ϕ-chained ring with at most one nonnil prime ideal and every nonnil ideal of R is principal. Recall from [4] that a ring $R \in \mathcal{H}$ is said to be a ϕ-Krull ring if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i.

Theorem 7.1 ([4, Theorem 3.1]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if $R/\operatorname{Nil}(R)$ is a Krull domain.*

We have the following pullback characterization of ϕ-Krull rings.

Theorem 7.2 ([4, Theorem 3.2]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a Krull subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Example 7.3 ([4, Example 3.3]). Let D be a Krull domain with quotient field K, and let L be a ring extension of K. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-Krull ring which is not a Krull domain.

It is well known [29, Theorem 3.6] that an integral domain R is a Krull domain if and only if R is a completely integrally closed Mori domain. We have a similar characterization for ϕ-Krull rings.

Theorem 7.4 ([4, Theorem 3.4]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if R is a ϕ-completely integrally closed ϕ-Mori ring.*

Theorem 7.5 ([4, Theorem 3.5]). *Let $R \in \mathcal{H}$ be a ϕ-Krull ring which is not zero-dimensional. Then the following statements are equivalent:*

(i) *R is a ϕ-Prüfer ring;*

(ii) *R is a ϕ-Dedekind ring;*

(iii) *R is one-dimensional.*

It is well known that if R is a Noetherian domain, then R' is a Krull domain. In particular, an integrally closed Noetherian domain is a Krull domain. We have the following analogous result for nonnil-Noetherian rings.

Theorem 7.6 ([4, Theorem 3.6]). *Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then $\phi(R)'$ is a ϕ-Krull ring. In particular, if R is a ϕ-integrally closed nonnil-Noetherian ring, then R is a ϕ-Krull ring.*

It is known [40, Problem 8, page 83] that if R is a Krull domain in which all prime ideals of height ≥ 2 are finitely generated, then R is a Noetherian domain. We have the following analogous result for nonnil-Noetherian rings.

Theorem 7.7 ([4, Theorem 3.7]). *Let $R \in \mathcal{H}$ be a ϕ-Krull ring in which all prime ideals of R with height ≥ 2 are finitely generated. Then R is a nonnil-Noetherian ring.*

For a ring $R \in \mathcal{H}$, let ϕ_R denotes the ring-homomorphism $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$. It is well known [29, Proposition 1.9, page 8] that an integral domain R is a Krull domain if and only if R satisfies the following three conditions:

(i) R_P is a discrete valuation domain for every height-one prime ideal P of R;

(ii) $R = \cap R_P$, the intersection being taken over all height-one prime ideals P of R;

(iii) Each nonzero element of R is in only a finite number of height-one prime ideals of R, i.e., each nonzero element of R is a unit in all but finitely many R_P, where P is a height-one prime ideal of R.

The following result is an analog of [29, Proposition 1.9, page 8].

Theorem 7.8 ([4, Theorem 3.9]). *Let $R \in \mathcal{H}$ with* $\dim(R) \geq 1$. *Then R is a ϕ-Krull ring if and only if R satisfies the following three conditions:*

(i) *R_P is a discrete ϕ-chained ring for every height-one prime ideal P of R;*

(ii) *$\phi_R(R) = \cap \phi_{R_P}(R_P)$, the intersection being taken over all height-one prime ideals P of R;*

(iii) *Each nonnilpotent element of R lies in only a finite number of height-one prime ideals of R, i.e., each nonnilpotent element of R is a unit in all but finitely many R_P, where P is a height-one prime ideal of R.*

Recall that a ring R is called a *Marot ring* if each regular ideal of R is generated by its set of regular elements. A Marot ring is called a *Krull ring* in the sense of [38, page 37] if either $R = T(R)$ or if there exists a family $\{V_i\}$ of discrete rank-one valuation rings such that:

(i) R is the intersection of the valuation rings $\{V_i\}$;

(ii) Each regular element of $T(R)$ is a unit in all but finitely many V_i.

The following is an example of a ring $R \in \mathcal{H}$ which is a Krull ring but not a ϕ-Krull ring.

Example 7.9 ([4, Example 3.12]). Let D be a non-Krull domain with (proper) quotient field K. Set $R = D(+)K/D$. Then $R \in \mathcal{H}$ and $R = T(R)$. Hence R is a Krull ring. Since $R/\mathrm{Nil}(R)$ is ring-isomorphic to D, R is not a ϕ-Krull ring by Theorem 7.1.

8 ϕ-Mori rings

According to [46], a ring R is called a *Mori ring* if it satisfies a.c.c. on divisorial regular ideals. Let $R \in \mathcal{H}$. A nonnil ideal I of R is ϕ-divisorial if $\phi(I)$ is a divisorial ideal of $\phi(R)$, and R is a *ϕ-Mori ring* if it satisfies a.c.c. on ϕ-divisorial ideals.

The following is a characterization of ϕ-Mori rings in terms of Mori rings in the sense of [46].

Theorem 8.1 ([17, Theorem 2.2]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if $\phi(R)$ is a Mori ring.*

The following is a characterization of ϕ-Mori rings in terms of Mori domains.

Theorem 8.2 ([17, Theorem 2.5]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if $R/\operatorname{Nil}(R)$ is a Mori domain.*

Theorem 8.3 ([17, Theorem 2.7]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring. Then R satisfies a.c.c. on nonnil divisorial ideals of R. In particular, R is a Mori ring.*

The converse of Theorem 8.3 is not valid as it can be seen by the following example.

Example 8.4 ([17, Example 2.8]). Let D be an integral domain with quotient field L which is not a Mori domain and set $R = D(+)(L/D)$, the idealization of L/D over D. Then $R \in \mathcal{H}$ is a Mori ring which is not a ϕ-Mori ring.

Example 8.18 shows how to construct a nontrivial Mori ring (i.e., where $R \neq T(R)$) in $\mathcal{H}$ which is not ϕ-Mori.

Theorem 8.5 ([17, Theorem 2.10]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Then R is both a ϕ-Mori ring and a Mori ring.*

Given a Krull domain of the form $E = L + M$, where L is a field and M a maximal ideal of E, any subfield K of L gives rise to a Mori domain $D = K + M$. If L is not a finite algebraic extension of K, then D cannot be Noetherian (see [19, Section 4]). We make use of this in our next example to build a ϕ-Mori ring which is neither an integral domain nor a ϕ-Noetherian.

Example 8.6 ([17, Example 2.11]). Let K be the quotient field of the ring $D = \mathbb{Q} + X\mathbb{R}[[X]]$ and set $R = D(+)K$, the idealization of K over D. It is easy to see that $\operatorname{Nil}(R) = \{0\}(+)K$ is a divided prime ideal of R. Hence $R \in \mathcal{H}$. Now since $R/\operatorname{Nil}(R)$ is ring-isomorphic to D and D is a Mori domain but not a Noetherian domain, we conclude that R is a ϕ-Mori ring which is not a ϕ-Noetherian ring.

In light of Example 8.6, ϕ-Mori rings can be constructed as in the following example.

Example 8.7 ([17, Example 2.12]). Let D be a Mori domain with quotient field K and let L be an extension ring of K. Then $R = D(+)L$, the idealization of L over D, is in $\mathcal{H}$. Moreover, R is a ϕ-Mori ring since $R/\operatorname{Nil}(R)$ is ring-isomorphic to D which is a Mori domain.

The following result is a generalization of [54, Theorem 1]. An analogous result holds for Mori rings when the chains under consideration are restricted to regular divisorial ideals whose intersection is regular [46, Theorem 2.22].

Theorem 8.8 ([17, Theorem 2.13]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if whenever $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq \mathrm{Nil}(R)$, then $\{I_m\}$ is a finite set.*

Let D be an integral domain with quotient field K. If I is an ideal of D, then $(D : I) = \{x \in K \mid xI \subseteq D\}$. Mori domains can be characterized by the property that for each nonzero ideal I, there is a finitely generated ideal $J \subset I$ such that $(D : I) = (D : J)$ (equivalently, $I_v = J_v$) ([51, Theorem 1]). Our next result generalizes this result to ϕ-Mori rings.

Theorem 8.9 ([17, Theorem 2.14]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if for any nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$, equivalently, $\phi(J)_v = \phi(I)_v$.*

In the following theorem we combine all of the different characterizations of ϕ-Mori rings stated in this section.

Theorem 8.10 ([17, Corollary 2.15]). *Let $R \in \mathcal{H}$. The following statements are equivalent:*

(i) *R is a ϕ-Mori ring;*

(ii) *$R/\mathrm{Nil}(R)$ is a Mori domain;*

(iii) *$\phi(R)/\mathrm{Nil}(\phi(R))$ is a Mori domain;*

(iv) *$\phi(R)$ is a Mori ring.*

(v) *If $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq \mathrm{Nil}(R)$, then $\{I_m\}$ is a finite set;*

(vi) *For each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$;*

(vii) *For each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)_v = \phi(I)_v$.*

The following result is a generalization of [54, Theorem 5].

Theorem 8.11 ([17, Theorem 3.1]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and I be a nonzero ϕ-divisorial ideal of R. Then I contains a power of its radical.*

We recall a few definitions regarding special types of ideals in integral domains. For a nonzero ideal I of an integral domain D, I is said to be strong if $II^{-1} = I$, strongly divisorial if it is both strong and divisorial, and v-invertible if $(II^{-1})_v = D$. We will extend these concepts to the rings in $\mathcal{H}$.

Let I be a nonnil ideal of a ring $R \in \mathcal{H}$. We say that I is *strong* if $II^{-1} = I$, *ϕ-strong* if $\phi(I)\phi(I)^{-1} = \phi(I)$, *strongly divisorial* if it is both strong and divisorial, *strongly ϕ-divisorial* if it is both ϕ-strong and ϕ-divisorial, *v-invertible* if $(II^{-1})_v = R$

and *ϕ-v-invertible* if $(\phi(I)\phi(I)^{-1})_v = \phi(R)$. Obviously, I is ϕ-strong, strongly ϕ-divisorial or ϕ-v-invertible if and only if $\phi(I)$ is, respectively, strong, strongly divisorial or v-invertible.

In [51, Proposition 1], J. Querré proved that if P is a prime ideal of a Mori domain D, then P is divisorial when it is height one. In the same proposition, he incorrectly asserted that if the height of P is larger than one and P^{-1} properly contains D, then P is strongly divisorial. While it is true that such a prime must be strong, a (Noetherian) counterexample to the full statement can be found in [34]. What one can say is that P_v will be strongly divisorial (see [5]).

Theorem 8.12 ([17, Theorem 3.3]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and P be a (nonnil) prime ideal of R. If* $\operatorname{ht}(P) = 1$*, then P is ϕ-divisorial. If* $\operatorname{ht}(P) \geq 2$*, then either $\phi(P)^{-1} = \phi(R)$ or $\phi(P)_v$ is strongly divisorial.*

For a ϕ-Mori ring $R \in \mathcal{H}$, let $D_m(R)$ denote the *maximal ϕ-divisorial ideals of R*; i.e., the set of nonnil ideals of R maximal with respect to being ϕ-divisorial. The following result generalizes [25, Theorem 2.3] and [19, Proposition 2.1].

Theorem 8.13 ([17, Theorem 3.4]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring such that* $\operatorname{Nil}(R)$ *is not the maximal ideal of R. Then the following hold:*

(a) *The set $\mathcal{D}_m(R)$ is nonempty. Moreover, $M \in \mathcal{D}_m(R)$ if and only if $M/\operatorname{Nil}(R)$ is a maximal divisorial ideal of $R/\operatorname{Nil}(R)$.*

(b) *Every ideal of $\mathcal{D}_m(R)$ is prime.*

(c) *Every nonnilpotent nonunit element of R is contained in a finite number of maximal ϕ-divisorial ideals.*

As with a nonempty subset of R, a nonempty set of ideals $\mathbb{S}$ is *multiplicative* if (i) the zero ideal is not contained in $\mathbb{S}$, and (ii) for each I and J in $\mathbb{S}$, the product IJ is in $\mathcal{S}$. Such a set $\mathbb{S}$ is referred to as a multiplicative system of ideals and it gives rise to a generalized ring of quotients $R_{\mathbb{S}} = \{t \in T(R) \mid tI \subset R \text{ for some } I \in \mathcal{S}\}$. For each prime ideal P, $R_{(P)} = \{t \in T(R) \mid st \in R \text{ for some } s \in R\backslash P\} = R_{\mathbb{S}}$, where $\mathbb{S}$ is the set of ideals (including R) that are not contained in P. Note that in general a localization of a Mori ring need not be Mori (see Example 8.18 below). On the other hand, if $\mathbb{S}$ is a multiplicative system of regular ideals, then $R_{\mathbb{S}}$ is a Mori ring whenever R is Mori ring ([46, Theorem 2.13]).

Theorem 8.14 ([17, Theorem 3.5], and [17, Theorem 2.2]). *Let R be a ϕ-Mori ring. Then*

(a) *R_S is a ϕ-Mori ring for each multiplicative set S.*

(b) *R_P is a ϕ-Mori ring for each prime P.*

(c) *$R_{\mathcal{S}}$ is a ϕ-Mori ring for each multiplicative system of ideals $\mathcal{S}$.*

(d) *$R_{(P)}$ is a ϕ-Mori ring for each prime ideal P.*

One of the well-known characterizations of Mori domains is that an integral domain D is a Mori domain if and only if (i) D_M is a Mori domain for each maximal divisorial ideal M, (ii) $D = \cap D_M$ where the M range over the set of maximal divisorial ideals of D, and (iii) each nonzero element is contained in at most finitely many maximal divisorial ideals ([52, Théorème 2.1] and [54, Théorème I.2]). A similar statement holds for ϕ-Mori rings. Note that in condition (ii), if D has no maximal divisorial ideals, the intersection is assumed to be the quotient field of D. For the equivalence, that means that D is its own quotient field. The analogous statement is that if $\mathcal{D}_m$ is empty, then we have $R = T(R) = R_{\mathrm{Nil}(R)}$ with $\mathrm{Nil}(R)$ the maximal ideal.

Theorem 8.15 ([17, Theorem 3.6]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Mori ring;*

(ii) (a) *R_M is a ϕ-Mori ring for each maximal ϕ-divisorial M,*

(b) *$\phi(R) = \cap \phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and*

(c) *each nonnil element (ideal) is contained in at most finitely many maximal ϕ-divisorial ideals;*

(iii) (a) *$R_{(M)}$ is a ϕ-Mori ring for each maximal ϕ-divisorial M,*

(b) *$\phi(R) = \cap \phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and*

(c) *each nonnil element (ideal) is contained in at most finitely many maximal ϕ-divisorial ideals.*

In [19], V. Barucci and S. Gabelli proved that if P is a maximal divisorial ideal of a Mori domain D, then the following three conditions are equivalent: (1) D_P is a discrete rank-one valuation domain, (2) P is v-invertible, and (3) P is not strong [19, Theorem 2.5]. A similar result holds for ϕ-Mori rings.

Theorem 8.16 ([17, Theorem 3.9]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and $P \in \mathcal{D}_m(R)$. Then the following statements are equivalent:*

(i) *R_P is a discrete rank-one ϕ-chained ring;*

(ii) *P is ϕ-v-invertible;*

(iii) *P is not ϕ-strong.*

Recall from [38] that if $f(x) \in R[x]$, then $c(f)$ denotes the ideal of R generated by the coefficients of $f(x)$, and $R(x)$ denotes the quotient ring $R[x]_S$ of the polynomial ring $R[x]$, where S is the set of $f \in R[x]$ such that $c(f) = R$.

Theorem 8.17 ([17, Theorem 4.5]). *Let R be an integrally closed ring with* $\mathrm{Nil}(R) = Z(R) \neq \{0\}$. *Then the following statements are equivalent:*

(i) *R is ϕ-Mori and the nilradical of $T(R[x])$ is an ideal of $R(x)$;*

(ii) *$R(x)$ is ϕ-Mori;*

(iii) *$R(x)$ is ϕ-Noetherian;*

(iv) *R is ϕ-Noetherian and the nilradical of $T(R[x])$ is an ideal of $R(x)$;*

(v) *Each regular ideal of R is invertible;*

(vi) *$R/\operatorname{Nil}(R)$ is a Dedekind domain;*

(vii) *R is a ϕ-Dedekind ring.*

As mentioned above, a Mori ring is said to be nontrivial if it is properly contained in its total quotient ring. Our next example is of a nontrivial Mori ring that is in the set $\mathcal{H}$ but is not a ϕ-Mori ring.

Example 8.18 ([17, Example 5.3]). Let E be a Dedekind domain with a maximal ideal M such that no power of M is principal (equivalently, M generates an infinite cyclic subgroup of the class group) and let $D = E + xF[x]$, where F is the quotient field of E. Let $\mathcal{P} = \{ND \mid N \in \operatorname{Max}(E)\backslash\{M\}\}$, $B = \sum F/D_{P_\alpha}$ where each $P_\alpha \in \mathcal{P}$, and let $R = D(+)B$. Then the following hold:

(a) If J is a regular ideal, then $J = I(+)B = IR$ for some ideal I that contains a polynomial in D whose constant term is a unit of E. Moreover, the ideal I is principal and factors uniquely as $P_1^{r_1} \cdots P_n^{r_n}$, where the P_i are the height-one maximal ideals of D that contain I.

(b) $R \neq T(R)$ since, for example, the element $(1 + x, 0)$ is a regular element of R that is not a unit.

(c) R is a nontrivial Mori ring but R is not ϕ-Mori.

(d) MR is a maximal ϕ-divisorial ideal of R, but R_{MR} is not a Mori ring.

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Prüfer-like conditions in pullbacks

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Abstract. In this work, we consider five possible extensions of the Prüfer domain concept to arbitrary commutative rings. We investigate the transfer of these Prüfer-like properties to pullbacks, and then generate new families of rings with zero divisors subject to some given Prüfer conditions.

Keywords. Prüfer ring, Gaussian ring, arithmetical ring, weak global dimension, semihereditary ring, trivial ring extension, total ring, pullback.

AMS classification. 13A15, 13D05, 13B02.

Dedicated to Alain Bouvier

1 Introduction

All rings considered in this paper are commutative with identity element and all modules are unital. Prüfer domains were defined in 1932 by H. Prüfer, as domains in which every finitely generated ideal is invertible [30]. In 1936, Krull [27] named these rings in Prüfer's honor and stated equivalent conditions that make a domain Prüfer. Since then, many of these conditions have been extended to the case of rings with zero divisors and gave rise to five classes of Prüfer-like rings ([6] and [7]), namely:

(1) R is semihereditary, i.e., every finitely generated ideal is projective (Cartan–Eilenberg [10]).

(2) The weak global dimension of R is at most one (Glaz [17]).

(3) R is an arithmetical ring, i.e., every finitely generated ideal is locally principal (Fuchs [12] and Jensen [25]).

(4) R is a Gaussian ring, i.e., $c(fg) = c(f)c(g)$ for any polynomials f, g with coefficients in R, where $c(f)$ is the ideal of R generated by the coefficients of f called the content ideal of f (Tsang [33]).

(5) R is a Prüfer ring, i.e., every finitely generated regular ideal is invertible (equivalently, every two generated regular ideal is invertible); (Butts–Smith [8] and Griffin [21]).

In [19] it was proved that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) and examples are given to show that, in general, the implications cannot be reversed.

Recall, from Bazzoni and Glaz [7, Theorem 3.12], that if the total ring of quotients $\operatorname{Tot}(R)$ of R is von Neumann regular, then all the five conditions above are equivalent on R; and a Prüfer ring R satisfies one of the five Prüfer conditions if and only if $\operatorname{Tot}(R)$ satisfies the same condition (see for instance [6, 7]).

The goal of this work is to study the transfer of the Prüfer-like properties to pullbacks. For this purpose, we show in Theorem 2.1 that, if $T = K + M$ is a local ring (which is not a field) where M is its maximal ideal and K its residue field and D is a subring of T/M such that $\mathrm{qf}(D) = T/M$, then $R = D + M$ satisfies one of the five Prüfer conditions if and only if T and D satisfy the same condition.

In this spirit, our main result (Theorem 2.5) studies the transfer of Gaussian property to the general pullbacks, where the ambient local ring (T, M) is principal or satisfies $M^2 = 0$.

A special application of these constructions is the notion of trivial ring extension. Let A be a ring, E an A-module and $R = A \propto E$, the set of pairs (a, e) with $a \in A$ and $e \in E$, under coordinatewise addition and under an adjusted multiplication defined by $(a, e)(a', e') = (aa', ae' + a'e)$, for all $a, a' \in A, e, e' \in E$. Then R is called the trivial ring extension of A by E. It is clear that the trivial ring extension $R = K \propto E$, where K is a field, has the form $R = (K \propto 0) + (0 \propto E)$. Trivial ring extensions have been studied extensively; the work is summarized in Glaz's book [17] and Huckaba's book [24]. These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory (see for instance [17, 24, 26]).

Our results generate new and original examples which enrich the current literature with new families of Prüfer-like rings with zerodivisors.

2 Main results

Throughout this section, we adopt the following riding assumptions and notations: (T, M) is a local ring of the form $T = K + M$ where K is a field, $h : T \to T/M$ is the canonical surjection, D is a subring of K such that $\mathrm{qf}(D) = K$, and $R = D + M$. It is easy to see that M is D-flat (since M is a K-vector space), $T = S^{-1}R$ where $S = D - \{0\}$ and $T = R_M$; in particular T is R-flat. For more details on properties of such pullbacks, see [9, 11, 15].

Theorem 2.1. *Let T, M, D, and R as above. Then:*

(1) *R is a Prüfer ring if and only if T and D are Prüfer rings.*

(2) *R is a Gaussian ring if and only if T and D are Gaussian rings.*

(3) *R is an arithmetical ring if and only if T and D are arithmetical rings.*

(4) $\mathrm{wdim}(R) \leq 1$ *if and only if* $\mathrm{wdim}(T) \leq 1$ *and* $\mathrm{wdim}(D) \leq 1$.

(5) *R is a semihereditary ring if and only if T and D are semihereditary rings.*

The proof uses the following useful lemma.

Lemma 2.2. *Let T, M, D, and R as above. Then R and T have the same total ring of quotients, that is,* $\mathrm{Tot}(R) = \mathrm{Tot}(T)$.

Proof. Let $S_1 = R - Z(R)$ and $S = D - \{0\}$. Let us remark first that $S \subseteq S_1$ and $R - Z(R) \subseteq T - Z(T)$. Indeed, let $d + m \in R$ and $0 \neq d' \in D$ such that

$d'(d+m)=0$. Then $d'd+d'm=0$ implies that $d=0$ and $m=0$ since M, as a flat D-module, is torsion free. Also, let $0\neq r\in R-Z(R)$ and $r'/d\in T=S^{-1}R$ such that $rr'/d=0$. Then there exists an element $t'\in S(\subseteq R-Z(R))$ such that $t'rr'=0$, and so $r'/d=0$. Then, $\mathrm{Tot}(R)=S_1^{-1}R=S_1^{-1}S^{-1}R$ since $S\subseteq S_1$. So, $\mathrm{Tot}(R)=S_1^{-1}T$ (as $T=S^{-1}R$) which is contained in $\mathrm{Tot}(T)$ since $S_1=R-Z(R)\subseteq T-Z(T)$. But the fact that $S_1^{-1}T$ is a total ring of quotients means that $S_1^{-1}T(=\mathrm{Tot}(R))=\mathrm{Tot}(T)$, as desired. □

Proof of Theorem 2.1. (1) Assume that R is a Prüfer ring. By the proof of Lemma 2.2, $S=D-\{0\}\subseteq R-Z(R)$. Hence T, as localization of R, is a Prüfer ring by [5, Theorem 2.1]. Also, from the construction of $R=D+M$, we can see that D is a module retract of R, and so, by [4, Theorem 2.2(1)] D is Prüfer since it is torsion free.

Conversely, assume that D is a Prüfer domain and T is a Prüfer ring. We wish to show that R is a Prüfer ring. For that, let us consider a finitely generated regular ideal I of R and prove that I is invertible. By [6, Theorem 2.5(3)], it suffices to show that I is R-projective, that is, $I\otimes_R T$ is T-projective and $I\otimes_R(R/M)$ is (R/M)-projective by [17, Theorem 5.1.1(1)]. Since T is R-flat, $I\otimes_R T=IT$ which is T-projective since it is a finitely generated regular ideal of T; also, $IT=xT$ for some $x\in T$ as it is a free ideal in the local ring T. On the other hand, $I\otimes_R(R/M)=I/IM\subseteq(IT)/(IMT)=(IT)\otimes_T(T/M)=(xT)\otimes_T K\cong K$. Hence $I\otimes_R(R/M)$ is a D-submodule of $K(=\mathrm{qf}(D))$ which is invertible as a fractional ideal of D and therefore $I\otimes_R(R/M)$ is D-projective by [6, Theorem 2.5(1)]. Hence R is a Prüfer ring.

(2) If R is a Gaussian ring, then so is T since it is a localization of R by [5, Theorem 2.5(1)]. Also, R Gaussian implies R is a Prüfer ring, and by Theorem 2.1(1), D is a Prüfer ring and therefore Gaussian as it is a domain.

Conversely, assume that T and D are Gaussian rings. So R is a Prüfer ring by (1) since T and D are, in particular, Prüfer rings. But $\mathrm{Tot}(T)(=\mathrm{Tot}(R))$ is a Gaussian ring by [7, Theorem 3.12] since T is a Gaussian ring. Therefore, R is a Gaussian ring by [6, Theorem 5.7(1)].

The proof of the assertions (3), (4) and (5) is similar to (2), and this completes the proof of Theorem 2.1. □

The following example ensures the necessity of the condition "$\mathrm{qf}(D)=T/M$" imposed in Theorem 2.1.

Example 2.3. Let $T=K[[X]]=K+M$ be the ring of formal power series over a field K, where $M:=XT$ is the maximal ideal of the valuation domain T, D be a subring of K such that $\mathrm{qf}(D)\neq K$ and $R:=D+M$. Then R is not a Gaussian (Prüfer) domain.

Proof. By [17, Theorem 5.2.10]. □

Now, we can construct a new example of an arithmetical ring R with $\mathrm{wdim}(R)>1$.

Example 2.4. Let D be a valuation domain, $K=\mathrm{qf}(D)$, $T=K[[X]]/(X^n)=K+M$ be a local ring, where $M=XT$ is its maximal ideal and $n>2$ be a positive integer.

Set $R = D + M$. Then:

(1) R is arithmetical.

(2) $\mathrm{wdim}(R) > 1$.

Proof. (1) By Theorem 2.2(3).

(2) Denote by x the image of X in T. We claim that the ideal xR is not flat. Assuming the opposite, the ideal xR is free since R is local, which is a contradiction since $x \cdot x^{n-1} = x^n = 0$. Hence xR is not flat and so $\mathrm{wdim}(R) > 1$. □

The following theorem investigates the transfer of the Gaussian property to the general pullbacks and provides some original examples satisfying this property.

Theorem 2.5. *Let (T, M) be a local ring, $h : T \to T/M$, D a subring of T/M such that* $\mathrm{qf}(D) = T/M$, *and let $R := h^{-1}(D)$. Assume that $M^2 = 0$ or T is principal. Then D is Gaussian if and only if R is Gaussian.*

Before proving Theorem 2.5, we establish the following lemma.

Lemma 2.6. *Let (T, M), D and R as in Theorem 2.5, f be a polynomial of $R[X]$ such that $c(f)T = xT$ for some $x \in M$. Then there exists $x' \in M$ and $g \in R[X]$ such that*

(1) *$f = x'g$ and $c(f) = x'c(g)$,*

(2) *$c(g)T = T$.*

Proof. Let $f = \sum_{i=0}^{n} a_i X^i$ be a polynomial of $R[X]$ such that $c(f)T = xT$ for some $x \in M$. We wish to construct a polynomial $g \in R[X]$ and find an element $x' \in M$ such that $f = x'g$ and $c(f) = x'c(g)$.

(1) We have $a_i \in c(f) \subseteq c(f)T = xT$ for each $i = 0, \ldots, n$. Then there exists $b_i \in R$ and $s_i \in S$ such that $a_i = x(b_i/s_i)$. Thus for $x' = x/(\prod_{i=0}^{n} s_i) \in M$, we have $a_i = x'a'_i$, where $a'_i = (\prod_{j=0, j\neq i}^{n} s_j)b_i \in R$. For $g = \sum_{i=0}^{n} a'_i X^i \in R[X]$, we have $f = x'g$ and so $c(f) = x'c(g)$.

(2) We have $f = x'g$ and $x' \in M$. Our aim is to show that $c(g)T = T$.

We have $x \in xT = c(f)T = x'c(g)T = xc(g)T = xS^{-1}c(g)$ since $xT = x'T$. Hence, $x = xa/s$ for some $a \in c(g)$ and $s \in S$ and so $x((a/s) - 1) = 0$. Therefore, $(a/s) - 1 \in \mathrm{Ann}_T(x) \subseteq M$ (since (T, M) local) and then $a/s \notin M$. This means that a/s is invertible in T since (T, M) is local and so $c(g)T = T$ since $(a/s) \in c(g)T$. □

Proof of Theorem 2.5. If R is Gaussian then, by [5, Theorem 3.1(1)], D is Gaussian as an homomorphic image of R.

Conversely, assume that D is a Gaussian (Prüfer) domain and let f and g be two polynomials of $R[X]$. Our aim is to show that $c(fg) = c(f)c(g)$. Two cases are then possible:

Case 1: $c(f) \subsetneq M$ or $c(g) \subsetneq M$.

Assume for example that $c(f) \subsetneq M$. Hence, it suffices to show that $c(f)$ is locally principal in R (since in this case, f is Gaussian and so $c(fg) = c(f)c(g)$).

Therefore, it suffices to show that $c(f)$ is R-projective, that is $c(f) \otimes_R T$ is T-projective and $c(f) \otimes_R (R/M)$ is R/M-projective by [17, Theorem 5.1.1(1)]. But $c(f) \otimes_R T = c(f)T$ (since T is R-flat) $= T$ is T-projective. On the other hand, $c(f) \otimes_R (R/M) = c(f)/MC(f) = c(f)/MTc(f) = c(f)/MT = c(f)/M$ which is a finitely generated ideal of $R/M(= D)$ which is supposed to be a Prüfer domain, so $c(f) \otimes_R (R/M)$ is R/M-invertible and so projective. Therefore $c(f)$ is R-projective and so $c(fg) = c(f)c(g)$.

If $c(g) \notin M$, the same argument shows that $c(fg) = c(f)c(g)$.

Case 2: $c(f) \subseteq M$ and $c(g) \subseteq M$.

If $M^2 = 0$, then $c(fg) \subseteq c(f)c(g) \subseteq M^2 = 0$, and so $c(fg) = c(f)c(g)(= 0)$.

If T is principal, there exists then $x \in M$ such that $c(f)T = xT$. By Lemma 2.6, there exists $f' \in R[X]$ and $x' \in M$ such that $c(f) = x'c(f')$ and $c(f')T = T$. Hence, $c(f')$ is locally principal by case 1 and so $c(f)$ is locally principal. Therefore f is Gaussian and so $c(fg) = c(f)c(g)$. □

Note that the same Example 2.3 proves the necessity of the condition "$\mathrm{qf}(D) = T/M$" imposed in Theorem 2.5.

Now, we are able to construct a non arithmetical Gaussian ring and an arithmetical ring with $\mathrm{wdim}(R) > 1$.

Corollary 2.7. *Let D be a Prüfer domain, $K = \mathrm{qf}(D)$, E be a nonzero K-vector space and $R = D \propto E$. Then:*

(1) *R is a Gaussian ring.*

(2) *R is an arithmetical ring if and only if* $\dim_K(E) = 1$.

(3) $\mathrm{wdim}(R) > 1$.

Proof. (1) By Theorem 2.5.

(2) Assume that $\dim_K(E) = 1$, we may assume then $E = K$. Set $T = K \propto K$. The ring T is arithmetical since $M := 0 \propto K$ is the unique proper ideal of T. Hence R is an arithmetical ring by Theorem 2.5 since D is a Prüfer domain.

Conversely, assume that $\dim_K(E) \neq 1$. It may be proved similarly as [3, Example 2.3] that T is not arithmetical; and so R is not arithmetical too by Theorem 2.1 since $M := 0 \propto E$ is a common ideal of R and S.

(3) By Theorem 2.1 since $\mathrm{wdim}(T) > 1$ ($T(0,1)(= 0 \propto K)$ is not flat (as T is local and $(0,1)T(0,1) = (0,0)$). □

Now, we are ready to give, using Corollary 2.7, a new class of non arithmetical Gaussian rings.

Example 2.8. Let D be a Prüfer domain, $K = \mathrm{qf}(D)$, E be a nonzero K-vector space such that $\dim_K(E) \neq 1$, and $R = D \propto E$. Then:

(1) R is a Gaussian ring.

(2) R is not an arithmetical ring.

Also we give, by Corollary 2.7, a new class of arithmetical rings with $\mathrm{wdim}(R) > 1$.

Example 2.9. Let D be a Prüfer domain which is not a field, $K = \mathrm{qf}(D)$ and $R = D \propto K$. Then:

(1) R is a an arithmetical ring.

(2) $\mathrm{wdim}(R) > 1$.

Finally, we close this paper by constructing an example of non arithmetical Gaussian rings.

Example 2.10. Let $T = \mathbb{Q}[[X]] \propto (\mathbb{Q}[[X]]/(X)) = (\mathbb{Q} + X\mathbb{Q}[[X]]) \propto (\mathbb{Q}[[X]]/(X))$ and $R = (\mathbb{Z} + X\mathbb{Q}[[X]]) \propto (\mathbb{Q}[[X]]/(X))$. Then:

(1) T is a non arithmetical Gaussian ring.

(2) R is a non arithmetical Gaussian ring.

Proof. (1) By [4, Theorem 2.1(2) and Example 2.6(2)]. (2) Follows from Theorem 2.1 and (1). □

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On propinquity of numerical semigroups and one-dimensional local Cohen Macaulay rings

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Abstract. The paper contains some known and unknown results for numerical semigroups. They show how a numerical semigroup is close to a one-dimensional local Cohen Macaulay ring.

Keywords. Numerical semigroup, canonical ideal, irreducible ideal.

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1 Introduction

This paper deals mainly with numerical semigroups, i.e. subsemigroups of $\mathbb{N}$, with zero and with finite complement in $\mathbb{N}$. The literature on this subject is wide and rich. However our interest is focused on those aspects of the theory which are more related to commutative ring theory. If S is a numerical semigroup and k is a field, the semigroup ring $k[[S]]$ inherits several properties of S. More generally, the behaviour of ideals in numerical semigroups is very close to that of ideals in one-dimensional local Cohen-Macaulay rings or, even more generally, in local rings possessing an m-canonical ideal (cf. [1]). The main original results of the paper are Theorems 4.2 and 5.2. Theorem 4.2 shows how a principal integral ideal of a numerical semigroup is in a unique way a finite intersection of t irreducible integral ideals, where t is the type of the semigroup. It is not surprising that this is stronger than the corresponding result for one-dimensional local Cohen Macaulay rings, where only the number of components is invariant. Moreover, Proposition 4.4 points out how the decomposition of a principal ideal in irreducible relative ideals is completely different. Theorem 5.2 characterizes almost symmetric semigroups. Its ring theoretic version appears in [1], but the proof for numerical semigroups is more direct. Almost symmetric semigroups and their corresponding rings, the almost Gorenstein rings were introduced in [3] and recently used in [8]. The paper contains also several results which are in large part well known. However for some of them an explicit proof is not published anywhere or exists only in ring theoretic version. The arguments for a numerical semigroup are not always an additive version of the arguments for the corresponding ring. Sometimes they are simpler or different. The paper includes a theorem due to Marco La Valle, which gives a partial answer to the Wilf problem.

2 Generalities

We fix for all the paper the following notation. S is a *numerical semigroup*, i.e. a subsemigroup of $\mathbb{N}$, with zero and with finite complement in $\mathbb{N}$. The numerical semigroup generated by $d_1, \dots, d_h \in \mathbb{N}$ is $S = \langle d_1, \dots, d_h \rangle = \{\sum_{i=1}^{h} n_i d_i;\ n_i \in \mathbb{N}\}$. $M = S \setminus \{0\}$ is the *maximal ideal* of S, e is the *multiplicity* of S, that is the smallest positive integer of S, g is the *Frobenius number* of S, that is the greatest integer which does not belong to S.

A *relative ideal* of S is a nonempty subset I of $\mathbb{Z}$ (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal which is contained in S is an *integral ideal* of S.

If I, J are relative ideals of S, then the following are relative ideals too:

$$I + J = \{i + j;\ i \in I, j \in J\},$$
$$I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\},$$
$$I \cap J,$$
$$I \cup J.$$

If $z \in \mathbb{Z}$, $z + S = \{z + s;\ s \in S\}$ is the principal relative ideal generated by z and it is easy to check that $I - (z + S) = I - z = \{i - z;\ i \in I\}$.

Moreover the ideal generated by $z_1, \dots, z_h$ is

$$(z_1 + S) \cup \cdots \cup (z_h + S).$$

Proposition 2.1. *If I, J are relative ideals of S, then*

$$I \subseteq J - (J - I) = \bigcap_{I \subseteq z+J} (z + J).$$

Proof. The first inclusion is trivial. To show the equality, assume $x \in J - (J - I)$, i.e. $x + (J - I) \subseteq J$ and let $z \in \mathbb{Z}$ such that $I \subseteq z + J$, i.e. $-z \in J - I$, then $x - z \in J$, i.e. $x \in z + J$ and so $J - (J - I) \subseteq \bigcap_{I \subseteq z+J}(z + J)$. Conversely, if $x \in z + J$ for each $z \in \mathbb{Z}$ such that $I \subseteq z + J$, i.e. for each z such that $-z \in J - I$, then $x + y = x - z \in J$ for each $y \in J - I$, that is $x \in J - (J - I)$. □

In particular we have

$$I \subseteq S - (S - I) = \bigcap_{I \subseteq z+S} (z + S)$$

and, if $I = S - (S - I)$, I is *bidual*.

If I is a relative ideal of S, and $s \in S$, $s \neq 0$, then $\mathrm{Ap}_s(I) = I \setminus (s + I)$ is the set of the s smallest elements in I in the s congruence classes mod s and is called the *Apery set* of I (with respect to s). In particular $\mathrm{Ap}_e(S)$ is the Apery set of S with respect to the multiplicity e. Since g is the greatest gap of S, $g + s$ is the largest element in $\mathrm{Ap}_s(S)$.

The following lemma, corresponds to Nakayama's lemma for local rings. For numerical semigroups the proof is very easy.

Lemma 2.2. *If I is a relative ideal of S, then the unique minimal set of generators of I is $I \setminus (M + I)$.*

Proof. An element $x \in I$ is superfluous as generator of I if and only if $x = z + s$, for some $z \in I$ and some $s \in S$, $s \neq 0$. Hence x is superfluous if and only if $x \in M + I$ □

Since $e + I \subseteq M + I$, then $I \setminus (M + I) \subseteq I \setminus (e + I) = \mathrm{Ap}_e(I)$ and by Lemma 2.2 each relative ideal I of S needs at most e generators.

In particular $M \setminus 2M$ is the minimal set of generators of M and its cardinality is the *embedding dimension* ν of S. Of course a minimal set of generators of M coincide with a minimal set of generators of $S = M \cup \{0\}$. We have $\nu \leq e$ and if equality holds the semigroup S is called of *maximal embedding dimension*.

3 Canonical ideal

A particular relative ideal of S plays a special role. It is the *canonical ideal* $\Omega = \{g - x;\ x \in \mathbb{Z} \setminus S\}$. Thus, calling an integer z *symmetric* to x if $z = g - x$, Ω consists of the integers which are symmetric to the gaps of the semigroup. The contents of the following lemma can be found in [7] and [3].

Lemma 3.1. (i) $S \subseteq \Omega \subseteq \mathbb{N}$.

(ii) *For each relative ideal I of S, $\Omega - I = \{g - x;\ x \in \mathbb{Z} \setminus I\}$.*

(iii) *If $I \subseteq J$ are relative ideals of S, then* $\mathrm{Card}(J \setminus I) = \mathrm{Card}((\Omega - I) \setminus (\Omega - J))$.

Proof. (i) If $s \in S$ then $x = g - s \notin S$, so $s = g - (g - s) = g - x \in \Omega$, hence $S \subseteq \Omega$. Moreover, if $z \in \mathbb{Z}$, $z < 0$, then $g - z > g$, so $g - z \in S$ and $z \notin \Omega$, hence $\Omega \subseteq \mathbb{N}$.

In order to prove (ii), we show that, for each relative ideal I of S,

$$x \in I \Rightarrow g - x \notin \Omega - I,$$
$$x \notin I \Rightarrow g - x \in \Omega - I.$$

In fact, let $x \in I$. If $g - x \in \Omega - I$, then $g - x + I \subseteq \Omega$. In particular $g - x + x = g \in \Omega$, a contradiction.

Now let $x \notin I$. If $g - x \notin \Omega - I$, then $g - x + i \notin \Omega$, for some $i \in I$. It follows that $g - x + i$ is symmetric to an element of S, i.e. $g - (g - x + i) = x - i \in S$, hence $x \in i + S \subseteq I$, a contradiction.

(iii) Let $z \in \mathbb{Z}$. By (ii), $z \in J \setminus I$ if and only if $g - z \in (\Omega - I) \setminus (\Omega - J)$, thus the two sets are in one to one correspondence and they have the same cardinality. □

Example 3.2. Consider the following semigroup:

$$S = \{0, 4, 7, 8, 11, 12, \rightarrow\} = \langle 4, 7, 13\rangle.$$

Here the Frobenius number g is 10, the multiplicity e is 4 and the embedding dimension ν is 3. The canonical ideal is

$$\Omega = \{0, 1, 4, 5, 7, 8, 9, 11, \rightarrow\}.$$

The role of the canonical ideal Ω is well known (cf. [7]) and recalled in next proposition, point (1). Points (2) and (3) are proved for rings in [6].

Proposition 3.3. (1) *For each relative ideal I of S, $\Omega - (\Omega - I) = I$. In particular $\Omega - (\Omega - S) = \Omega - \Omega = S$.*

(2) *For each set $\{I_h\}_{h\in H}$ of relative ideals of S,*

$$\Omega - \bigcap_{h\in H} I_h = \bigcup_{h\in H} (\Omega - I_h).$$

(3) *Ω is an irreducible relative ideal, i.e., Ω is not the intersection of any set of relative ideals properly containing Ω.*

Proof. (1) Let $z \in \Omega - (\Omega - I)$. Then by Lemma 3.1 (ii) $z = g - y$, with $y \notin \Omega - I$. Since $y = g - x$, for some $x \in I$ (again by Lemma 3.1 (ii)), we have $z = g - y = g - (g - x) = x \in I$. The opposite inclusion is by Proposition 2.1.

(2) Set $\bigcup_{h\in H}(\Omega - I_h) = U$. For each ideal I_{h_0} of the set $\{I_h\}_{h\in H}$, we have $(\Omega - I_{h_0}) \subseteq U$, thus $(\Omega - U) \subseteq \Omega - (\Omega - I_{h_0}) = I_{h_0}$, so $(\Omega - U) \subseteq \bigcap_{h\in H} I_h$. It follows that $\Omega - \bigcap_{h\in H} I_h \subseteq \Omega - (\Omega - U) = U$. The other inclusion is trivial.

(3) Assume that Ω is the intersection of a set of relative ideals, $\Omega = \bigcap_{h\in H} I_h$. We want to show that $\Omega = I_{h_0}$ for some $h_0 \in H$.

Applying (2) we have that $S = \Omega - \Omega = \Omega - \bigcap_{h\in H} I_h = \bigcup_{h\in H}(\Omega - I_h)$, thus $\Omega - I_{h_0} = S$, for some $h_0 \in H$ and so $I_{h_0} = \Omega - (\Omega - I_{h_0}) = \Omega - S = \Omega$. □

Properties (1) and (3) characterize Ω in the following sense:

Proposition 3.4. *Let Ω' be a relative ideal of S. Then*

$$\Omega' - (\Omega' - I) = I$$

for each relative ideal I of S if and only if $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$.

Proof. Since

$$\Omega' - (\Omega' - \Omega) = \Omega = \bigcap_{\Omega\subseteq z+\Omega'} (z + \Omega')$$

and Ω is irreducible, we have $\Omega = z + \Omega'$, for some $z \in \mathbb{Z}$. The converse is trivial. □

Proposition 3.5. *Let Ω' be a relative ideal of S. Then Ω' is an irreducible relative ideal if and only if $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$.*

Proof. Since

$$\Omega' = \Omega - (\Omega - \Omega') = \bigcap_{\Omega' \subseteq z+\Omega} (z + \Omega)$$

and Ω' is irreducible, we have $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$. The converse is trivial. □

We now show how the Apery set of S is strictly connected to the Apery set of Ω This fact was observed for $s = e$ in [11].

Proposition 3.6. *Let $s \in S$, $s \neq 0$. If $\mathrm{Ap}_s(S) = \{p_0, \ldots, p_{s-1}\}$, where $p_0 = 0 < p_1 < \cdots < p_{s-1}$, then $\mathrm{Ap}_s(\Omega) = \{p_{s-1} - p_{s-1}, p_{s-1} - p_{s-2}, \ldots, p_{s-1} - p_0\}$.*

Proof. Recall that $g + s = p_{s-1}$.

We have $p_i \in \mathrm{Ap}_s(S) = S \setminus (s + S)$. Thus

$$p_i \in S \;\Rightarrow\; g - p_i \notin \Omega \;\Rightarrow\; g + s - p_i = p_{s-1} - p_i \notin \Omega + s.$$

On the other hand

$$p_i \notin S + s \;\Rightarrow\; p_i - s \notin S \;\Rightarrow\; g - (p_i - s) = g + s - p_i \in \Omega.$$

Therefore, for each i, $p_{s-1} - p_i \in \Omega \setminus (s + \Omega) = \mathrm{Ap}_s(\Omega)$. □

Remark 3.7. (a) Note that by Proposition 3.6, $\mathrm{Ap}_s(S)$ can be obtained from $\mathrm{Ap}_s(\Omega)$ in a similar way.

(b) Proposition 3.6 shows in particular that the biggest element in $\mathrm{Ap}_s(\Omega)$ is the same as the biggest element in $\mathrm{Ap}_s(S)$, it is in fact $p_{s-1} = g + s$.

4 Type and decompositions of principal ideals in irreducible ideals

The *type* t of a numerical semigroup S is the minimal number of generators of the canonical ideal, that is $t = \mathrm{Card}(\Omega \setminus (\Omega + M))$.

Consider the partial order on S given by

$$s_1 \preceq s_2 \;\Leftrightarrow\; s_1 + s_3 = s_2, \;\; \text{for some } \; s_3 \in S. \qquad (\star)$$

If $s \in S$, $s \neq 0$, the number of maximal elements in $\mathrm{Ap}_s(S)$, with respect to the order $(\star)$ turns out to be the type of the semigroup (cf. [5]). We include this fact in the following proposition, which is all well known, cf. [5].

Proposition 4.1. *Let t be the type of S. Then*

(i) $t = \mathrm{Card}((S - M) \setminus S)$.

(ii) *If $s \in S$, $s \neq 0$, then $x \in (S - M) \setminus S$ if and only if $x + s$ is maximal in $\mathrm{Ap}_s(S)$ with respect to the order $(\star)$.*

(iii) $1 \leq t \leq e - 1$.

Proof. (i) Applying 3.1 (iii) and Proposition 3.3 (1), we get $t = \mathrm{Card}(\Omega \setminus (\Omega + M)) = \mathrm{Card}(\Omega - (\Omega + M) \setminus (\Omega - \Omega)) = \mathrm{Card}(((\Omega - \Omega) - M) \setminus S) = \mathrm{Card}((S - M) \setminus S)$.

(ii) If $x \in (S - M) \setminus S$ then $x \notin S$, but $x + m \in S$ for each $m \in M$. In particular $x \notin S$ and $x + s \in S$, thus $x + s \in S \setminus (s + S) = \mathrm{Ap}_s(S)$. Moreover, for each $u \in M$, both $x + u$ and $x + s + u$ are in S, so that $x + s + u \notin \mathrm{Ap}_s(S)$, this means that $x + s$ is maximal in $\mathrm{Ap}_s(S)$, with respect to the order $(\star)$. Conversely, if $x + s$ is maximal in $\mathrm{Ap}_s(S)$ then $x \notin S$, $x + s \in S$ and $x + s + m \notin \mathrm{Ap}_s(S) = S \setminus (s + S)$, for each $m \in M$. Since $x + s + m \in S$, this last condition is equivalent to $x + s + m \in s + S$, i.e. to $x + m \in S$, for each $m \in M$. Thus $x \in (S - M) \setminus S$.

(iii) $\mathrm{Ap}_e(S)$ has e elements and the element $0 \in \mathrm{Ap}_e(S)$ is not maximal. Thus there are at most $e - 1$ maximal elements, so that $t \leq e - 1$ by (ii). □

It is well known that, if (R, m) is a one-dimensional local Cohen Macaulay ring, and $r \in m$ is a non-zerodivisor, then the number of components in a decomposition of the principal ideal rR as irredundant intersection of irreducible integral ideals is invariant and independent on r. This number is in fact the CM-type of the ring R. In case of numerical semigroups not only the number of components is invariant, but such irredundant intersection is unique. This is shown in next theorem.

If $s \in S$, set $B(s) = \{z \in S \mid z \preceq s\}$. It turns out (cf. [10]) that $I = S \setminus B(s)$ is an irreducible integral ideal of S, i.e., I is not the intersection of any set of integral ideals properly containing I. As a matter of fact each integral ideal containing I contains s.

Theorem 4.2. *Let $s \in S$, $s \neq 0$ and let $p_{i_1}, \ldots, p_{i_t}$ be the maximal elements in $\mathrm{Ap}_s(S)$ with respect to the order $(\star)$. Setting, for $j = 1, \ldots, t$, $I_{i_j} = S \setminus B(p_{i_j})$, then*

$$I_{i_1} \cap \cdots \cap I_{i_t}$$

is the unique irredundant decomposition of the principal ideal $s + S$ in irreducible integral ideals.

Proof. Observe first that, for each element $p \in \mathrm{Ap}_s(S)$, $B(p) \subseteq \mathrm{Ap}_s(S)$. In fact, if $z \in S$ and $z \preceq p$, i.e. $z + u = p \in \mathrm{Ap}_s(S)$, for some $u \in S$, then $z + u \notin s + S$. It follows that $z - s + u \notin S$ and so $z - s \notin S$. Thus $z \in \mathrm{Ap}_s(S)$. Therefore the ideals I_{i_j} are integral irreducible ideals containing $s + S$. If $p \in \mathrm{Ap}_s(S)$, then $p \preceq p_{i_j}$, for some maximal element p_{i_j}, so $p \notin I_{i_j}$. Thus we have an equality $s + S = \bigcap_{j=1}^{t} I_{i_j}$. To show that the intersection is irredundant, observe that each component is necessary: deleting I_{i_h}, we have that $\bigcap_{j \neq h} I_{i_j}$ contains p_{i_h} and is not equal to $s + S$.

We have to show that the decomposition is unique. Suppose that $s + S = \bigcap_{\alpha=1}^{m} I_\alpha$, where I_α are irreducible integral ideals. Consider a maximal element of $\mathrm{Ap}_s(S)$, say p_{i_1}. We have that $p_{i_1} \notin I_\alpha$, for some α, otherwise $p_{i_1} \in s + S = \bigcap_{\alpha=1}^{m} I_\alpha$, say $p_{i_1} \notin I_1$. So $q \notin I_1$, for each $q \preceq p_{i_1}$. Moreover, if $q \in \mathrm{Ap}_s(S)$, $q \npreceq p_{i_1}$, then $q \in I_1$, otherwise $I_1 = (I_1 \cup \{p_{i_1}\}) \cap (I_1 \cup E(q))$, where $E(q) = \{z \in S \mid q \preceq z\}$ and I_1 is not irreducible, a contradiction. Therefore $I_1 = S \setminus B(p_{i_1}) = I_{i_1}$. In the same way, for each maximal element p_{i_j} of $\mathrm{Ap}_s(S)$, we get the unique irreducible integral ideal I_{i_j}. □

Example 4.3. (a) Let $S = \{0, 5, 6, 8, 10, \rightarrow\} = \langle 5, 6, 8\rangle$. If we take $s = e = 5$, the maximal elements in $\mathrm{Ap}_5(S) = \{0, 6, 8, 12, 14\}$ are 12 and 14 and the corresponding irreducible integral ideals are

$$I_1 = S \setminus B(12) = S \setminus \{0, 6, 12\} = \{5, 8, 10, 11, 13, \rightarrow\} = (5 + S) \cup \{8, 14\},$$
$$I_2 = S \setminus B(14) = S \setminus \{0, 6, 8, 14\} = \{5, 10, 11, 12, 13, 15, \rightarrow\} = (5 + S) \cup \{12\}.$$

Hence $I_1 \cap I_2$ is the unique irredundant decomposition of the principal ideal $5 + S$ in irreducible integral ideals.

(b) If in the same semigroup $S = \langle 5, 6, 8\rangle$, we take $s = 6$, we get 13 and 15 as maximal elements in $\mathrm{Ap}_6(S)$. So the corresponding ideals are

$$J_1 = S \setminus B(13) = (6 + S) \cup \{10, 15\},$$
$$J_2 = S \setminus B(15) = (6 + S) \cup \{8, 13\}$$

and $J_1 \cap J_2$ is the unique irredundant decomposition of the principal ideal $6 + S$ in irreducible integral ideals.

We show now that the type t of a semigroup S is also the number of components of an irredundant intersection of a principal ideal in irreducible relative ideals. A similar ring theoretic result is in [4, Proposition 2.6].

Proposition 4.4. (i) *If Ω is minimally generated by $z_1, \ldots, z_t$, then*

$$S = (\Omega - z_1) \cap \cdots \cap (\Omega - z_t)$$

is the unique irredundant decomposition of S in irreducible relative ideals.

(ii) *Each relative principal ideal of S is in a unique way an irredundant intersection of t irreducible relative ideals.*

Proof. (i) We have:

$$\Omega - S = \Omega = \bigcup_{i=1}^{t} (z_i + S) = \bigcup_{i=1}^{t} (z_i + (\Omega - \Omega)) = \bigcup_{i=1}^{t} (\Omega - (\Omega - z_i)) = \Omega - \bigcap_{i=1}^{t} (\Omega - z_i)$$

where we applied Proposition 3.3 (2) for the last equality. So that, dualizing

$$S = (\Omega - (\Omega - S)) = \bigcap_{i=1}^{t} (\Omega - z_i).$$

Moreover the intersection is irredundant: if $\bigcap_{i \neq h} (\Omega - z_i) \subseteq (\Omega - z_h)$, then $z_h \in \Omega - (\Omega - z_h) \subseteq \Omega - \bigcap_{i \neq h} (\Omega - z_i) = \bigcup_{i \neq h} (z_i + S)$, a contradiction with the minimality of the set of generators for Ω. By Proposition 3.5 the components of the intersection are irreducible relative ideals. To show that such a decomposition is unique, recall that, again by Proposition 3.5, each irreducible relative ideal is of the form $y + \Omega$, for some $y \in \mathbb{Z}$. Thus, if $S = \bigcap_i (y_i + \Omega)$ then, by Proposition 3.3 (2),

$$\Omega = \Omega - S = \bigcup_i (\Omega - (y_i + \Omega)) = \bigcup_i ((\Omega - \Omega) + (y_i + S)) = \bigcup_i (-y_i + S).$$

Thus $\{-y_i\}$ is a set of generators for Ω. Requesting that the decomposition is irredundant, $\{-y_i\}$ has to be the minimal set of generators of Ω.

(ii) Since, by (i), $S = (\Omega - z_1) \cap \cdots \cap (\Omega - z_t)$, if $z \in \mathbb{Z}$, we have also

$$z + S = (z - z_1 + \Omega) \cap \cdots \cap (z - z_t + \Omega)$$

where each component of the intersection is an irreducible relative ideal by Proposition 3.5. □

Example 4.5. Consider again the semigroup $S = \{0, 5, 6, 8, 10, \rightarrow\} = \langle 5, 6, 8 \rangle$. Here the canonical ideal $\Omega = \{0, 2, 5, 6, 7, 8, 10, \rightarrow\}$ is minimally generated by 0 and 2, in fact $\Omega \setminus (\Omega + M) = \{0, 2\}$. Applying Proposition 4.4 (i), we get $S = \Omega \cap (\Omega - 2)$. Thus, taking for example the principal ideal $5 + S$, we have that

$$5 + S = (5 + \Omega) \cap (3 + \Omega)$$

is its unique decomposition in irreducible relative ideals.

5 Almost symmetric semigroups

If $t = 1$ or equivalently $\Omega = S$, then the numerical semigroup S is classically called *symmetric*.

S is *almost symmetric* if $\Omega \subseteq S - M$, equivalently if $t = (g + 1 - 2n) + 1$ (cf. [3]).

Example 5.1. If $S = \langle 5, 8, 9, 12 \rangle = \{0, 5, 8, 9, 10, 12, \rightarrow\}$, then

$$\Omega = \{0, 4, 5, 7, 8, 9, 10, 12, \rightarrow\},$$

thus $\Omega \subseteq S - M = \{0, 4, 5, 7, \rightarrow\}$ and S is almost symmetric.

Theorem 5.2. *The following conditions are equivalent for a numerical semigroup S of maximal ideal M:*

(i) *S is almost symmetric.*

(ii) *Each ideal of $M - M$ is bidual as ideal of S.*

(iii) *$M - e$ is the canonical ideal of $M - M$.*

Proof. If $S = \mathbb{N}$, the three conditions trivially hold. Suppose $S \subsetneq \mathbb{N}$, so that $M - M = S - M$ is a semigroup which properly contains S.

(i) $\Rightarrow$ (ii). Let I be an ideal of $M - M$. Then: $\Omega + I \subseteq (M - M) + I \subseteq I$. Since $0 \in \Omega$, also the opposite inclusion $\Omega + I \supseteq I$ holds, thus $\Omega + I = I$.

If I is not bidual in S then $I \subsetneq S - (S - I) \subseteq \Omega - (S - I)$, hence

$$S - I \subsetneq \Omega - I = \Omega - (\Omega + I) = (\Omega - \Omega) - I = S - I,$$

a contradiction.

(ii) $\Rightarrow$ (iii). We have to prove that $M - (M - I) \subseteq I$, for each ideal I of $M - M$. First notice that

$$M - I = S - I.$$

If $M - I \subsetneq S - I$, then $z + I \subseteq S$ and $z + I \not\subseteq M$ for some $z \in \mathbb{Z}$, thus $z + I = S$ and S is an ideal of $M - M$, a contradiction. Thus

$$M - (M - I) \subseteq S - (M - I) = S - (S - I) = I.$$

By Proposition 3.4 a translation $M + z$ of M, for some $z \in \mathbb{Z}$ is the canonical ideal of $M - M$. To obtain 0 as minimal element in $z + M$, the right choice is $z = -e$.

(iii) $\Rightarrow$ (i). Suppose S is not almost symmetric, i.e., there is $b \in \mathbb{Z}$, $b \notin M$ (a gap of S), $g - b \notin M$ (b is symmetric to another gap of S), $b \notin M - M$.

The Frobenius number of $M - M$ is $g - e$. We claim that $x = g - e - b$ is in the canonical ideal of $M - M$ but is not in $M - e$. In fact:

$$(g - e) - x = (g - e) - (g - e - b) = b \notin M - M$$

since $(g - e) - x$ is a gap of $M - M$, x is in the canonical ideal of $M - M$. On the other hand $x = g - e - b \notin M - e$, because $g - b \notin M$. □

Example 5.3. If $S = \langle 5, 8, 9, 12\rangle = \{0, 5, 8, 9, 10, 12, \rightarrow\}$ then

$$M - M = \{0, 4, 5, 7, \rightarrow\}$$

is a semigroup of Frobenius number 6 and the canonical ideal of $M - M$ is

$$M - e = M - 5 = \{0, 3, 4, 5, 7, \rightarrow\}.$$

By Theorem 5.2, we reobtain a result of [3].

Corollary 5.4. *The following conditions are equivalent for a numerical semigroup S of maximal ideal M:*

(i) *S is almost symmetric of maximal embedding dimension.*

(ii) *$M - M$ is a symmetric semigroup.*

Proof. Assume $S \neq \mathbb{N}$, to avoid the trivial case. Recall that a semigroup S is of maximal embedding dimension if and only if $M - M = M - e$ (cf. [2, Proposition I.2.9]). Thus, if S is almost symmetric of maximal embedding dimension, then by Theorem 5.2 ((i) $\Rightarrow$ (iii)) $M - M$ is a symmetric semigroup because coincides with its canonical ideal. Conversely, if $M - M$ is a symmetric semigroup, then each ideal I of $M - M$ is bidual, thus I is bidual also as ideal of S and by Theorem 5.2 ((ii) ⇒ (i)) S is almost symmetric. Moreover, by (iii) of Theorem 5.2, $M - e$ is the canonical ideal of $M - M$, hence $M - e = M - M$ and S is of maximal embedding dimension. □

Wilf in [12] asked whether the inequality $g + 1 \leq n\nu$ always holds in a numerical semigroup. The following is a partial positive answer. The proof sketched here (cf. [9]) is due to Marco La Valle, an old student of mine, who does not work anymore in mathematics. I consider his result interesting and worthwhile to publish.

Theorem 5.5 (M. La Valle). *If S is an almost symmetric semigroup, then*

$$g + 1 \leq n\nu.$$

Proof. Since S is almost symmetric, $t = g + 2 - 2n$, thus $n = \frac{g-t}{2} + 1$. Hence the Wilf inequality becomes

$$g + 1 \leq \Big(\frac{g-t}{2} + 1\Big)\nu$$

or equivalently

$$2 + \nu t \leq (\nu - 2)g + 2\nu.$$

It is enough to prove

$$\nu t \leq (\nu - 2)g$$

equivalently

$$\nu(g - t) \geq 2g.$$

Since $t < e$ (cf. Proposition 4.1 (iii)), it is enough to prove

$$\nu(g - e) \geq 2g \quad \text{that is} \quad \nu \geq \frac{2g}{g - e}.$$

Case 1. If $e < g/2$, then $g - e > g/2$ and

$$\frac{2g}{g - e} < \frac{2g}{g/2} = 4.$$

Since, for $\nu \leq 3$, Wilf's inequality is easily checked, in this case the theorem is proved.

Case 2. if $e = g + 1$, it is easy: $\nu = e$ and $n = 1$ thus $g + 1 = e \leq 1e$.

Case 3. If $g/2 < e < g$, we have to compute the number ν of (necessary) generators of S.

We have at least: $n - 1$ generators smaller than g, $2e - (g + 1)$ generators between $g + 1$ and $2e$, $(e + s) - 2e - 1$ generators between $2e$ and $e + s$, where $S = \{s_0 = 0, s_1 = e, s_2 = s, \dots (s_i < s_{i+1})\}$. Therefore:

$$\nu \geq (n - 1) + (2e - g - 1) + (s - e - 1) = n + e + s - g - 3$$

and

$$\nu n \geq n^2 + ne + ns - ng - 3n = f(n).$$

We claim that

$$f(n) \geq g + 1.$$

By induction on $n \geq 3$. For $n = 3$, it is

$$f(n) = 3(e + s - g).$$

But for $n = 3$ an almost symmetric semigroup is of one of the following forms:

$$S = \{0, g - 3, g - 1, g + 1, \rightarrow\},$$
$$S = \{0, g - 2, g - 1, g + 1, \rightarrow\}.$$

In the first case

$$f(3) = 3g - 12 \geq g + 1 \Leftrightarrow g \geq 7.$$

In the second case

$$f(3) = 3g - 9 \geq g + 1 \Leftrightarrow g \geq 5.$$

Thus, taking care of a finite small number of easy cases ($g < 7$ in the first case and $g < 5$ in the second case) the inductive hypothesis is verified.

Now the inductive step:

$$f(n + 1) - f(n) = (2n - 2) + (s + e - g) > 0.$$

So if $f(n) > g + 1$, then also $f(n + 1) > g + 1$. □

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n-perfectness in pullbacks

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Abstract. A ring is called n-perfect ($n \geq 0$), if every flat module has projective dimension at most n. The n-perfect rings have a homological characterization using the cotorsion global dimension of rings, to the effect that R is n-perfect if and only if R has cotorsion global dimension at most n. This paper continues the investigation of n-perfectness initiated by the authors for pullback constructions. It leads to further examples of n-perfect rings and allows to compute the cotorsion global dimension of some special rings. A result involving flatness in pullbacks is also stated.

Keywords. n-perfect ring, cotorsion dimension, pullback, flat epimorphism.

AMS classification. 13D02, 13D05, 13D07, 13B10, 13B30, 13B02.

Dedicated to Alain Bouvier

1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unitary. For a ring R and an R-module M, we use $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective and flat dimensions of M. It is convenient to use "m-local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal m.

A commutative square of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R' \end{array} \qquad (\square)$$

is said to be a pullback square, if given $r_1 \in R_1$ and $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there exists a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. The ring R is called a pullback of R_1 and R_2 over R'. we shall refer to the diagram ($\square$) as a pullback diagram of type ($\square$).

The much useful particular case of pullback rings are constructed as follows: Let I be an ideal of a ring T. A subring D of the quotient ring T/I is of the form R/I where R is a subring of T, which contained I as an ideal. Then, R is a pullback ring of T and D over T/I issued from the following pullback diagram of canonical homomorphisms:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi_{|R}} & D = R/I \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow[\pi]{} & T/I \end{array}$$

Following [2], R is called the ring of the (T, I, D) construction. This construction includes the well-known $D + M$-construction and in general the $D + I$-construction (for more details about these constructions, please see [2, 4, 8, 9, 11]).

These constructions have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. In the same direction, the pullback rings are used in [1] to give some interesting examples of particular n-perfect rings. Recall that a ring is called n-perfect ($n \geq 0$), if every flat module has projective dimension at most n [7]. These rings have a homological characterization using their cotorsion global dimension, such that the cotorsion global dimension of a ring R, $\mathrm{Cgldim}(R)$, is the supremum of the cotorsion dimensions of all R-modules, where the cotorsion dimension of an R-module M, $\mathrm{cd}_R(M)$, is the least positive integer n for which $\mathrm{Ext}_R^{n+1}(F, C) = 0$ for all flat R-modules F [5]. Namely, the modules of cotorsion dimension 0 are the known cotorsion modules (see [16, Definition 3.1.1]). For a positive integer n, we have [5, Theorem 19.2.5(1)]:

$$\mathrm{Cgldim}(R) \leq n \text{ if and only if } R \text{ is } n\text{-perfect.}$$

In this paper, we continue the investigation of n-perfectness in more general context of pullbacks. Namely, in our main result (Theorem 2.1), we study the transfer of n-perfectness in the following pullback diagram. In other words, we compute the cotorsion global dimension of the pullback ring R defined bellow in terms of its associated rings T and D:

Let T be a ring of the form $S + I$, where S is a subring of T and I is a nonzero ideal of T such that $S \cap I = 0$. Consider a ring homomorphism $\phi : D \to S$. Then, we may show, via ϕ, that I is a D-module. Thus, we may define on the D-module $R := D \oplus_D I$ a structure of ring with multiplication given by: $(r, i)(r', i') = (rr', ri' + r'i + ii')$ for $r, r' \in R$ and $i, i' \in I$. Then, R is a pullback ring of T and D over S associated to the following pullback diagram:

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & D = R/I \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ T & \xrightarrow[\beta]{} & S = T/I \end{array} \qquad (\square\square)$$

where α and β are the natural surjections and φ is defined by: $\varphi((d, i)) = \phi(d) + i$ for $d \in D$ and $i \in I$. We shall refer to the diagram ($\square\square$) as a pullback diagram of type

(□□). This construction generalizes the $D + I$-construction in which T is assumed to be an integral domain, I is a prime ideal of T, and φ is the natural injection. Then, the construction (□□) enriches the commutative ring theory by new examples of particular pullback rings (see Corollaries 2.6 and 2.9 and Examples 2.7, 2.8, 2.10, and 2.11).

2 Main results

First, recall that a ring homomorphism $\phi : R \to S$ is called a flat epimorphism of R, if S is a flat R-module and ϕ is an epimorphism; that is, for any two ring homomorphisms $S \overset{f}{\underset{g}{\rightrightarrows}} T$, T a ring, satisfying $f\phi = g\phi$ we have $f = g$. We may also say that S is a flat epimorphism of R (please see [11, pages 13–14] and [14]). Particularly, for every multiplicative set W of R, $W^{-1}R$ is a flat epimorphism of R. Also, the quotient ring R/I is a flat epimorphism of R for every pure ideal I of R; that is, R/I is a flat R-module [11, Theorem 1.2.15].

The main result is the following generalization of [1, Theorem 5.1].

Theorem 2.1. *Consider a pullback diagram of type* (□□). *Let n be a positive integer. If T and D are n-perfect, then R is n-perfect.*

Furthermore, if ϕ is a flat epimorphism of D, then we get an equivalence: T and D are n-perfect if and only if R is n-perfect.

In other words, $\mathrm{Cgldim}(R) = \sup\{\mathrm{Cgldim}(T), \mathrm{Cgldim}(D)\}$.

The proof of the theorem involves the following results.

Lemma 2.2. *Consider a pullback diagram of type* (□) *and assume that j_2 is surjective. Then, for any flat R-module F, we have*

$$\mathrm{pd}_R(F) = \sup\{\mathrm{pd}_{R_1}(F \otimes_R R_1), \mathrm{pd}_{R_2}(F \otimes_R R_2)\}.$$

Proof. Consider an exact sequence of R-modules

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to F \to 0,$$

where each P_j, for $j = 0, \ldots, n-1$, is projective. Since F is flat, we obtain the following exact sequences of R_i-modules for $i = 1$ and 2:

$$0 \to P_n \otimes_R R_i \to P_{n-1} \otimes_R R_i \to \cdots \to P_0 \otimes_R R_i \to F \otimes_R R_i \to 0.$$

From [15, Chapter 2], P_n is an R-module projective if and only if $P_n \otimes_R R_i$ is an R_i-module projective for $i = 1$ and 2. This implies the desired equality. □

The following is a generalization of [1, Lemma 5.2].

Lemma 2.3. *Consider a pullback diagram of type* (□) *and assume that j_2 is surjective. Then, for a positive integer n, R is n-perfect if R_1 and R_2 are n-perfect.*

In other words, $\mathrm{Cgldim}(R) \leq \sup\{\mathrm{Cgldim}(T), \mathrm{Cgldim}(D)\}$.

Proof. Follows from Lemma 2.2. □

Recall that, for a ring homomorphism $\psi : R \to S$, we say that R is a subring retract of S, if there exists a ring homomorphism $\phi : S \longrightarrow R$ satisfying $\phi\psi = \mathrm{id}_{|R}$. In this case, ψ is injective and the R-module S contains R as a direct summand [11, page 111].

Lemma 2.4 ([1, Lemma 3.4]). *Let R be a subring retract of a ring S. If S is n-perfect for some positive integer n, then R is n-perfect.*

The following is a generalization of [1, Lemma 3.5], [5, Proposition 19.3.1(1)], and [16, Proposition 3.3.3].

Lemma 2.5. *For any ring homomorphism $\phi : R \to S$, and any S-module M, we have the inequality* $\mathrm{cd}_R(M) \leq \mathrm{cd}_S(M)$.

Furthermore, if ϕ is a flat epimorphism, then we have equality: $\mathrm{cd}_R(M) = \mathrm{cd}_S(M)$. *Consequently, if ϕ is a flat epimorphism, then* $\mathrm{Cgldim}(S) \leq \mathrm{Cgldim}(R)$. *In other words, if R is n-perfect for some positive integer n, then S is n-perfect.*

Proof. We may assume that $\mathrm{cd}_S(M) = n$ for some positive integer n. We have $\mathrm{Tor}_p^R(S, F) = 0$ for every $p > 0$ and every flat R-module F. Thus, we may apply [3, Proposition 4.1.3] which gives $\mathrm{Ext}_R^{n+1}(F, M) \cong \mathrm{Ext}_S^{n+1}(F \otimes_R S, M) = 0$. Therefore, $\mathrm{cd}_R(M) \leq n$.

Now, suppose that ϕ is a flat epimorphism. From the first inequality, it remains to prove the inequality $\mathrm{cd}_S(M) \leq \mathrm{cd}_R(M)$. For that, we may assume that $\mathrm{cd}_R(M) = n$ for some positive integer n. Let F be a flat S-module (then it is also flat as an R-module). We have:

$$\begin{aligned} S \otimes_R F \cong S \otimes_R (S \otimes_S F) &\cong (S \otimes_R S) \otimes_S F \\ &\cong S \otimes_S F && \text{from [11, Theorem 1.2.19]} \\ &\cong F. \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{Ext}_S^{n+1}(F, M) &= \mathrm{Ext}_S^{n+1}(S \otimes_R F, M) \\ &\cong \mathrm{Ext}_R^{n+1}(F, M) && \text{from [3, Proposition 4.1.3]} \\ &= 0 && \text{since } \mathrm{cd}_R(M) = n. \end{aligned}$$

Therefore, $\mathrm{cd}_S(M) \leq n$, as desired. □

Proof of Theorem 2.1. The first implication follows immediately from Lemma 2.3.

Now, if we assume that ϕ is a flat epimorphism of D, then φ is a flat epimorphism of R (by [13, page 13]). Also, we have that D is a subring retract of R. Thus, the converse implication is a simple consequence of Lemmas 2.4 and 2.5. □

As an application of Theorem 2.1, we set the following first (general) example of a pullback ring satisfying the hypotheses of Theorem 2.1.

Recall the trivial extension of R by an R-module M is the ring denoted by $R \propto M$ whose underling group is $A \times M$ with multiplication given by $(r, m)(r', m') = (rr', rm' + r'm)$ (see for instance [11, Chapter 4, Section 4] and [10]).

Corollary 2.6. *Let $f : D \to S$ be a ring homomorphism and let M be an S-module. For a positive integer n, if $T := S \propto M$ and D are n-perfect, then $R := D \propto M$ is n-perfect.*

Furthermore, if f is a flat epimorphism of D, then we get an equivalence: T and D are n-perfect if and only if R is n-perfect.

Proof. Note that $R \propto M$ is a pullback ring of R and $S \propto M$ over S associated to the following pullback diagram of type (□□):

$$\begin{array}{ccc} R \propto M & \twoheadrightarrow & R \\ \downarrow & & \downarrow f \\ S \propto M & \twoheadrightarrow & S \end{array}$$

Then, the result is a simple application of Theorem 2.1. □

In [10, Proposition 1.15], we have that R is perfect if and only if $R \propto M$ is perfect. In the following examples we extend this result to n-perfect rings in particular cases.

Example 2.7. Let R be a ring, let M be an R-module, and let n be a positive integer. For a multiplicative set U of R, we have: R and $U^{-1}R \propto U^{-1}M$ are n-perfect if and only if $R \propto U^{-1}M$ is n-perfect.

Particularly:

(1) If R is a domain with quotient field Q and $U = R \setminus \{0\}$, then R is n-perfect if and only if $R \propto U^{-1}M$ is n-perfect.

(2) If R is von Neumann regular, then, for every prime ideal p of R, R is n-perfect if and only if $R \propto M_p$ is n-perfect.

Proof. Use Corollary 2.6, [10, Proposition 1.15], and the fact that $U^{-1}R$ is a flat epimorphism of R. □

Example 2.8. Let R be a ring, let M be an R-module, and let n be a positive integer. We have: For every ideal I of R such that $IM = 0$, if $R/I \propto M$ and R are n-perfect, then $R \propto M$ is n-perfect.

Particularly, if $I = \mathrm{Ann}(M)$ is maximal, then R is n-perfect if and only if $R \propto M$ is n-perfect.

Namely, for every maximal ideal m of R, R is n-perfect if and only if $R \propto R/m$ is n-perfect.

Proof. Use Corollary 2.6 and [10, Proposition 1.15]. □

Now, similarly to Corollary 2.6, we give the second (general) example of a pullback ring satisfying the hypothesis of Theorem 2.1.

Corollary 2.9. *Let $f : D \to S$ be a ring homomorphism. For a positive integer n, if $T := S[X]$ and D are n-perfect, then the ring $R := D \oplus_D XS[X]$ (with multiplication defined as in the pullback diagram of type (□□)) is n-perfect.*

Furthermore, if f is a flat epimorphism of D, then we get an equivalence: T and D are n-perfect if and only if R is n-perfect.

Also, as Examples 2.7 and 2.8, we give two concrete examples of the preceding construction. Note that, from [1, Theorem 4.1], every polynomial ring in one indeterminate $R[X]$ over a ring R may not be 0-perfect.

Example 2.10. Let R be a ring and let n be a positive integer. For a multiplicative set U of R, we have: $U^{-1}R[X]$ and R are n-perfect if and only if $R \oplus_R X(U^{-1}R[X])$ is n-perfect.

Particularly:

(1) If R is a domain with quotient field Q and $U = R \setminus \{0\}$, then, for $n \geq 1$, R is n-perfect if and only if $R \oplus_R X(U^{-1}R[X])$ is n-perfect.

(2) If R is von Neumann regular, then, for every prime ideal p of R and $n \geq 1$, R is n-perfect if and only if $R \oplus_R XR_p[X]$ is n-perfect.

Example 2.11. Let R be a ring, let I be an ideal of R, and let n be a positive integer. We have: if $R/I[X]$ and R are n-perfect, then $R \oplus_R X(R/I[X])$ is n-perfect.

Particularly, if I is maximal, then, for $n \geq 1$, R is n-perfect if and only if $R \oplus_R X(R/I[X])$ is n-perfect.

Finally, we turn to a question involving flatness in pullbacks. Namely, we give a generalization of [6, Theorem 3.4].

Theorem 2.12. *Consider the following pullback diagram:*

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi_{|R}} & D = R/M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\pi} & K = T/M \end{array}$$

where M is a maximal ideal of T, i and j are the natural injections, and π is the natural surjection.

(1) *If $K = \mathrm{qf}(D)$, then T is a flat epimorphism of R, where $\mathrm{qf}(D)$ denotes the quotient field of D.*

(2) *If either T is M-local or K is a subring retract of T, then $K = \mathrm{qf}(D)$ if T is a flat R-module and M contains a regular element m.*

Proof. (1) Assume that $K = \mathrm{qf}(D)$, then j is a flat epimorphism of D, and so is i (by [13, page 13]), as desired.

(2) Now suppose that T is a flat R-module and assume that $\mathrm{qf}(D) \subsetneq K$. Then, there exists $y \in K \setminus \{0\}$ with $yD \cap D = 0$, hence:

$$\pi^{-1}(yD) \cap \pi^{-1}(D) = \pi^{-1}(yD \cap D) = M. \qquad (*)$$

With either of the hypotheses, we can choose an invertible element x of T which satisfies $\pi(x) = y$. Indeed, if T is M-local then every $x \in T$ such that $\pi(x) = y$ is invertible (since $y \neq 0$ implies $x \in T \setminus \{M\}$). And if there exists a ring homomorphism $\phi : K \to T$ such that $\pi\phi = 1_K$, then we can choose $x = \phi(y)$.

Thus, $\pi^{-1}(yD) = xR + M = xR$, and the equality $(*)$ becomes $xR \cap R = M$. Then, since m is regular, we obtain easily that $xmR \cap mR = mM$. Now, from [11, Theorem 1.2.7(1)], we have $(xmR \cap mR)T = xmRT \cap mRT = xmT \cap mT = mT$ (since x is invertible in T). Then, $mMT = (mxR \cap mR)T = mT$. So, from Nakayama's lemma, there exist an element m' of M such that $(1 - m')m = 0$. Then, since m is regular, $m' = 1 \in M$, which is absurd, and therefore $K = \mathrm{qf}(D)$. □

Let T and R be as in Theorem 2.12 above. If T is flat as an R-module and M contains a regular element m, then it is a localization of R under each of the conditions: T is M-local or K is a subring retract of T, as shown by the following, which shows that Theorem 2.12 is also a generalization of [12, Proposition 3.2].

Corollary 2.13. *Consider the pullback diagram of Theorem 2.12, such that M contains a regular element.*

(1) *If T is a flat R-module and M-local, then $T = R_M$.*

(2) *If T is a flat R-module and K is a subring retract of T, then $T = W^{-1}R$, where $W = D \setminus \{0\}$.*

Proof. (1) From Theorem 2.12 (2), $K = \mathrm{qf}(D)$. Then, T is a flat epimorphism R (by Theorem 2.12 (1)). Therefore, by [11, Theorem 1.2.21(4)], $R_M = T_M = T$.

(2) Assume that K is a subring retract of T, then $T = K + M$ and so $R = D + M$. Therefore, $T = W^{-1}R$, where $W = D \setminus \{0\}$ (since, by Theorem 2.12 (2), $\mathrm{qf}(D) = K$). □

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On flatness of factor rings and Krull dimension of tensor products

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Abstract. This paper mainly discusses Sharma's paper [12] on Krull dimension of tensor products of commutative rings. Actually, it characterizes the family of algebras A over an arbitrary ring R such that $\frac{A}{P}$ is (faithfully) flat over R for each prime ideal P of A and thus proves that the main theorem of [12] is trivial. As an alternative result to Sharma's theorem, we provide a satisfactory lower bound of the Krull dimension of $A \otimes_R B$ in terms of geometrical invariants of the R-algebras A and B and the connections intertwining their respective algebra structures over R. Finally, we compute $\dim(A \otimes_R B)$ in the case where either A or B is a field.

Keywords. Krull dimension, faithfully flat module, prime ideal, AF-domain, tensor product.

AMS classification. 13C15, 13B24, 13F05.

Dedicated to Alain Bouvier

1 Introduction

All rings and algebras considered in this paper are commutative with identity element and all ring homomorphisms are unital. All standard notations and definitions are as in [2] and [12]. Throughout, we denote by $\mathrm{Spec}(R)$ (resp., $\mathrm{Max}(R)$) the set of prime ideals (resp., maximal ideals) of a ring R. Also, we use $k_R(p)$ to denote the quotient field of $\frac{R}{p}$ for each prime ideal p of R. Further, given a ring R and an R-algebra A, we denote by $\lambda_A : R \longrightarrow A$, with $\lambda_A(r) := r.1_A$ for each $r \in R$, the associated ring homomorphism defining the R-algebra structure on A, and if $P \in \mathrm{Spec}(A)$, when no confusion is likely, we denote by the prime ideal p_A of R the inverse image $\lambda_A^{-1}(P)$.

It is well known that if I is an ideal of a ring R and S is a multiplicative subset of R such that $I \cap S = \varnothing$, then there exists a prime ideal P of R such that $I \subseteq P$ and $P \cap S = \varnothing$. This is a useful tool for finding prime ideals of R with specified properties. Motivated by this, in [2], G. Bergman led a profound analysis of the following problem: Let J be a partially ordered set and R be a ring. Let $\{(I_j, S_j)\}_{j\in J}$ be a collection of pairs where I_j is an ideal and S_j is a multiplicative subset of R such that $I_j \cap S_j = \varnothing$.

Under what conditions does there exist $P_j \in \mathrm{Spec}(R)$, for all $j \in J$, such that $P_j \supseteq I_j$, $P_j \cap S_j = \varnothing$ satisfying $P_j \subseteq P_r$ whenever $j \leq r \in J$?

As an application, some theorems of [2] are used to prove the well-known result stating that for any two algebras A and B over a field k,

$$\dim(A \otimes_k B) \geq \dim(A) + \dim(B).$$

In [12], P. Sharma discusses [2] and proves the following theorem that stands for the main result of [12]:

> If R is a ring and A, B are two R-algebras such that $\frac{A}{P}$ and $\frac{B}{Q}$ are faithfully flat R-algebras whenever $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$, then, by [12, Theorem 2.5],
> $$\dim(A \otimes_R B) \geq \dim(A) + \dim(B).$$

This theorem yields the following result: If R is a zero-dimensional quasilocal ring, then for any two R-algebras A, B, by [12, Corollary 2.6],

$$\dim(A \otimes_R B) \geq \dim(A) + \dim(B).$$

The principal goal of the present paper is to lead an in-depth study of Sharma's paper [12], precisely its major theorem. First, we seek the algebras A over a ring R satisfying the strong hypotheses of this theorem. In this regard, we prove that, given a ring R and an R-algebra A, $\frac{A}{P}$ is faithfully flat over R whenever $P \in \mathrm{Spec}(A)$ if and only if R is a field (cf. Proposition 2.1). In other words, the above-mentioned theorem of [12] turns out to be merely the well-known result on $\dim(A \otimes_k B)$ for a field k. Also, we characterize the algebras A over a domain R satisfying the condition $\frac{A}{P}$ is flat over R for each prime ideal P of A. In section 3, we give an alternative result to Sharma's theorem by providing a satisfactory lower bound of $\dim(A \otimes_R B)$ in terms of geometrical invariants of A and B and of the connections intertwining their respective algebra structures over R, and compute $\dim(A \otimes_R B)$ in the case where either A or B is a field. First, Proposition 3.1 determines the algebras A and B over a ring R such that $A \otimes_R B \neq \{0\}$. The main theorem of this section states the following:

> Let R be a nonzero ring and A, B be two R-algebras such that $A \otimes_R B \neq \{0\}$. Then
> $$\dim(A \otimes_R B) \geq \max \Big\{ \mathrm{ht}\Big(\frac{P}{pA}\Big) + \mathrm{ht}\Big(\frac{Q}{pB}\Big) : P \in \mathrm{Spec}(A) \text{ and } Q \in \mathrm{Spec}(B) \text{ such that } \lambda_A^{-1}(P) = \lambda_B^{-1}(Q) := p \Big\}.$$

By virtue of this theorem, we prove that if R is a nonzero zero-dimensional ring and A, B are R-algebras such that $A \otimes_R B \neq \{0\}$, then

$$\dim(A \otimes_R B) \geq \max \Big\{ \mathrm{ht}(M) + \mathrm{ht}(M') : M \in \mathrm{Max}(A) \text{ and } M' \in \mathrm{Max}(B) \text{ such that } \lambda_A^{-1}(M) = \lambda_B^{-1}(M') \Big\},$$

extending [12, Corollary 2.6].

2 Flatness of factor rings

This section discusses Sharma's main theorem in [12]. More precisely, we characterize the algebras A over a ring R such that $\frac{A}{P}$ is (faithfully) flat over R for each prime ideal

P of A. Proposition 2.1 allows to prove that [12, Theorem 2.5] turns out to be merely the well-known result $\dim(A \otimes_k B) \geq \dim(A)+\dim(B)$ for two algebras A and B over a field k.

First, it is convenient to remind the reader of the following natural ring isomorphisms related to tensor products of algebras over an arbitrary ring R that will be used throughout this paper: Let R be a ring and let A, B be two R-algebras. Let $\mathbf{a}$ be an ideal of R and S a multiplicative subset of R. Let I be an ideal of A and J an ideal of B. Then

$$\begin{cases} S^{-1}R \otimes_R A \cong S^{-1}A, \\ \frac{R}{\mathbf{a}} \otimes_R A \cong \frac{A}{\mathbf{a}A}, \\ \frac{A\otimes_R B}{I\otimes_R B + A\otimes_R J} \cong \frac{A}{I} \otimes_R \frac{B}{J}, \end{cases}$$

where $I \otimes_R B$ (resp., $A \otimes_R J$) denotes the canonical image of $I \otimes_R B$ (resp., $A \otimes_R J$) in $A \otimes_R B$.

Our first result proves that Sharma's main theorem in [12] is trivial.

Proposition 2.1. *Let R be a nonzero ring and A a nonzero R-algebra. Then the following assertions are equivalent:*

(1) *$\frac{A}{P}$ is faithfully flat over R for each prime ideal P of A;*

(2) *$\frac{A}{M}$ is faithfully flat over R for each maximal ideal M of A;*

(3) *R is a field.*

Proof. Clearly, (3) $\Rightarrow$ (1) $\Rightarrow$ (2). Now, assume that (2) holds. Choose $x \neq 0 \in R$. Let M be a maximal ideal of A and $k(M) := \frac{A}{M}$. Since faithfully flat maps are injective (see [10, Theorem 7.5 (i)]), the image of x in $k(M)$ is nonzero, so a unit in $k(M)$. Hence $\frac{R}{xR} \otimes_R k(M) \cong \frac{k(M)}{xk(M)} = \{0\}$. So, by faithful flatness, we get $\frac{R}{xR} = \{0\}$. Therefore, $xR = R$, so that x is a unit in R. Consequently R is a field, establishing (3), as desired. □

Remark 2.2. We present here, for convenience, an alternate proof of (2) $\Rightarrow$ (3) of Proposition 2.1.

Consider the given ring homomorphism $\lambda_A : R \to A$, with $M \in \operatorname{Max}(A)$ and $p := \lambda_A^{-1}(M)$. As, by faithful flatness, the map $\lambda_{\frac{A}{M}}^{-1} : \operatorname{Spec}(\frac{A}{M}) \to \operatorname{Spec}(R)$ is surjective, we get $\operatorname{Spec}(R) = \{p\}$ is a singleton. Since $\frac{A}{M}$ is faithfully flat over R, we can conclude that $\frac{R}{p}$ is faithfully flat over R because

$$\frac{R}{p} \otimes_R \frac{A}{M} \cong \frac{A/M}{p(A/M)} = \frac{A/M}{(0)} \cong \frac{A}{M}$$

is faithfully flat over $\frac{A}{M}$. But faithfully flat maps are injective [10, Theorem 7.5 (i)], and so the canonical map $R \to \frac{R}{p}$ is an isomorphism. It follows that R is a field, as desired.

In light of Proposition 2.1, for a given ring R and an R-algebra A, it is then natural to ask when $\frac{A}{P}$ is flat over R for each prime ideal P of A. The following result characterizes such a case. It is worthwhile to note that the case $A = R$ of Proposition 2.3 is included in [8, Remark 2.6(e)].

Proposition 2.3. *Let R be a nonzero ring and K its total quotient ring. Suppose that K is a Von Neumann regular ring. Let A be a nonzero R-algebra. Then the following statements are equivalent:*

(1) *$\frac{A}{I}$ is flat over R for each ideal I of A;*

(2) *$\frac{A}{P}$ is flat over R for each prime ideal P of A;*

(3) *$\lambda_A^{-1}(P)$ is a minimal prime ideal of R for each prime ideal P of A;*

(4) *$\lambda_A^{-1}(M)$ is a minimal prime ideal of R for each maximal ideal M of A;*

(5) *A is a K-algebra.*

The proof relies on the following easy result. First, given a ring R and an R-algebra A, recall that the going-down theorem holds between R and A if for each pair $p \subseteq p'$ of prime ideals of R and for each prime ideal P' of A lying over p', there exists $P \in \mathrm{Spec}(A)$ such that $P \subseteq P'$ and $\lambda_A^{-1}(P) = p$.

Lemma 2.4. *Let R be a nonzero ring and A a nonzero R-algebra. Let $P \in \mathrm{Spec}(A)$ and $p := \lambda_A^{-1}(P)$. If $\frac{A}{P}$ is flat over R, then p is a minimal prime ideal of R.*

Proof. Assume that $\frac{A}{P}$ is flat over R. Then, applying [10, Theorem 9.5], the going-down theorem holds between R and $\frac{A}{P}$. Notice that $\lambda_{\frac{A}{P}} = s \circ \lambda_A$, where s is the canonical surjection $s : A \to \frac{A}{P}$. Therefore $\lambda_{\frac{A}{P}}^{-1}((\bar{0})) = \lambda_A^{-1}(s^{-1}((\bar{0}))) = \lambda_A^{-1}(P) = p$ is a minimal prime ideal of R. □

Proof of Proposition 2.3. (1) ⇒ (2) and (3) ⇒ (4) are straightforward. (2) ⇒ (3) holds by Lemma 2.4.

(4) ⇒ (5). Assume that (4) holds. Let $\mathbf{Z}$ denote the set of zero-divisor elements of R and let $r \in R \setminus \mathbf{Z}$. Then, in particular, r does not lie in any minimal prime ideal p of R. Thus, applying (4), for each maximal ideal M of A, we get $\lambda_A(r) \notin M$. Hence $\lambda_A(r)$ is invertible in A. Applying [11, definition, p. 97], it follows, by the universal property of the localization $K := S^{-1}R$ of R, where $S := R \setminus \mathbf{Z}$, that there exists a ring homomorphism φ such that the following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{i} & K \\ \lambda_A \searrow & & \swarrow \varphi \\ & A & \end{array}$$

where i is the canonical injective ring homomorphism. Hence A is a K-algebra. Thus (5) holds.

(5) ⇒ (1). Suppose that (5) holds. Then $\frac{A}{I}$ is flat over K for each ideal I of A since K is Von Neumann regular. Being a localization of R, K is flat over R. Hence,

by transitivity of flatness, $\frac{A}{I}$ is flat over R for each ideal I of A (cf. [10, p. 46]), establishing (1). This completes the proof. $\square$

3 Krull dimension of tensor products

The main goal of this section is to give an alternative result to Sharma's main theorem of [12]. In fact, we seek a satisfactory lower bound of $\dim(A \otimes_R B)$ in terms of geometrical invariants of A and B and of the connections intertwining their respective algebra structures over R. This enables us to provide a family of algebras A and B over a ring R satisfying the inequality $\dim(A \otimes_R B) \geq \dim(A)+\dim(B)$. Also, we compute $\dim(A \otimes_R B)$ in the case where either A or B is a field.

Throughout this section, we use $\text{t.d.}(A : k)$ to denote the transcendence degree of an algebra A over a field k (for nondomains, $\text{t.d.}(A : k) := \sup\left\{\text{t.d.}(\frac{A}{p} : k) : p \in \text{Spec}(A)\right\}$).

First, it is worthwhile pointing out that, in the context of computing the Krull dimension of tensor products of algebras, the case which has been most deeply investigated is when R is a field k. The initial impetus for these investigations was a result of R. Sharp in [13], and Grothendieck some 10 years earlier, that, for any two extension fields K and L of k, by [13, Theorem 3.1],

$$\dim(K \otimes_k L) = \min\Big(\text{t.d.}(K : k),\ \text{t.d.}(L : k)\Big).$$

Subsequently, A. Wadsworth extended this result to AF-domains. Recall that a domain A that is a k-algebra of finite transcendence degree over k is said to be an AF-domain if it satisfies the altitude formula over k, that is,

$$\text{ht}(p) + \text{t.d.}\Big(\frac{A}{p} : k\Big) = \text{t.d.}(A : k)$$

for all prime ideals p of A. The class of AF-domains contains the most basic rings of algebraic geometry, including finitely generated k-algebras that are domains. Let us recall at this point some notation. For a k algebra A and integers $0 \leq d \leq s$, put

$$D(s, d, A) := \max\left\{\text{ht}(p[X_1, \dots, X_s]) + \min\Big(s, d + \text{t.d.}\Big(\frac{A}{p} : k\Big)\Big) : p \in \text{Spec}(A)\right\},$$

where $X_1, X_2, \dots, X_s$ are indeterminates over A. Wadsworth proved that if A_1 and A_2 are AF-domains, then, by [14, Theorem 3.8],

$$\dim(A_1 \otimes_k A_2) = \min\Big(\dim(A_1) + \text{t.d.}(A_2 : k),\ \text{t.d.}(A_1 : k) + \dim(A_2)\Big).$$

He also stated a formula for $\dim(A \otimes_k B)$ which holds for an AF-domain A, with no restriction on B; namely, by [14, Theorem 3.7],

$$\dim(A \otimes_k B) = D\Big(\text{t.d.}(A : k), \dim(A), B\Big)$$

(this formula holds even when $\text{t.d.}(B : k) = \infty$; see [14, p. 400]).

Recent developments on height and grade of (prime) ideals as well as on dimension theory in tensor products of algebras over a field k are to be found in [3], [4], [5] and [6].

Our first result characterizes the algebras A and B over a ring R such that $A \otimes_R B \neq \{0\}$. Recall that if A is an R-algebra and $P \in \mathrm{Spec}(A)$, then we denote by p_A the inverse image $\lambda_A^{-1}(P)$, where $\lambda_A : R \to A$ is the canonical ring homomorphism.

Proposition 3.1. *Let R be a nonzero ring and A, B be two nonzero R-algebras. Then*

(a) *Let $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$. Then*

$$\lambda_A^{-1}(P) = \lambda_B^{-1}(Q) \Longrightarrow \frac{A}{P} \otimes_R \frac{B}{Q} \neq \{\bar{0}\}.$$

(b) *$A \otimes_R B \neq \{0\}$ if and only if there exists $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $\lambda_A^{-1}(P) = \lambda_B^{-1}(Q)$.*

Proof. Both assertions follow from [9, Corollaire 3.2.7.1 (i), p. 235]. □

The following result is a direct consequence of Proposition 3.1.

Corollary 3.2. *Let R be a nonzero ring and K, L be two fields that are R-algebras. Let $\lambda_K : R \longrightarrow K$ and $\lambda_L : R \longrightarrow L$ be the two ring homomorphisms defining the R-algebra structures on K and L, respectively, and let $p := \mathrm{Ker}(\lambda_K)$, $q := \mathrm{Ker}(\lambda_L)$. Then $K \otimes_R L \neq \{0\}$ if and only if $p = q$.*

Next, we extend the above-mentioned Sharp's theorem [13, Theorem 3.1] and, partially, Wadsworth's theorem [14, Theorem 3.7] to the general context of tensor products of algebras over an arbitrary ring R.

Theorem 3.3. *Let R be a nonzero ring. Let K, L be two fields that are R-algebras and A a nonzero R-algebra. Let $p := \lambda_K^{-1}(0)$. Then*

(1) *Assume that $K \otimes_R L \neq \{0\}$. Then*

$$\dim(K \otimes_R L) = \min\Big(\mathrm{t.d.}\big(K : k_R(p)\big),\ \mathrm{t.d.}\big(L : k_R(p)\big)\Big).$$

(2) *Assume that $K \otimes_R A \neq \{0\}$ and $\mathrm{t.d.}\big(K : k_R(p)\big)$ is finite. Then*

$$\dim(K \otimes_R A) = D\Big(\mathrm{t.d.}\big(K : k_R(p)\big), 0, \overline{S}^{-1}\frac{A}{pA}\Big),$$

where $\overline{S} := \frac{R}{p} \setminus \{\bar{0}\}$.

We need the following technical lemma.

Lemma 3.4. *Let R be a nonzero ring and A an R-algebra. Let p be a prime ideal of R. Then*

$$k_R(p) \otimes_R A \cong \overline{S}^{-1} \frac{A}{pA},$$

where $\overline{S} := \frac{R}{p} \setminus \{\overline{0}\}$.

Proof. It is straightforward from the next natural ring isomorphisms:

$$\begin{aligned} k_R(p) \otimes_R A &\cong k_R(p) \otimes_{\frac{R}{p}} \left(\frac{R}{p} \otimes_R A\right) \\ &\cong k_R(p) \otimes_{\frac{R}{p}} \frac{A}{pA} \\ &\cong \overline{S}^{-1} \frac{A}{pA}, \end{aligned}$$

as desired. □

Proof of Theorem 3.3. (1) First, observe that, by Corollary 3.2, $\lambda_K^{-1}(0) = \lambda_L^{-1}(0) = p$, as $K \otimes_R L \neq \{0\}$. Then, we may view $\frac{R}{p}$ as a common subring of K and L, so that $k_R(p)$ is a common subfield of K and L. Hence

$$\begin{aligned} K \otimes_R L &\cong K \otimes_{k_R(p)} (k_R(p) \otimes_R L) \\ &\cong K \otimes_{k_R(p)} \overline{S}^{-1} \frac{L}{pL} && \text{by Lemma 3.4} \\ &\cong K \otimes_{k_R(p)} L. \end{aligned}$$

Thus, applying [13, Theorem 3.1], we get the formula.

(2) Notice that, by virtue of Lemma 3.4,

$$K \otimes_R A \cong K \otimes_{k_R(p)} (k_R(p) \otimes_R A) \cong K \otimes_{k_R(p)} \overline{S}^{-1} \frac{A}{pA}.$$

Then, as $K \otimes_R A \neq \{0\}$, $\overline{S}^{-1} \frac{A}{pA} \neq \{\overline{0}\}$. Therefore, by [14, Theorem 3.7],

$$\dim(K \otimes_R A) = D\Big(\text{t.d.}\Big(K : k_R(p)\Big), 0, \overline{S}^{-1} \frac{A}{pA}\Big),$$

as desired. □

Next, we announce the main theorem of this section.

Theorem 3.5. *Let R be a nonzero ring and A, B be two R-algebras such that $A \otimes_R B \neq \{0\}$. Then*

$$\dim(A \otimes_R B) \geq \max\Big\{\text{ht}\Big(\frac{P}{pA}\Big) + \text{ht}\Big(\frac{Q}{pB}\Big) : P \in \text{Spec}(A) \textit{ and } Q \in \text{Spec}(B) \\ \textit{with } \lambda_A^{-1}(P) = \lambda_B^{-1}(Q) := p\Big\}.$$

Proof. Fix a prime ideal p of R. Let $k_R(p) \otimes_R A$ (resp., $k_R(p) \otimes_R B$) denote the fibre ring of A (resp., of B) over p. By Lemma 3.4,

$$\begin{cases} k_R(p) \otimes_R A \cong \overline{S}^{-1} \frac{A}{pA}, \\ k_R(p) \otimes_R B \cong \overline{S}^{-1} \frac{B}{pB}, \end{cases}$$

where $\overline{S} := \frac{R}{p} \setminus \{\overline{0}\}$. Then there exists a one-to-one order preserving correspondence between the spectrum of $k_R(p) \otimes_R A$ and the set of prime ideals of $\frac{A}{pA}$ of the form $\frac{P}{pA}$ such that $\lambda_A^{-1}(P) = p$. In light of this, it is clear that to prove the theorem we are reduced to proving that

$$\dim(A \otimes_R B) \geq \dim(k_R(p) \otimes_R A) + \dim(k_R(p) \otimes_R B).$$

In view of the following canonical surjective ring homomorphism:

$$A \otimes_R B \xrightarrow{\varphi} \frac{A}{pA} \otimes_{\frac{R}{p}} \frac{B}{pB},$$

we get

$$\begin{aligned} \dim(A \otimes_R B) &\geq \dim\Big(\frac{A}{pA} \otimes_{\frac{R}{p}} \frac{B}{pB}\Big) \\ &\geq \dim\Big(\overline{S}^{-1}\Big(\frac{A}{pA} \otimes_{\frac{R}{p}} \frac{B}{pB}\Big)\Big) \\ &= \dim\Big(\Big(\overline{S}^{-1} \frac{A}{pA}\Big) \otimes_{k_R(p)} \Big(\overline{S}^{-1} \frac{B}{pB}\Big)\Big) \\ &= \dim\Big((k_R(p) \otimes_R A) \otimes_{k_R(p)} (k_R(p) \otimes_R B)\Big) \\ &\geq \dim(k_R(p) \otimes_R A) + \dim(k_R(p) \otimes_R B), \end{aligned}$$

as desired. □

Theorem 3.5 yields the following interesting consequences. Recall that a ring A is said to be equidimensional if all its maximal ideals have the same height.

Corollary 3.6. *Let R be a nonzero ring and A, B be two R-algebras such that $A \otimes_R B \neq \{0\}$. Assume that $\frac{A}{P}$ and $\frac{B}{Q}$ are flat over R for each prime ideal P of A and each prime ideal Q of B. Then:*

(a)

$$\dim(A \otimes_R B) \geq \max\Big\{\operatorname{ht}(P) + \operatorname{ht}(Q) : P \in \operatorname{Spec}(A) \text{ and } Q \in \operatorname{Spec}(B) \text{ such that } \lambda_A^{-1}(P) = \lambda_B^{-1}(Q)\Big\}.$$

(b) *If either A and B are equidimensional or R contains exactly one minimal prime ideal, then*

$$\dim(A \otimes_R B) \geq \dim(A) + \dim(B).$$

Proof. (a) Let $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $p_A = q_B := p$. As $\frac{A}{P}$ is flat over R, we get, by Lemma 2.4, p is a minimal prime ideal of R. Hence $\mathrm{ht}(P) = \mathrm{ht}(\frac{P}{pA})$ and $\mathrm{ht}(Q) = \mathrm{ht}(\frac{Q}{pB})$. Applying Theorem 3.5, we obtain

$$\dim(A \otimes_R B) \geq \mathrm{ht}\left(\frac{P}{pA}\right) + \mathrm{ht}\left(\frac{Q}{pA}\right) = \mathrm{ht}(P) + \mathrm{ht}(Q),$$

as we wish to show.

(b) Assume that A and B are equidimensional. As $A \otimes_R B \neq \{0\}$, by Proposition 3.1, there exists $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $p_A = q_B := p$. Let $M \in \mathrm{Max}(A)$ and $M' \in \mathrm{Max}(B)$ such that $P \subseteq M$ and $Q \subseteq M'$. Applying Lemma 2.4, we get $p_A = m_A = m'_B = q_B := p$, as p, m_A, m'_B are minimal primes in R. Hence

$$\dim(A \otimes_R B) \geq \mathrm{ht}(M) + \mathrm{ht}(M') = \dim(A) + \dim(B),$$

as A and B are equidimensional.

Now, suppose that R contains exactly one minimal prime ideal. Then, by Lemma 2.4, $\lambda_A^{-1}(P) = \lambda_B^{-1}(Q)$ for each prime ideal P of A and each prime ideal Q of B. Thus, applying (a), we get $\dim(A \otimes_R B) \geq \mathrm{ht}(P) + \mathrm{ht}(Q)$ for each prime ideal P of A and each prime ideal Q of B, so that

$$\dim(A \otimes_R B) \geq \dim(A) + \dim(B),$$

completing the proof. □

Applying Theorem 3.5 to the case where R is a zero-dimensional ring R, we get the following result extending [12, Corollary 2.6].

Corollary 3.7. *Let R be a nonzero zero-dimensional ring and let A, B be two R-algebras such that $A \otimes_R B \neq \{0\}$. Then*

$$\dim(A \otimes_R B) \geq \max\Big\{\mathrm{ht}(M) + \mathrm{ht}(M') : M \in \mathrm{Max}(A) \textit{ and } M' \in \mathrm{Max}(B) \textit{ with } \lambda_A^{-1}(M) = \lambda_B^{-1}(M')\Big\}.$$

Moreover, if either R is quasilocal or A and B are equidimensional, then

$$\dim(A \otimes_R B) \geq \dim(A) + \dim(B).$$

Proof. First, note that if $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ with $p_A = q_B$, then for any $P \subseteq P' \in \mathrm{Spec}(A)$ and any $Q \subseteq Q' \in \mathrm{Spec}(B)$, we have $p'_A = q'_B = p_A$, as R is zero-dimensional. It follows that

$$\max\Big\{\mathrm{ht}(P)+\mathrm{ht}(Q) : P \in \mathrm{Spec}(A) \text{ and } Q \in \mathrm{Spec}(B) \text{ such that } p_A = q_B\Big\}$$
$$= \max\Big\{\mathrm{ht}(M)+\mathrm{ht}(M') : M \in \mathrm{Max}(A) \text{ and } M' \in \mathrm{Max}(B) \text{ such that } m_A = m'_B\Big\}.$$

Taking account of this, the result easily follows from Theorem 3.5. □

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Bouvier's conjecture

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Abstract. This paper deals with Bouvier's conjecture which sustains that finite-dimensional non-Noetherian Krull domains need not be Jaffard.

Keywords. Noetherian domain, Krull domain, factorial domain, affine domain, Krull dimension, valuative dimension, Jaffard domain, fourteenth problem of Hilbert.

AMS classification. 13C15, 13F05, 13F15, 13E05, 13F20, 13G05, 13B25, 13B30.

Dedicated to Alain Bouvier

1 Introduction

All rings and algebras considered in this paper are commutative with identity element and, unless otherwise specified, are assumed to be non-zero. All ring homomorphisms are unital. If k is a field and A a domain which is a k-algebra, we use $\mathrm{qf}(A)$ to denote the quotient field of A and $\mathrm{t.d.}(A)$ to denote the transcendence degree of $\mathrm{qf}(A)$ over k. Finally, recall that an affine domain over a ring A is a finitely generated A-algebra that is a domain [28, p. 127]. Any unreferenced material is standard as in [17, 23, 25].

A finite-dimensional integral domain R is said to be Jaffard if there holds $\dim(R[X_1, \ldots, X_n]) = n + \dim(R)$ for all $n \geq 1$; equivalently, if $\dim(R) = \dim_v(R)$, where $\dim(R)$ denotes the (Krull) dimension of R and $\dim_v(R)$ its valuative dimension (i.e., the supremum of dimensions of the valuation overrings of R). As this notion does not carry over to localizations, R is said to be locally Jaffard if R_p is a Jaffard domain for each prime ideal p of R (equiv., $S^{-1}R$ is a Jaffard domain for each multiplicative subset S of R). The class of Jaffard domains contains most of the well-known classes of rings involved in Krull dimension theory such as Noetherian domains, Prüfer domains, universally catenarian domains, and universally strong S-domains. We assume familiarity with these concepts, as in [3, 5, 7, 8, 13, 20, 21, 22, 24].

It is an open problem to compute the dimension of polynomial rings over Krull domains in general. In this vein, Bouvier conjectured that "finite-dimensional Krull (or more particularly factorial) domains need not be Jaffard" [8, 15]. In Figure 1, a diagram of implications places this conjecture in its proper perspective and hence shows how it naturally arises. In particular, it indicates how the classes of (finite-dimensional) Noetherian domains, Prüfer domains, UFDs, Krull domains, and PVMDs [17] interact with the notion of Jaffard domain as well as with the (strong) S-domain properties of Kaplansky [22, 23, 24].

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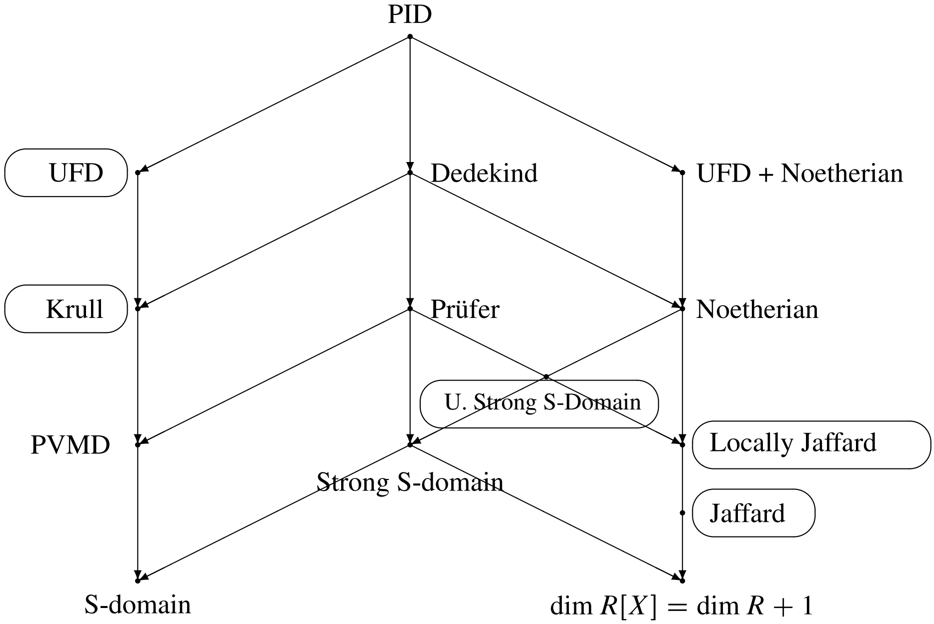

Figure 1. Diagram of Implications

This paper scans all known families of examples of non-Noetherian finite dimensional Krull (or factorial) domains existing in the literature. In Section 2, we show that most of these examples are in fact locally Jaffard domains. One of these families which arises from David's second example [12] yields examples of Jaffard domains but it is still open whether these are locally Jaffard. Further, David's example turns out to be the first example of a 3-dimensional factorial domain which is not catenarian (i.e., prior to Fujita's example [16]). Section 3 is devoted to the last known family of examples which stem from the generalized fourteenth problem of Hilbert (also called Hilbert–Zariski problem): Let k be a field of characteristic zero, T a normal affine domain over k, and F a subfield of $\mathrm{qf}(T)$. The Hilbert–Zariski problem asks whether $R := F \cap T$ is an affine domain over k. Counterexamples on this problem were constructed by Rees [30], Nagata [27] and Roberts [31, 32] where R wasn't even Noetherian. In this vein, Anderson, Dobbs, Eakin, and Heinzer [4] asked whether R and its localizations inherit from T the Noetherian-like main behavior of having Krull and valuative dimensions coincide (i.e., Jaffard). This problem will be addressed within the more general context of subalgebras of affine domains over Noetherian domains; namely, let $A \subseteq R$ be an extension of domains where A is Noetherian and R is a subalgebra of an affine domain over A. It turns out that R is Jaffard but it is still elusively open whether R is locally Jaffard.

2 Examples of non-Noetherian Krull domains

Obviously, Bouvier's conjecture (mentioned above) makes sense beyond the Noetherian context. As the notion of Krull domain is stable under formation of rings of fractions and adjunction of indeterminates, it merely claims "the existence of a Krull domain R and a multiplicative subset S (possibly equal to $\{1\}$) such that $1+\dim(S^{-1}R) \gneqq \dim(S^{-1}R[X])$." However, finite-dimensional non-Noetherian Krull domains are scarce in the literature and one needs to test them and their localizations as well for the Jaffard property.

Next, we show that most of these families of examples are subject to the (locally) Jaffard property. This reflects the difficulty of proving or disproving Bouvier's conjecture.

Example 2.1. Nagarajan's example [26] arises as the ring R_0 of invariants of a finite group of automorphisms acting on $R := k[[X, Y]]$, where k is a field of characteristic $p \neq 0$. It turned out that R is integral over R_0. Therefore [24, Theorem 4.6] forces R_0 to be a universally strong S-domain, hence a locally Jaffard domain [3, 23].

Example 2.2. Nagata's example [28, p. 206] and David's example [11] arise as integral closures of Noetherian domains, which are necessarily universally strong S-domains by [24, Corollary 4.21] (hence locally Jaffard).

Example 2.3. Gilmer's example [18] and Brewer–Costa–Lady's example [9] arise as group rings (over a field and a group of finite rank), which are universally strong S-domains by [2] (hence locally Jaffard).

Example 2.4. Fujita's example [16] is a 3-dimensional factorial quasilocal domain (R, M) that arises as a directed union of 3-dimensional Noetherian domains, say $R = \bigcup R_n$. We claim R to be a locally Jaffard domain.

Indeed, the localization with respect to any height-one prime ideal is a DVR (i.e., discrete valuation ring) and hence a Jaffard domain. As, by [13, Theorem 2.3], R is a Jaffard domain, then R_M is locally Jaffard. Now, let P be a prime ideal of R with $\mathrm{ht}(P) = 2$. Clearly, there exists $Q \in \mathrm{Spec}(R)$ such that $(0) \subset Q \subset P \subset M$ is a saturated chain of prime ideals of R. As, $\mathrm{ht}(M[n]) = \mathrm{ht}(M) = 3$ for each positive integer n, we obtain $\mathrm{ht}(P[n]) = \mathrm{ht}(P) = 2$ for each positive integer n. Then R_P is locally Jaffard, as claimed.

Example 2.5. David's second example [12] is a 3-dimensional factorial domain $J := \bigcup J_n$ which arises as an ascending union of 3-dimensional polynomial rings J_n in three indeterminates over a field k. We claim that J is a Jaffard domain. Moreover, J turns out to be non catenarian. Thus, David's example is the first example of a 3-dimensional factorial domain which is not catenarian (prior to Fujita's example).

Indeed, we have $J_n := k[X, \beta_{n-1}, \beta_n]$ for each positive integer n, where the indeterminates β_n satisfy the following condition: For $n \geq 2$,

$$\beta_n = \frac{-\beta_{n-1}^{s(n)} + \beta_{n-2}}{X} \tag{1}$$

where the $s(n)$ are positive integers. Also, $J_n \subseteq J \subseteq J_n[X^{-1}]$ for each positive integer n. By [13, Theorem 2.3], J is a Jaffard domain, as the J_n are affine domains. Notice, at this point, we weren't able to prove or disprove that J is locally Jaffard.

Next, fix a positive integer n. We have $\frac{J_n}{XJ\cap J_n} = k[\overline{\beta_{n-1}}, \overline{\beta_n}]$. On account of (1), we get

$$\overline{\beta_{n-1}} = \overline{\beta_n}^{\,s(n+1)}. \tag{2}$$

Therefore

$$\frac{J_n}{XJ \cap J_n} = k[\overline{\beta_n}].$$

Iterating the formula in (2), it is clear that for each positive integers $n \leq m$, there exists a positive integer r such that $\overline{\beta_n} = \overline{\beta_m}^{\,r}$ with respect to the integral domain $\frac{J}{XJ}$. It follows that $\frac{J}{XJ}$ is integral over $k[\overline{\beta_n}]$ for each positive integer n. Surely, $\overline{\beta_n}$ is transcendental over k, for each positive integer n, since $(0) \subset XJ \subset M := (X, \beta_0, \beta_1, \ldots, \beta_n, \ldots)$ is a chain of distinct prime ideals of J. Then $\dim(\frac{J}{XJ}) = 1$ and thus $(0) \subset XJ \subset M := (X, \beta_0, \beta_1, \ldots, \beta_n, \ldots)$ is a saturated chain of prime ideals of J. As $\mathrm{ht}(M) = 3$, it follows that J is not catenarian, as desired.

Example 2.6. Anderson–Mulay's example [6] draws from a combination of techniques of Abhyankar [1] and Nagata [28] and arises as a directed union of polynomial rings over a field. Let k be a field, d an integer ≥ 1, and $X, Z, Y_1, \ldots, Y_d$ $d+2$ indeterminates over k. Let $\{\beta_i := \sum_{n\geq 0} b_{in}X^n \mid 1 \leq i \leq d\} \subset k[[X]]$ be a set of algebraically independent elements over $k(X)$ (with $b_{in} \neq 0$ for all i and n). Define $\{U_{in} \mid 1 \leq i \leq d,\ 0 \leq n\}$ by

$$U_{i0} := Y_i,$$
$$U_{in} := \frac{Y_i + Z(\sum_{0\leq k\leq n-1} b_{ik}X^k)}{X^n}, \quad \text{for } n \geq 1.$$

For any i, n we have

$$U_{in} = XU_{i(n+1)} - b_{in}Z. \tag{3}$$

Let $R_n := k[X, Z, U_{1n}, \ldots, U_{dn}]$ be a polynomial ring in $d+2$ indeterminates (by (3)); and let $R := \bigcup R_n = k[X, Z, \{U_{1n}, \ldots, U_{dn} \mid n \geq 0\}]$. Anderson and Mulay proved that R is a $(d+2)$-dimensional non-Noetherian Jaffard and factorial domain. We claim that R is locally Jaffard. For this purpose, we envisage two cases.

Case 1: *k* is algebraically closed. Let P be a prime ideal of R. We may suppose $\mathrm{ht}(P) \geq 2$ (since R is factorial). Assume $X \notin P$. Clearly, $R_0 \subset R \subset R_0[X^{-1}]$, then $R_P \cong (R[X^{-1}])_{PR[X^{-1}]} = (R_0[X^{-1}])_{PR_0[X^{-1}]}$ is Noetherian (hence Jaffard). Assume $X \in P$. By (3), $\frac{R}{XR} \cong k[Z]$. Then $P = (X, f)$ for some irreducible polynomial f in $k[Z]$. As k is algebraically closed, we get $f = Z - \alpha$ for some $\alpha \in k$. For any positive integer n and $i = 1, \ldots, d$, define

$$V_{in} := U_{in} + b_{in}\alpha.$$

Observe that, for each n and i, we have

$$R_n = k[X, Z-\alpha, V_{1n}, \dots, V_{dn}],$$
$$V_{in} = XU_{i(n+1)} - b_{in}(Z-\alpha).$$

Then $P \cap R_n = (X, Z-\alpha, \{V_{1n}, \dots, V_{dn}\})$ is a maximal ideal of R_n for each positive integer n. For each $0 \le i \le d$, set

$$P_i := (Z-\alpha, \{V_{rn}\}_{1\le r\le i,\ 0\le n})R.$$

Each P_i is a prime ideal of R since $P_i \cap R_n = (Z-\alpha, V_{1n}, \dots, V_{in})$ is a prime ideal of R_n. This gives rise to the following chain of prime ideals of R

$$0 \subset (Z-\alpha)R = P_0 \subset P_1 \subset \cdots \subset P_d \subset P.$$

Each inclusion is proper since the P_i's contract to distinct ideals in each R_n. Hence $\mathrm{ht}(P) \ge d+2$, whence $\mathrm{ht}(P) = d+2$ as $\dim(R) = d+2$. Since R is a Jaffard domain, we get $\mathrm{ht}(P[n]) = \mathrm{ht}(P)$ for each positive integer n. Therefore, R is locally Jaffard, as desired.

Case 2: k is an arbitrary field. Let K be an algebraic closure of k. Let $T_n = K[X, Z, U_{1n}, \dots, U_{dn}]$ for each positive integer n and let

$$T := \bigcup_{n\ge 0} T_n = K[X, Z, \{U_{1n}, \dots, U_{dn} : n \ge 0\}].$$

Let Q be a minimal prime ideal of PT. Then $Q = (X, Z-\beta)$ with $\beta \in K$, as $\frac{T}{XT} \cong K[Z]$. By the above case, we have $\mathrm{ht}(Q) = d+2$. Hence $\mathrm{ht}(PT) = d+2$. As $T_n \cong K \otimes_k R_n$, we get,

$$T = \bigcup_{n\ge 0} T_n = \bigcup_{n\ge 0} K \otimes_k R_n = K \otimes_k \bigcup_{n\ge 0} R_n = K \otimes_k R.$$

Then T is a free and hence faithfully flat R-module. A well-known property of faithful flatness shows that $PT \cap R = P$. Further, T is an integral and flat extension of R. It follows that $\mathrm{ht}(PT) = \mathrm{ht}(P) = d+2$, and thus R_P is a Jaffard domain.

Example 2.7. Eakin–Heinzer's 3-dimensional non-Noetherian Krull domain, say R, arises – via [30] and [14, Theorem 2.2] – as the symbolic Rees algebra with respect to a minimal prime ideal P of the 2-dimensional homogeneous coordinate ring A of a nonsingular elliptic cubic defined over the complex numbers. We claim that this construction, too, yields locally Jaffard domains. Indeed, let $K := \mathrm{qf}(A)$, t be an indeterminate over A, and $P^{(n)} := P^n A_P \cap A$, the nth symbolic power of P, for $n \ge 2$. Set $R := A[t^{-1}, Pt, P^{(2)}t^2, \dots, P^{(n)}t^n, \dots]$, the 3-dimensional symbolic Rees algebra with respect to P. We have

$$A \subset A[t^{-1}] \subset R \subset A[t, t^{-1}] \subset K(t^{-1}).$$

Let Q be a prime ideal of R, $Q' := Q \cap A[t^{-1}]$, and $q := Q \cap A = Q' \cap A$. We envisage three cases.

Case 1: ht(Q) = 1. Then R_Q is a DVR hence a Jaffard domain.

Case 2: ht(Q) = 3. Then $3 = \dim(R_Q) \leq \dim_v(R_Q) \leq \dim_v(A[t^{-1}]_{Q'}) = \dim(A[t^{-1}]_{Q'}) \leq \dim(A[t^{-1}] = 1 + \dim(A) = 3$. Hence R_Q is a Jaffard domain.

Case 3: ht(Q) = 2. If $t^{-1} \notin Q$, then R_Q is a localization of $A[t, t^{-1}]$, hence a Jaffard domain. Next, assume that $t^{-1} \in Q$. If Q is a homogeneous prime ideal, then $Q \subset M := (m[t^{-1}] + t^{-1}A[t^{-1}]) \oplus pt \oplus \cdots \oplus p^{(n)}t^n \oplus \cdots$ and $\text{ht}(M) = 3$, where m is the unique maximal ideal of A. As R is a Jaffard domain, we get $\text{ht}(M[X_1, \ldots, X_n]) = \text{ht}(M) = 3$ for each positive integer n. Hence $\text{ht}(Q[X_1, \ldots, X_n]) = \text{ht}(Q) = 2$ for each positive integer n, so that R_Q is Jaffard. Now, assume that Q is not homogeneous. As $t^{-1} \in Q$ and $\text{ht}(Q) = 1 + \text{ht}(Q^*)$, where Q^* is the ideal generated by all homogeneous elements of Q, we get $Q^* = t^{-1}R$ which is a height one prime ideal of the Krull domain R. Also, for each positive integer n, note that $Q[X_1, X_2, \ldots, X_n]^* = Q^*[X_1, \ldots, X_n]$. Therefore, for each positive integer n, we have

$$\begin{aligned}
\text{ht}(Q[X_1, \ldots, X_n]) &= 1 + \text{ht}(Q[X_1, \ldots, X_n]^*) \\
&= 1 + \text{ht}(Q^*[X_1, \ldots, X_n]) \\
&= 1 + \text{ht}(t^{-1}R[X_1, \ldots, X_n]) \\
&= 1 + \text{ht}(t^{-1}R) = 2 \\
&= \text{ht}(Q).
\end{aligned}$$

It follows that R_Q is Jaffard, completing the proof. Notice that Anderson–Dobbs–Eakin–Heinzer's example [4, Example 5.1] is a localization of R (by a height 3 maximal ideal), then locally Jaffard.

Also, Eakin–Heinzer's second example [14] is a universally strong S-domain; in fact, it belongs to the same family as Example 2.1. Another family of non-Noetherian finite-dimensional Krull domains stems from the generalized fourteenth problem of Hilbert (also called Hilbert–Zariski problem). This is the object of our investigation in the following section.

3 Krull domains issued from the Hilbert–Zariski problem

Let k be a field of characteristic zero and let T be a normal affine domain over k. Let F be a subfield of the field of fractions of T. Set $R := F \cap T$. The Hilbert–Zariski problem asks whether R is an affine domain over k. Counterexamples on this problem were constructed by Rees [30], Nagata [27] and Roberts [31, 32], where it is shown that R does not inherit the Noetherian property from T in general. In this vein, Anderson, Dobbs, Eakin, and Heinzer [4] asked whether R inherits from T the Noetherian-like main behavior of being locally Jaffard. We investigate this problem within a more general context; namely, extensions of domains $A \subseteq R$, where A is Noetherian and R is a subalgebra of an affine domain over A.

The next result characterizes the subalgebras of affine domains over a Noetherian domain. It allows one to reduce the study of the prime ideal structure of these constructions to those domains R between a Noetherian domain B and its localization $B[b^{-1}]$ $(0 \neq b \in B)$.

Proposition 3.1. *Let $A \subseteq R$ be an extension of domains where A is Noetherian. Then the following statements are equivalent:*

(1) *R is a subalgebra of an affine domain over A;*

(2) *There is $r \neq 0 \in R$ such that $R[r^{-1}]$ is an affine domain over A;*

(3) *There is an affine domain B over A and $b \neq 0 \in B$ such that $B \subseteq R \subseteq B[b^{-1}]$.*

Proof. (1) $\Rightarrow$ (2). This is [19, Proposition 2.1(b)].

(2) $\Rightarrow$ (3). Let $r \neq 0 \in R$ and $x_1, \dots, x_n \in R[r^{-1}]$ with $R[r^{-1}] = A[x_1, \dots, x_n]$. For each $i = 1, \dots, n$, write $x_i = \sum_{j=0}^{n_i} r_{ij} r^{-j}$ with $r_{ij} \in R$ and $n_i \in \mathbb{N}$. Let $B := A[\{r_{ij} : i = 1, \dots, n \text{ and } j = 0, \dots, n_i\}]$ and let $b := r$. Clearly, B is an affine domain over A such that $B \subseteq R \subseteq B[b^{-1}]$.

The implication (3) $\Rightarrow$ (1) is trivial, completing the proof of the proposition. □

Corollary 3.2. *Let $A \subseteq R$ be an extension of domains where A is Noetherian and R is a subalgebra of an affine domain over A. Then there exists an affine domain T over A such that $R \subseteq T$ and R_p is Noetherian (hence Jaffard) for each prime ideal p of R that survives in T.*

Proof. By Proposition 3.1, there exists an affine domain B over A and a nonzero element b of B such that $B \subseteq R \subseteq B[b^{-1}]$. Put $T = B[b^{-1}]$. Let p be a prime ideal of R that survives in T (i.e., $b \notin p$). Then it is easy to see that

$$R_p \cong R[b^{-1}]_{pR[b^{-1}]} = B[b^{-1}]_{pB[b^{-1}]} = T_{pT}$$

is a Noetherian domain, as desired. □

Corollary 3.3. *Let R be a subalgebra of an affine domain T over a field k. Then:*

(1) $\dim(R) = \text{t.d.}(R)$ *and R is a Jaffard domain.*

(2) $\dim(R) = \text{ht}(P \cap R) + \text{t.d.}(\frac{R}{P \cap R})$ *for each prime ideal P of T. In particular,* $\dim(R) = \text{ht}(M)$ *for each maximal ideal M of R that survives in T.*

Proof. (1) This is [10, Proposition 5.1] which is a consequence of a more general result on valuative radicals [10, Théorème 4.4]. Also the statement "$\dim(R) = \text{t.d.}(R)$" is [29, Corollary 1.2]. We offer here an alternate proof: By Proposition 3.1, there exists an affine domain B over k and a nonzero element b of B such that $B \subseteq R \subseteq B[b^{-1}]$. By [28, Corollary 14.6], $\dim(B[b^{-1}]) = \dim_v(B[b^{-1}]) = \dim_v(B) = \dim(B) = \text{t.d.}(B) = \text{t.d.}(R)$. Further, observe that $B[b^{-1}] = R[b^{-1}]$ is a localization of R. Hence $\dim(B[b^{-1}]) = \dim(R[b^{-1}]) \leq \dim(R) \leq \dim_v(R) \leq \dim_v(B)$. Consequently, $\dim(R) = \dim_v(R) = \text{t.d.}(R)$, as desired.

(2) Let P be a prime ideal of T with $p := P \cap R$. By [10, Théorème 1.2], the extension $R \subseteq T$ satisfies the altitude inequality formula. Hence

$$\mathrm{ht}(P) + \mathrm{t.d.}\Big(\frac{T}{P} : \frac{R}{p}\Big) \leq \mathrm{ht}(p) + \mathrm{t.d.}(T : R).$$

By [28, Corollary 14.6], we obtain

$$\mathrm{t.d.}(T : k) - \mathrm{t.d.}\Big(\frac{R}{p} : k\Big) \leq \mathrm{ht}(p) + \ \mathrm{t.d.}(T : k) - \mathrm{t.d.}(R : k).$$

Then $\mathrm{t.d.}(R) \leq \mathrm{ht}(p) + \mathrm{t.d.}(\frac{R}{p} : k)$. Moreover, it is well known that ([33, p. 10])

$$\mathrm{ht}(p) + \mathrm{t.d.}\Big(\frac{R}{p} : k\Big) \leq \mathrm{t.d.}(R).$$

Applying (1), we get

$$\dim(R) = \mathrm{t.d.}(R : k) = \mathrm{ht}(p) + \mathrm{t.d.}\Big(\frac{R}{p} : k\Big).$$

Finally, notice that if $M \in \mathrm{Spec}(R)$ with $MT \neq T$, then there exists $M' \in \mathrm{Spec}(T)$ contracting to M, so that ([28, Corollary 14.6])

$$\mathrm{t.d.}\Big(\frac{R}{M}\Big) \leq \mathrm{t.d.}\Big(\frac{T}{M'}\Big) = 0,$$

completing the proof. □

The above corollaries shed some light on the dimension and prime ideal structure of the non-Noetherian Krull domains emanating from the Hilbert–Zariski problem. In particular, these are necessarily Jaffard. But we are unable to prove or disprove if they are locally Jaffard. An in-depth study is to be carried out on (some contexts of) subalgebras of affine domains over Noetherian domains in line with Rees, Nagata, and Roberts constructions.

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Elastic properties of some semirings defined by positive systems

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Abstract. We consider two semirings motivated by the study of positive systems in control theory and consider their basic factorization properties. The first is the semiring $\mathbb{R}^+[X]$ of polynomials with nonnegative real coefficients. The second is a semiring of algebraic integers having the form $\mathbb{N}_0[\tau] = \{x + y\tau :$ where x, y are nonnegative integers$\}$ for an appropriately chosen real quadratic integer τ. In each case, we show that the semiring has full infinite elasticity and that the Δ-set is $\{1, 2, 3, \ldots\}$. The proof in the latter case uses results of Hans Rademacher on the distribution of primes in quadratic extensions which may be of independent interest.

Keywords. Algebraic integer, non-unique factorization, elasticity of factorization, delta set.

AMS classification. 20M14, 20D60, 11B75.

1 Introduction

Let $\mathbb{Z}$ denote the ring of integers, $\mathbb{N}$ the set of positive integers, and $\mathbb{N}_0$ the set of nonnegative integers. At the recent PREP Conference on the theory of non-unique factorization, Ulrich Krause [10] posed several questions to the second author of this note which were motivated by Krause's familiarity with the theory of positive systems in control theory (see [5] for recent Conference Proceedings on this subject). These questions are as follows:

(1) If $\mathbb{R}^+[X] = \{ f(X) : f(X) = \sum_{i=0}^{t} a_i x^i \in \mathbb{R}[X]$ with $a_i \geq 0$ for every $i\}$, then what are the relative factorization properties of this multiplicative monoid?

(2) Let $d > 0$ be a squarefree integer. If $\mathbb{N}_0[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{N}_0\}$ then what factorization properties does this multiplicative monoid inherit from the regular ring of integers in $\mathbb{Q}(\sqrt{d})$?

We found these questions of interest, and we provide basic answers to both. In particular, we prove in Theorem 2.3 that any rational number $s \geq 1$ can serve as the elasticity of some $f(X) \in \mathbb{R}^+[X]$ and that any positive integer m may serve as a consecutive difference in the length set of some element of $\mathbb{R}^+[X]$. For question (2), we consider the set $\mathbb{N}_0[\tau] = \{x + y\tau : x, y \in \mathbb{N}_0\}$ where τ is an appropriately chosen integer of $\mathbb{Q}(\sqrt{d})$. Under our assumptions, $\mathbb{N}_0[\tau]$ is a semiring, and we repeat in Theorem 3.1 results analogous to those of Theorem 2.3 for $\mathbb{N}_0[\tau]$. A key step in the proof of this latter theorem originates in the work of Hans Rademacher on the distribution of primes among quadratic integers which may provide independent interest (see for example Corollary 3.12 and [13]). Theorems 2.3 and 3.1 demonstrate that in these monoids, unique factorization fails rather spectacularly.

Let us briefly review the notation and definitions which we will require. A very good reference for the theory of non-unique factorization is the monograph [9]. Given a ring or semiring R with no zero divisors, we let $R^\bullet$ represent the multiplicative monoid $R \setminus \{0\}$; by hypothesis this monoid is cancellative. For a commutative monoid M, let $\mathcal{A}(M)$ denote the set of all irreducible elements (atoms) of M, and let $M^\times$ denote the set of all invertible elements (units) of M. For $x \in M \setminus M^\times$, we define

$$\mathcal{L}(x) = \{n : \text{there are } \alpha_1, \ldots, \alpha_n \in \mathcal{A}(M) \text{ with } x = \alpha_1 \cdots \alpha_n\}$$

to be the *set of lengths of x in M*. For the M we are considering in Sections 2 and 3, we have always that $|\mathcal{L}(x)| < \infty$ (such a monoid is called a *finite factorization monoid* or FFM [9, Section 1.5]). The ratio $\sup \mathcal{L}(x)/\min \mathcal{L}(x)$ is called the *elasticity* of x and denoted $\rho(x)$. The elasticity of the monoid M is defined by

$$\rho(M) = \sup\{\rho(x) : x \in M \setminus M^\times\}$$

(see [9, Chapter 1.4]). M is said to be *fully elastic* if for every $q \in [1, \rho(M)] \cap \mathbb{Q}$, there exists an $x \in M \setminus M^\times$ such that $\rho(x) = q$. Full elasticity has been studied in great detail for various monoids in the papers [1], [3], [4], [7] and [8]. Accordingly, M is said to have *full infinite elasticity* if $\rho(M) = \infty$ and M is fully elastic; equivalently, every rational number $s \geq 1$ is the elasticity of some element of M.

The Δ-*set* of $x \in M \setminus M^\times$ is the set of consecutive differences in the set $\mathcal{L}(x)$. Explicitly, suppose that

$$\mathcal{L}(x) = \{n_1, n_2, \ldots, n_k\}$$

where $n_i < n_{i+1}$ for $1 \leq i \leq k-1$. Then $\Delta(x) = \{n_i - n_{i-1} : 2 \leq i \leq k\}$. As one might expect, we define the Δ-set of the monoid M by

$$\Delta(M) = \bigcup_{x \in M \setminus M^\times} \Delta(x)$$

(see again [9, Chapter 1.4]). As with elasticity, the study of the Δ-sets of particular monoids has an active history, and various calculations in specific cases can be found in [2] and [6].

2 Semirings of polynomials with nonnegative coefficients

Let $\mathbb{R}^+[X]$ be the semiring of univariate polynomials with nonnegative real coefficients. That is,

$$\mathbb{R}^+[X] = \left\{ f(X) \in \mathbb{R}[X] : f(X) = \sum_{i=0}^{t} a_i X^i \text{ with } a_i \geq 0 \text{ for every } i \right\}.$$

Lemma 2.1. *Let $n \in \mathbb{Z}$ and $b, c \in \mathbb{R}$ such that $n > 1$, $b > 0$, and $c \geq n$. Then $f(X) = (X+c)^n(X^2 - X + b) \in \mathbb{R}^+[X]$ if and only if $nb \geq c$.*

Proof. Let a_i be the coefficient of X^i in f. We will show that $a_i \geq 0$ for all $i \neq 1$ and that $a_1 \geq 0$ if and only if $nb \geq c$. The leading and trailing coefficients of f are $a_{n+2} = 1$ and $a_0 = bc^n$, respectively, and these are both positive by our hypotheses. We also have $a_{n+1} = nc - 1 \geq n^2 - 1 > 0$.

Let i satisfy $2 \leq i \leq n$. Expanding $(X + c)^n$ by the binomial theorem and multiplying by $X^2 - X + b$ we obtain

$$a_i = \binom{n}{i} bc^{n-i} - \binom{n}{i-1} c^{n-i+1} + \binom{n}{i-2} c^{n-i+2}.$$

By our assumptions on n, c, i, we have $a_i \geq 0$ if and only if

$$\frac{(i!)(n-i+2)!}{c^{n-i}(n!)} a_i = (n-i+1)(n-i+2)b - ci(n-i+2) + c^2 i(i-1)$$

is nonnegative. The first term on the right is nonnegative, so it suffices to show that

$$c^2 i(i-1) \geq ci(n-i+2).$$

This follows from observing that $c(i-1) \geq c \geq n \geq n - i + 2$ and that $c, i > 0$.

Finally, $a_1 = c^{n-1}(nb - c)$, so $a_1 \geq 0$ if and only if $nb \geq c$. □

Corollary 2.2. *If $b > 1/4$ and $nb \geq c$, but $(n-1)b < c$, then $(X + c)^n(X^2 - X + b)$ is irreducible in $\mathbb{R}^+[X]$.*

Proof. The lemma shows our polynomial is in $\mathbb{R}^+[X]$. We must show it is irreducible there. The discriminant of $X^2 - X + b$ is $1 - 4b < 0$, and hence $X^2 - X + b$ is irreducible in $\mathbb{R}[X]$. Thus, any nontrivial factorization of our polynomial in $\mathbb{R}^+[X]$ would include a factor of the form $(X + c)^m(X^2 - X + b)$ with $m < n$. However, since $mb \leq (n-1)b < c$, the lemma shows that this factor does not lie in $\mathbb{R}^+[X]$. □

Theorem 2.3. *The monoid $\mathbb{R}^+[X]^\bullet$ has full infinite elasticity and $\Delta(\mathbb{R}^+[X]^\bullet) = \mathbb{N}$.*

Proof. Consider

$$g(X) = (X + n)^n (X^2 - X + 1)(X + 1)^k,$$

where $n, k \geq 1$. Lemma 2.1 with $b = 1$ and $c = n$ implies that the polynomial

$$(X + n)^m (X^2 - X + 1)$$

is an element of $\mathbb{R}^+[X]$ if and only if $m \geq n$ and Corollary 2.2 guarantees that it is irreducible in $\mathbb{R}^+[X]$ when $m = n$. In particular, $g \in \mathbb{R}^+[X]$.

Using the fact that $X^2 - X + 1$ is an irreducible element of $\mathbb{R}[X]$, it is clear that $(X^2 - X + 1)(X + 1)$ is an irreducible element of $\mathbb{R}^+[X]$. The reader will have no trouble verifying that the only irreducible factors of g in $\mathbb{R}^+[X]$ are

$$(X + n)^n (X^2 - X + 1), \quad X + 1, \quad X + n, \quad \text{and} \quad (X + 1)(X^2 - X + 1),$$

and that there are only two irreducible factorizations of g in $\mathbb{R}^+[X]$:

$$g(X) = [(X+n)^n(X^2 - X + 1)] \cdot [X+1]^k, \quad \text{and}$$
$$g(X) = [X+n]^n \cdot [(X^2 - X + 1)(X+1)] \cdot [X+1]^{k-1}$$

which have lengths $1+k$ and $n+k$ respectively. Thus, in $\mathbb{R}^+[X]$ we have

$$\mathcal{L}(g) = \{1+k, n+k\}, \quad \rho(g) = \frac{n+k}{1+k} \quad \text{and} \quad \Delta(g) = \{n-1\}.$$

Given any rational number $s \geq 1$, one can find integers $n, k \geq 1$ such that s equals $(n+k)/(1+k)$, so we conclude that $\mathbb{R}[X]^\bullet$ has full infinite elasticity. Similarly, we obtain $\Delta(\mathbb{R}[X]^\bullet) = \mathbb{N}$. □

Before we conclude this section, we remark that $\mathbb{R}$ can be replaced in all of the above results and arguments with any ring A with $\mathbb{Z} \subseteq A \subset \mathbb{R}$.

3 Semirings of quadratic integers

In the last section, we studied the basic factorization properties of the polynomial semiring $\mathbb{R}^+[X]$ by exploiting the properties of the UFD $\mathbb{R}[X]$ in which it is contained. However, a few issues with the inclusion $\mathbb{R}^+[X] \subset \mathbb{R}[X]$ make further results challenging to obtain in this manner. For instance, $\mathbb{R}^+[X]^\bullet$ is not a *root-closed* submonoid of $\mathbb{R}[X]^\bullet$; that is, there do exist nonzero $f \in \mathbb{R}[X]$ such that $f, -f \notin \mathbb{R}^+[X]$ but $f^n \in \mathbb{R}^+[X]$ for some $n > 1$. As an example, $f(X) = X^4 + 2X^3 - X^2 + 2X + 1$ does not lie in $\mathbb{R}^+[X]$ but its square

$$f(X)^2 = X^8 + 4X^7 + 2X^6 + 11X^4 + 2X^2 + 4X + 1$$

certainly does.

Another problem that arises in treating $\mathbb{R}^+[X]$ via $\mathbb{R}[X]$ is, simply put, the transcendence of X. Given $f \in \mathbb{R}^+[X]$, the naïve method for computing the atomic factorizations of f in $\mathbb{R}^+[X]$ is to factor f in the UFD $\mathbb{R}[X]$ and then determine by brute force which factors (among *all* divisors of f) are irreducible factors of f in $\mathbb{R}^+[X]$. As deg f grows, computation using this method becomes prohibitive, and an improved algorithm is not immediately apparent.

It is therefore natural to ask what happens when we replace $\mathbb{R}$ with $\mathbb{Z}$ and replace X with an appropriate algebraic integer τ. In this section, we will show that analogous results hold for these *algebraic semirings* in the quadratic case.

Let α be an algebraic integer of degree n. The set

$$\mathbb{N}_0[\alpha] = \left\{ \beta \in \mathbb{Z}[\alpha] : \beta = \sum_{i=0}^{n-1} u_i \alpha^i \text{ with } u_i \in \mathbb{N}_0 \text{ for every } i \right\}$$

forms a semiring under the usual operations if and only if $\alpha^n \in \mathbb{N}_0[\alpha]$. Equivalently, the minimal polynomial of α is of the form $T^n - f(T)$ where $f \in \mathbb{N}_0[T]$ (here $\mathbb{N}_0[T]$

is the semiring of univariate polynomials in T with nonnegative integer coefficients) and $\deg f < n$.

For the rest of the article we fix a quadratic integer τ such that $\mathbb{N}_0[\tau]$ (as defined above) is a semiring (with $n = 2$). From our previous discussion, we easily see that $\mathbb{N}_0[\tau]$ is a semiring if and only if $\tau + \bar{\tau} \geq 0$ and $\tau\bar{\tau} < 0$. These two inequalities have a number of implications which we summarize below.

- Set $K = \mathbb{Q}(\tau)$. There is a unique squarefree $d \in \mathbb{Z}$ such that $K = \mathbb{Q}(\sqrt{d})$. We fix $q, r \in \mathbb{Q}$ such that $\tau = q + r\sqrt{d}$.
- Since $\tau\bar{\tau} = q^2 - dr^2 < 0$, we must have $d > 0$. As is standard, we take $\sqrt{d} > 0$ and note that K is a subfield of $\mathbb{R}$.
- Because $\tau + \bar{\tau} = 2q \geq 0$, $q \geq 0$. We may also assume that $r > 0$, possibly after replacing τ with its conjugate.

In this section we prove the following theorem.

Theorem 3.1. *With the notation and assumptions as above,* $\mathbb{N}_0[\tau]^\bullet$ *has full infinite elasticity and* $\Delta(\mathbb{N}_0[\tau]^\bullet) = \mathbb{N}$.

The plan of the proof is as follows. Just as we exploited the inclusion $\mathbb{R}^+[X] \subset \mathbb{R}[X]$, we will use the properties of $\mathbb{Z}[\tau]$ to study $\mathbb{N}_0[\tau]$. However, these two situations are strikingly different: while $\mathbb{R}[X]^\times = \mathbb{R}^\times$ and $\mathbb{R}^+[X]^\times = \mathbb{R}^+ \setminus \{0\}$, invertible elements of $\mathbb{Z}[\tau]$ need not remain so in $\mathbb{N}_0[\tau]$. From algebraic number theory, we know that $\mathbb{Z}[\tau]^\times/\{\pm 1\} \simeq \mathbb{Z}$, but exactly one of the elements of $\mathbb{Z}[\tau]$ lifting to a generator of this group lives in $\mathbb{N}_0[\tau]$. This element, the *fundamental unit* η of $\mathbb{Z}[\tau]$, is actually irreducible in $\mathbb{N}_0[\tau]$. Using a theorem of Hans Rademacher, we show that for any positive integer k there is a prime π of the ring $\mathbb{Z}[\tau]$ such that $\pi \in \mathbb{N}_0[\tau]$ and k is least with $\eta \mid \pi^k$ in $\mathbb{N}_0[\tau]$. Exploiting the unique factorization of π^k in $\mathbb{Z}[\tau]$, we will show that the length set of π^k is either $\{2, k\}$ or $\{3, k\}$ (with the latter occurring when $\tau = \eta$). Along with the existence of a prime element of $\mathbb{N}_0[\tau]$ this is adequate to prove the theorem.

We begin the proof by formulating a membership criterion for the semiring $\mathbb{N}_0[\tau]$. Let θ be the endomorphism of $K^\times$ given by $\alpha \mapsto \bar{\alpha}/\alpha$. The following basic facts about θ follow directly from the definition:

Proposition 3.2. *Let* $\alpha, \beta \in K^\times$. *Then the following hold:*

- $\theta(-\alpha) = \theta(\alpha)$, *and* $\theta(\bar{\alpha}) = \theta(1/\alpha) = 1/\theta(\alpha)$.
- *If* $\alpha > 0$, *then* $\bar{\alpha} > 0$ *if and only if* $\theta(\alpha) > 0$. *In particular,* $\theta(\tau) < 0$.
- $\theta(\alpha) = 1$ *if and only if* $\alpha \in \mathbb{Q}$.
- $\theta(\alpha) = -1$ *if and only if* α *is a rational multiple of* $\sqrt{d}$.
- $\theta(\alpha) = \theta(\beta)$ *if and only if* $\alpha/\beta \in \mathbb{Q}$.

Also, since $\tau = q + r\sqrt{d}$ with $q, r \in \mathbb{Q}$, $q \geq 0$, $r > 0$, and $\bar{\tau} = q - r\sqrt{d} < 0$ we have $\theta(\tau) < 0$ and $|\theta(\tau)| = (-q + r\sqrt{d})/(q + r\sqrt{d}) \leq 1$, so $-1 \leq \theta(\tau) < 0$.

Proposition 3.3 (Membership criterion). *If α in $\mathbb{Z}[\tau]$ is nonzero, then α is in $\mathbb{N}_0[\tau]$ if and only if $\alpha > 0$ and $\theta(\alpha)$ is in $[\theta(\tau), 1]$.*

Proof. For any real number λ not equal to $-1/\sqrt{d}$, let

$$f(\lambda) = \frac{1 - \lambda\sqrt{d}}{1 + \lambda\sqrt{d}}.$$

Since $f(\lambda) \to -1$ as $\lambda \to \pm\infty$, let $f(\pm\infty) = -1$.

Let $\alpha = x + y\tau$ be nonzero in $\mathbb{Z}[\tau]$ and assume for now that $x + yq \neq 0$. We have

$$\alpha = x + y\tau = (x + yq) + yr\sqrt{d} = (x + yq)(1 + \lambda_\alpha\sqrt{d}),$$

where $\lambda_\alpha = yr/(x + yq)$. Since λ_α is rational, it is not $-1/\sqrt{d}$.

Since $x+yq$ is rational, we see $\theta(\alpha) = \theta(1+\lambda_\alpha\sqrt{d}) = f(\lambda_\alpha)$. In particular, when $\alpha = \tau$ (so $(x, y) = (0, 1)$), we get $\theta(\tau) = f(\lambda_\tau) = f(r/q)$ (here, when $q = 0$, we take $r/q = \infty$, since $r > 0$). Since $f(0) = 1$, we have $\theta(\alpha)$ is in $[\theta(\tau), 1]$ if and only if $f(\lambda_\alpha)$ is in $[f(r/q), f(0)]$. However, away from its singularity $-1/\sqrt{d}$, the derivative of $f(\lambda)$ is always negative. That singularity is negative, and so on the nonnegative interval $[0, r/q]$, $f(\lambda)$ is monotonically decreasing. Thus $\theta(\alpha)$ is in $[\theta(\tau), 1]$ if and only if $0 \leq \lambda_\alpha \leq r/q$.

Now suppose that α is in $\mathbb{N}_0[\tau]$. Then x, y are nonnegative and not both zero. Obviously $\alpha = (x + yq) + yr\sqrt{d} > 0$. Also $0 \leq yr/(x + yq) \leq yr/yq = r/q$, showing $0 \leq \lambda_\alpha \leq r/q$, which by the above shows that $\theta(\alpha)$ is in $[\theta(\tau), 1]$, as desired. Conversely, suppose $\alpha > 0$ and $\theta(\alpha)$ is in $[\theta(\tau), 1]$. Then $0 \leq yr/(x + yq) \leq r/q$ (by the above). The first inequality shows that either yr and $x + yq$ have the same sign, or $yr = 0$. They cannot both be negative, as otherwise $\alpha = (x + yq) + yr\sqrt{d} < 0$.

On the other hand, suppose that both are positive. From $yr > 0$ we obtain $y > 0$, and since $yr/(x + yq) \leq r/q = yr/yq$, we must also have $x \geq 0$, so $\alpha \in \mathbb{N}_0[\tau]$. If $yr = 0$, then $y = 0$ and so $\alpha = x$ where x is a positive integer, so again $\alpha \in \mathbb{N}_0[\tau]$.

We now turn to the case that $x + yq = 0$. Assume first that $\alpha \in \mathbb{N}_0[\tau]$ so that $x, y \geq 0$ (not both zero). Since $x, y, q \geq 0$, $x + yq = 0$ implies $x = 0$ and therefore $q = 0$ since y must be nonzero. Then $y > 0$ so $\alpha = y\tau > 0$ and $\theta(\alpha) = \theta(\tau)$ lies in the interval $[\theta(\tau), 1]$, as desired.

Conversely, suppose that $\alpha = x + y\tau > 0$, $\theta(\alpha) \in [\theta(\tau), 1]$, and $x + yq = 0$. Then $\alpha = yr\sqrt{d}$, so $\theta(\alpha) = -1$ and hence $\theta(\tau) = -1$, which implies $q = 0$. Since $x + yq = 0$, x is also equal to zero, so $\alpha = y\tau$. Since $\alpha > 0$ and $\tau > 0$, $y > 0$, from which we finally conclude that $\alpha \in \mathbb{N}_0[\tau]$. □

Corollary 3.4 (Divisibility criterion). *If $\alpha, \delta \in \mathbb{N}_0[\tau]$ are nonzero then $\delta \mid \alpha$ in $\mathbb{N}_0[\tau]$ if and only if $\delta \mid \alpha$ in $\mathbb{Z}[\tau]$ and $\theta(\alpha/\delta) \in [\theta(\tau), 1]$.*

The above follows from the membership criterion since θ is an endomorphism of $K^\times$. In particular, if δ is a unit of the ring $\mathbb{Z}[\tau]$, whether or not δ divides α in $\mathbb{N}_0[\tau]$ depends entirely on the value $\theta(\alpha)$. This idea plays a central part in our arguments.

As stated in the summary of the proof of the main theorem, $\mathbb{Z}[\tau]^\times/\{\pm 1\}$ is isomorphic to $\mathbb{Z}$ (this is an application of the classical unit theorem of Dirichlet, see for example [12], Theorem I.12.12). There is a unique $\eta \in \mathbb{Z}[\tau]^\times$ with $\eta > 1$ which lifts to a generator of this group; this is called the *fundamental unit* of $\mathbb{Z}[\tau]$.

Proposition 3.5. *We have* $\eta \in \mathbb{N}_0[\tau]$, $|\theta(\eta)| < 1$, *and*

$$\mathbb{N}_0[\tau] \cap \mathbb{Z}[\tau]^\times = \{\eta^k : k \geq 0\} = \{1, \eta, \eta^2, \ldots\}.$$

In particular, η *is an irreducible element of* $\mathbb{N}_0[\tau]$, *and* $\mathbb{N}_0[\tau]^\times = \{1\}$.

Proof. Let $\eta = u + v\sqrt{d}$ where $u, v \in \mathbb{Q}$. Since η is the greatest element of the set $\{\pm\eta, \pm\bar{\eta}\}$ we have $u, v > 0$.

If $\theta(\eta) > 0$, then η and $\bar{\eta}$ are both positive and $\eta\bar{\eta} = 1$ since η is a unit. It follows that $\theta(\eta) = 1/\eta^2$, and so $\theta(\eta) < 1$ since $\eta > 1$. Thus $\eta \in \mathbb{N}_0[\tau]$ by Proposition 3.3.

On the other hand, if $\theta(\eta) < 0$, then $\eta\bar{\eta} = -1$ instead. Since $\eta \in \mathbb{Z}[\tau]$, there exist $x, y \in \mathbb{Z}$ with $\eta = x + y\tau$. Comparing coefficients, $u = x + yq$ and $v = yr$ so immediately $y > 0$. In addition,

$$0 > y^2\tau\bar{\tau} = y^2(q^2 - dr^2) = (u^2 - dv^2) + (-2ux + x^2) = (-2ux + x^2) - 1$$

from which it follows that $x \geq 0$ and $\eta \in \mathbb{N}_0[\tau]$.

Since $\eta \in \mathbb{N}_0[\tau]$, $|\theta(\eta)| \leq 1$. However, η is neither rational nor a rational multiple of $\sqrt{d}$, so neither $\theta(\eta) = 1$ nor $\theta(\eta) = -1$, respectively, and thus $|\theta(\eta)| < 1$.

If $w \in \mathbb{N}_0[\tau] \cap \mathbb{Z}[\tau]^\times$, then since η is the fundamental unit of $\mathbb{Z}[\tau]$ and w is positive, $w = \eta^k$ for some $k \in \mathbb{Z}$. Since $w \in \mathbb{N}_0[\tau]$, by Proposition 3.3 we have $|\theta(w)| = |\theta(\eta)|^k \leq 1$ showing $k \geq 0$ since we know $|\theta(\eta)| < 1$.

It follows that η is irreducible in $\mathbb{N}_0[\tau]$, for if $\eta = \alpha\beta$ for $\alpha, \beta \in \mathbb{N}_0[\tau]$, we would also have $\alpha, \beta \in \mathbb{Z}[\tau]^\times$, meaning α and β are positive powers of η, so $\{\alpha, \beta\} = \{1, \eta\}$. Since any unit of $\mathbb{N}_0[\tau]$ is also a unit of $\mathbb{Z}[\tau]$, it also follows that $\mathbb{N}_0[\tau]^\times = \{1\}$. □

Corollary 3.6. *The monoid* $\mathbb{N}_0[\tau]^\bullet$ *is atomic.*

Proof. Let $\alpha \in \mathbb{N}_0[\tau]$, assume that $\eta \nmid \alpha$, and let $\mathbf{N}$ be the function $\delta \mapsto |\delta\bar{\delta}|$ (the absolute norm). We will show that α has an atomic factorization in $\mathbb{N}_0[\tau]$ by induction on $\mathbf{N}(\alpha)$. If $\mathbf{N}(\alpha) = 1$, then α is a unit in $\mathbb{Z}[\tau]$, but $\eta \nmid \alpha$ so we must have $\alpha = 1$ (by Proposition 3.5).

On the other hand, if $\mathbf{N}(\alpha) > 1$ and α is not itself irreducible, then there are nonunits β and γ of $\mathbb{N}_0[\tau]$ which satisfy $\alpha = \beta\gamma$. Of course $\eta \nmid \beta, \gamma$, so β and γ are also nonunits of $\mathbb{Z}[\tau]$. It follows that $1 < \mathbf{N}(\beta), \mathbf{N}(\gamma) < \mathbf{N}(\alpha)$, so α has an atomic factorization by induction.

In the general case, $|\theta(\eta)| < 1$ implies that $|\theta(\alpha/\eta^m)| > 1$ for large m, and so by Corollary 3.4 there is a greatest m such that $\eta^m \mid \alpha$ in $\mathbb{N}_0[\tau]$. Thus $\alpha = \eta^m\alpha'$ where $\eta \nmid \alpha'$. Since η is irreducible and α' has an atomic factorization, α has an atomic factorization. □

If τ is a unit unusual behavior arises, and this leads to a case division in our proof.

Proposition 3.7. *Suppose that τ is a unit of $\mathbb{Z}[\tau]$. Then $\tau = \eta$ and for all $\alpha \in \mathbb{N}_0[\tau]$ with $\bar{\alpha} > 0$, $\tau \mid \alpha$ implies $\tau^2 \mid \alpha$ in $\mathbb{N}_0[\tau]$.*

Proof. If τ is a unit, then $\tau \in \mathbb{Z}[\tau]^\times \cap \mathbb{N}_0[\tau]$, so it follows from Proposition 3.5 that $\tau = \eta^k$ where $k > 0$ (nonzero since τ is irrational). Since $\theta(\tau) < 0$, we also have $\theta(\eta) < 0$, and thus the membership criterion yields $|\theta(\eta)| \le |\theta(\tau)| = |\theta(\eta)|^k$. But $|\theta(\eta)| < 1$, so we must have $k = 1$.

Since τ is a unit and $\theta(\tau) < 0$, a given nonzero $\alpha \in \mathbb{N}_0[\tau]$ is divisible by τ if and only if $\theta(\alpha) \in [\theta(\tau), \theta(\tau^2)]$. Similarly, α is divisible by τ^2 if and only if $\theta(\alpha) \in [\theta(\tau^3), \theta(\tau^2)]$. Hence, if α is divisible by τ but not divisible by τ^2,

$$\theta(\alpha) \in \big[\theta(\tau), \theta(\tau^3)\big) \subset [-1, 0)$$

so $\theta(\alpha) < 0$ whence $\bar{\alpha} < 0$. □

Proposition 3.8. *If τ is not a unit of $\mathbb{Z}[\tau]$, then $\theta(\eta) > \theta(\tau)$.*

Proof. Suppose instead that $\theta(\tau) = \theta(\eta)$ (this suffices by the membership criterion). Then $\tau/\eta \in \mathbb{Q}$ by Proposition 3.2, and since it is also a nonzero element of $\mathbb{Z}[\tau]$, $m = \tau/\eta$ is a positive rational integer. Of course, $\eta = u + v\tau$ where $u, v \in \mathbb{N}_0$, so $\tau = mu + mv\tau$, which implies $(1 - mv)\tau = mu$. Since τ is irrational but $mu \in \mathbb{Z}$ we must have $1 - mv = 0$, so $m = v = 1$ and therefore $\tau = m\eta = \eta$. □

The above results comprise the first ingredients in the main theorem. We will now show that for any nondiscrete subset I of $(0, 1]$ there exist infinitely many primes π of the ring $\mathbb{Z}[\tau]$ satisfying $\theta(\pi) \in I$. The proof of Theorem 3.1 requires that for certain I at least one such prime exists, so a simpler argument than the one we give here may exist.

We briefly review a few notions of classical algebraic number theory: The *ring of integers* of K, denoted $\mathcal{O}_K$, is the integral closure of $\mathbb{Z}$ in K (as usual, $K = \mathbb{Q}(\sqrt{d})$ here). A basic result of algebraic number theory states that

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \mod 4, \text{ and} \\ \mathbb{Z}[\sqrt{d}] & \text{otherwise.} \end{cases}$$

Incidentally, both $\mathbb{N}_0[\sqrt{d}]$ and $\mathbb{N}_0[\frac{1+\sqrt{d}}{2}]$ are both semirings, since $\sqrt{d}$ and $\frac{1+\sqrt{d}}{2}$ satisfy the inequalities given in the introduction to this section.

Like the unit group of $\mathbb{Z}[\tau]$, the group $\mathcal{O}_K^\times$ also admits a unique *fundamental unit* $\varepsilon > 1$ which lifts to a generator of $\mathcal{O}_K^\times/\{\pm 1\}$. Note that η must be a positive power of ε (in fact it is the least power of ε lying in $\mathbb{Z}[\tau]$).

Given an ideal $\mathfrak{a}$ of $\mathcal{O}_K$ we will set $\varepsilon_{\mathfrak{a}} = \varepsilon^m$ where m is the least positive integer such that $\bar{\varepsilon}^m > 0$ and $\varepsilon^m \equiv 1 \mod \mathfrak{a}$. Finally, we define the *conductor* of $\mathbb{Z}[\tau]$ by

$$\mathfrak{f} = \{\alpha \in \mathcal{O}_K : \alpha\mathcal{O}_K \subset \mathbb{Z}[\tau]\}.$$

This is a nonzero ideal of both $\mathcal{O}_K$ and $\mathbb{Z}[\tau]$; in fact, it is the largest ideal of $\mathcal{O}_K$ which is contained in $\mathbb{Z}[\tau]$. Furthermore, if $\alpha \in \mathbb{Z}[\tau]$, $\beta \in \mathcal{O}_K$, and $\alpha \equiv \beta \mod \mathfrak{f}$, then

$\beta \in \mathbb{Z}[\tau]$ as well. For example, $\varepsilon_{\mathfrak{f}} \equiv 1 \bmod \mathfrak{f}$ so $\varepsilon_{\mathfrak{f}}$ is an element of $\mathbb{Z}[\tau]$. Since, $1 - \varepsilon_{\mathfrak{f}}$ is an element of $\mathfrak{f}$, so is its associate (in $\mathcal{O}_K$), $\varepsilon_{\mathfrak{f}}^{-1} - 1$, so we have $\varepsilon_{\mathfrak{f}}^{-1} \in \mathbb{Z}[\tau]$ which shows $\varepsilon_{\mathfrak{f}} \in \mathbb{Z}[\tau]^{\times}$. It follows that $\varepsilon_{\mathfrak{f}}$ is a positive power of η.

To summarize, the following units of $\mathcal{O}_K$ are important in our proof:

- ε, the fundamental unit of $\mathcal{O}_K$.
- η, the fundamental unit of $\mathbb{Z}[\tau]$, which is a positive power of ε; it is also an irreducible element of the semiring $\mathbb{N}_0[\tau]$.
- $\varepsilon_{\mathfrak{a}}$, the least power of ε such that $\varepsilon_{\mathfrak{a}} \equiv 1 \bmod \mathfrak{a}$ and $\bar{\varepsilon}_{\mathfrak{a}} > 0$ (for $\mathfrak{a}$ an ideal of $\mathcal{O}_K$).

An important property of the conductor is that it preserves primality.

Proposition 3.9. *If π is a prime of $\mathcal{O}_K$ and $\pi \equiv 1 \bmod \mathfrak{f}$, then π is also a prime of $\mathbb{Z}[\tau]$.*

Proof. Suppose that $\pi \mid \alpha\beta$ with $\alpha, \beta \in \mathbb{Z}[\tau]$. Since π is prime in $\mathcal{O}_K$, $\alpha = \pi\delta$ for some $\delta \in \mathcal{O}_K$, say. Since $\delta \equiv \alpha \bmod \mathfrak{f}$ and $\alpha \in \mathbb{Z}[\tau]$, $\delta \in \mathbb{Z}[\tau]$ as well. □

Example 3.10. As an example, let $\tau = 1 + \sqrt{5}$. Since $\tau + \bar{\tau} = 2$ and $\tau\bar{\tau} = -4$, $\mathbb{N}_0[\tau]$ is a semiring. Clearly $\mathbb{Z}[\tau] = \mathbb{Z}[\sqrt{5}]$, and by a result stated above $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

Observe that $2\mathcal{O}_K \subset \mathbb{Z}[\sqrt{5}]$. Since the minimal polynomial of $\frac{1+\sqrt{5}}{2}$ is irreducible over $\mathbb{F}_2$, $2\mathcal{O}_K$ is a nonzero prime ideal and therefore maximal ($\mathcal{O}_K$ is Dedekind). It follows that $\mathfrak{f} = 2\mathcal{O}_K$. It is important to note that while $\mathfrak{f}$ is also an ideal of $\mathbb{Z}[\sqrt{5}]$, it is not principal; rather $\mathfrak{f} = (2, 1 + \sqrt{5})$ in $\mathbb{Z}[\sqrt{5}]$.

The fundamental unit of $\mathcal{O}_K$ is $\varepsilon = \frac{1+\sqrt{5}}{2}$, which obviously does not lie in $\mathbb{Z}[\sqrt{5}]$, nor does its square, $1 + \varepsilon$. However, $\varepsilon^3 = 2 + \sqrt{5}$ does lie in $\mathbb{Z}[\sqrt{5}]$, so $\eta = \varepsilon^3$. (This is expected, since $(\mathcal{O}_K/\mathfrak{f})^{\times}$ is cyclic of order 3 and $(\mathbb{Z}[\sqrt{5}]/\mathfrak{f})^{\times}$ is trivial).

Now, $\eta \equiv 1 \bmod \mathfrak{f}$, but $\bar{\eta} < 0$. Hence $\varepsilon_{\mathfrak{f}}$ is not equal to η, but rather $\varepsilon_{\mathfrak{f}} = \eta^2 = \varepsilon^6$.

The element $\pi = 12 + \sqrt{5} \in \mathbb{N}_0[1 + \sqrt{5}]$ is prime in $\mathcal{O}_K$ (its norm is 139, which is prime, and $\mathcal{O}_K$ is factorial for our choice of K). Moreover, $12 + \sqrt{5} \equiv 1 \bmod \mathfrak{f}$, so it remains prime when we move to the ring $\mathbb{Z}[\sqrt{5}]$.

Note that $\eta = 2 + \sqrt{5}$ does not divide $12 + \sqrt{5}$ in $\mathbb{N}_0[1 + \sqrt{5}]$:

$$(12 + \sqrt{5})(2 + \sqrt{5})^{-1} = -19 + 10\sqrt{5} = -29 + 10(1 + \sqrt{5}).$$

However, η ought to divide some power of π. Since $\theta(\eta) < 0$, the divisibility criterion shows $\eta \mid \pi^k$ if and only if $\theta(\pi^k)$ is in the interval $[\theta(\eta), \theta(\eta\tau)]$, but since $\theta(\pi) > 0$, that is equivalent to having $\theta(\pi^k) \leq \theta(\eta\tau)$. We have $\theta(\pi) \approx 0.6859$, so the smallest power of π divisible by η in $\mathbb{N}_0[1 + \sqrt{5}]$ is $k = 11$. Indeed

$$(12 + \sqrt{5})^{11}(2 + \sqrt{5})^{-1} = 81893205959 + 329760376010(1 + \sqrt{5}).$$

Using the fact that $12 + \sqrt{5}$ is prime in $\mathbb{Z}[\tau]$, we can argue that the only irreducible factors of $(12 + \sqrt{5})^{11}$ in $\mathbb{Z}[\tau]$ are $12 + \sqrt{5}$, the fundamental unit $2 + \sqrt{5}$, and the large factor given above. It follows that the length set of $(12+\sqrt{5})^{11}$ in $\mathbb{N}_0[1+\sqrt{5}]$ is $\{2, 11\}$. This is essentially the argument we will generalize to prove the main theorem.

As demonstrated in the example, we want to find primes π of $\mathbb{Z}[\tau]$ which lie in $\mathbb{N}_0[\tau]$ and with $\theta(\pi)$ approximating some specified value. The following result is a direct corollary of work done by Hans Rademacher on the distribution of primes in quadratic integer rings [13].

Proposition 3.11. *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$, let $\rho \in \mathcal{O}_K$ be relatively prime to $\mathfrak{a}$, and let I be a nondiscrete subset of $\big(\theta(\varepsilon_{\mathfrak{a}}), 1\big]$. There exist infinitely many primes π of $\mathcal{O}_K$ such that $\pi \equiv \rho \bmod \mathfrak{a}$ and $\theta(\pi) \in I$.*

Proof. Define $L : (0,\infty) \to \mathbb{R}$ by $L(t) = \log_{\varepsilon_{\mathfrak{a}}^2}(1/t)$. Following Rademacher, we also define $w : K^\times \to \mathbb{R}$ by $w(\alpha) = L(|\theta(\alpha)|)$. For $v \in (0,1]$ and $x \geq 2$ let $c(x, v)$ be the number of primes π of $\mathbb{Z}[\tau]$ such that $\pi\bar{\pi} \leq x$, $\pi \equiv \rho \bmod \mathfrak{a}$, $\pi, \bar{\pi} > 0$, and $w(\pi) \in [0, v)$. Note that since $\varepsilon_{\mathfrak{a}}, \bar{\varepsilon}_{\mathfrak{a}} > 0$ by definition, $\bar{\varepsilon}_{\mathfrak{a}} = \varepsilon_{\mathfrak{a}}^{-1}$, hence $\theta(\varepsilon_{\mathfrak{a}}) = 1/\varepsilon_{\mathfrak{a}}^2$ and $w(\varepsilon_{\mathfrak{a}}) = 1$.

The Hauptsatz of [13] states that there are constants $A, B, H > 0$ depending only on K and $\mathfrak{a}$ such that for $x \geq 2$

$$|c(x, v) - Hv\,\mathrm{Li}(x)| \leq Ax\exp(-B\sqrt{\log x}),$$

where $\mathrm{Li}(x) = \int_2^x dt/\log t$.

It suffices to prove the proposition for $I = (t_0, t_1]$ where $1/\varepsilon_{\mathfrak{a}}^2 < t_0 < t_1 < 1$. The function L is monotonically decreasing and the image of $(1/\varepsilon_{\mathfrak{a}}^2, 1]$ under L is $[0, 1)$. Let $v_i = L(t_i)$ so that $0 < v_1 < v_0 < 1$. Applying the triangle inequality to Rademacher's result, we have

$$|c(x, v_0) - c(x, v_1) - H(v_0 - v_1)\,\mathrm{Li}(x)| \leq 2Ax\exp(-B\sqrt{\log x}).$$

One verifies easily that $\mathrm{Li}(x)$ dominates $2Ax\exp(-B\sqrt{\log x})$ (using L'Hôpital's rule, for example), so dividing through by $\mathrm{Li}(x)$ and taking $x \to \infty$ yields

$$\lim \frac{c(x, v_0) - c(x, v_1)}{\mathrm{Li}(x)} = H(v_0 - v_1) > 0.$$

In particular $c(x, v_0) - c(x, v_1) \to \infty$ as $x \to \infty$. Recalling the definition of c, this means that there are infinitely many primes π of $\mathbb{Z}[\tau]$ such that $\pi \equiv \rho \bmod \mathfrak{a}$, $\pi, \bar{\pi} > 0$, and $w(\pi) \in [v_1, v_0)$. Since the primes being counted satisfy $\pi, \bar{\pi} > 0$ (and therefore $\theta(\pi) > 0$), $w(\pi) \in [v_1, v_0)$ if and only if $\theta(\pi) \in (t_0, t_1]$. □

Corollary 3.12. *If $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ and $\rho \in \mathcal{O}_K$ is relatively prime to $\mathfrak{a}$ then the set $\{\theta(\pi) : \pi$ is prime in $\mathcal{O}_K$, $\pi \equiv \rho \bmod \mathfrak{a}\} \cap (0, 1]$ is dense in $(0, 1]$.*

Proof. Since $\varepsilon_{\mathfrak{a}}, \bar{\varepsilon}_{\mathfrak{a}} > 0$ and $\varepsilon_{\mathfrak{a}} > 1$, we have $0 < \theta(\varepsilon_{\mathfrak{a}}) < 1$. It follows that

$$(0, 1] = \bigcup_{k=0}^{\infty} \big(\theta(\varepsilon_{\mathfrak{a}}^{k+1}), \theta(\varepsilon_{\mathfrak{a}}^{k})\big]$$

and this union is clearly disjoint.

Let U be an open subset of $(0,1]$ and fix k so that $U' = U \cap \big(\theta(\varepsilon_{\mathfrak{a}}^{k+1}), \theta(\varepsilon_{\mathfrak{a}}^{k})\big]$ is nonempty, hence nondiscrete. Then $\theta(\varepsilon_{\mathfrak{a}}^{-k}) \cdot U'$ is a nondiscrete subset of $\big(\theta(\varepsilon_{\mathfrak{a}}), 1\big]$. By the preceding proposition there exists a prime element π of $\mathcal{O}_K$ such that $\pi \equiv \rho \mod \mathfrak{a}$ and $\theta(\pi) \in \theta(\varepsilon_{\mathfrak{a}}^{-k}) \cdot U'$. Finally, let $\pi' = \varepsilon_{\mathfrak{a}}^{k}\pi$. π' is prime in $\mathcal{O}_K$, $\pi' \equiv \rho \mod \mathfrak{a}$ (since $\varepsilon_{\mathfrak{a}} \equiv 1 \mod \mathfrak{a}$), and $\theta(\pi') \in U$. □

Corollary 3.13. *If I is a nondiscrete subset of $(0,1]$ there exist infinitely many primes π of $\mathbb{Z}[\tau]$ satisfying $\theta(\pi) \in I$.*

Proof. Applying the preceding corollary, there exist infinitely many primes π of $\mathcal{O}_K$ satisfying $\theta(\pi) \in I$ and $\pi \equiv 1 \mod \mathfrak{f}$. The congruence condition guarantees that π lies in $\mathbb{Z}[\tau]$ and that it remains prime in this subring (see Proposition 3.9). □

Note also that if the union of intervals in the proof of Corollary 3.12 is instead taken over all $k \in \mathbb{Z}$, we obtain all of $(0,\infty)$, so $(0,1]$ may be replaced with $(0,\infty)$ in both the above results. Another way to see this is to use the fact that if π is prime, then so is $\bar{\pi}$ and $\theta(\bar{\pi}) = 1/\theta(\pi)$.

Proposition 3.14. *Let k be an integer such that $k \geq 3$. There exists $\xi \in \mathbb{N}_0[\tau]^{\bullet}$ such that $\mathcal{L}(\xi) = \{2+\chi, k\}$ where $\chi = 0$ when $\tau \neq \eta$ and $\chi = 1$ when $\tau = \eta$.*

Proof. Suppose we have $\alpha \in \mathbb{N}_0[\tau]^{\bullet}$ with $\theta(\alpha) > 0$ and let

$$I_m = \big(\theta(\eta^m) \cdot [\theta(\tau), 1]\big) \cap (0,\infty) = \big(0, \max\{\theta(\eta^m), \theta(\eta^m\tau)\}\big].$$

Then α is divisible by η^m if and only if $\theta(\alpha) \in I_m$ (by Corollary 3.4).

We set $J = I_1 \setminus I_{2+\chi}$ and compute

$$J = \begin{cases} \big(\theta(\eta^2), \theta(\eta)\big] & \text{if } \tau \neq \eta \text{ and } \theta(\eta) > 0, \\ \big(\theta(\eta^2), \theta(\eta\tau)\big] & \text{if } \tau \neq \eta \text{ and } \theta(\eta) < 0, \\ \big(\theta(\eta^4), \theta(\eta^2)\big] & \text{if } \tau = \eta. \end{cases}$$

In all cases, J is a subinterval of $(0,1)$, open on the left and closed on the right. For any integer $i > 0$ let J_i denote the image of J under the map $(0,\infty) \to (0,\infty) : x \mapsto x^{1/i}$. Since $J \subset (0,1)$ and this map is continuous and monotonically increasing on $(0,1)$, we have $\max J_k > \max J_{k-1}$, so $J_k \setminus J_{k-1}$ is a nondiscrete subset of $(0,1]$. By Corollary 3.13 we may fix a prime π of $\mathbb{Z}[\tau]$ such that $\theta(\pi) \in J_k \setminus J_{k-1}$. By replacing π with its negative, we may assume that $\pi > 0$ and therefore $\pi \in \mathbb{N}_0[\tau]$ by Proposition 3.3.

Since $\theta(\pi) \in J_k$, $\theta(\pi^k) \in J \subset I_1$, so $\eta \mid \pi^k$ in $\mathbb{N}_0[\tau]$. On the other hand, $\theta(\pi) > \max J_{k-1}$, so $\theta(\pi^{k-1}) > \max J = \max I_1$ and therefore $\eta \nmid \pi^{k-1}$ in $\mathbb{N}_0[\tau]$. That is, k is the least positive integer such that $\eta \mid \pi^k$. Similarly, since $\theta(\pi^k) \in J$ and J is disjoint with $I_{2+\chi}$, $\eta^{2+\chi} \nmid \pi^k$ in $\mathbb{N}_0[\tau]$.

Let $\xi = \pi^k$. As shown above, $\eta \mid \xi$ but $\eta^{2+\chi} \nmid \xi$ in $\mathbb{N}_0[\tau]$. Let δ be an irreducible factor of ξ in $\mathbb{N}_0[\tau]$. Since then $\delta \mid \pi^k$ in $\mathbb{Z}[\tau]$ and π is prime in $\mathbb{Z}[\tau]$, $\delta = \eta^i \pi^j$ where $i, j \in \mathbb{Z}$ and $0 \leq j \leq k$. We have the following possibilities:

- If $i > 0$, then we must have $\delta = \eta$. This is irreducible by Proposition 3.5.

- If $i = 0$, then $\delta = \pi$. It is easily verified that π is irreducible in $\mathbb{N}_0[\tau]$ since $\eta \nmid \pi$ in $\mathbb{N}_0[\tau]$ and π is a prime element of $\mathbb{Z}[\tau]$.
- Assume that $i < 0$. Then $\eta^{|i|}$ divides π^j in $\mathbb{N}_0[\tau]$. However, since $\eta^{2+\chi} \nmid \pi^j$, $|i| \le 1 + \chi$. Hence, if $\tau \ne \eta$, we must have $i = -1$ and consequently also $j = k$ since k is the least positive integer such that $\eta \mid \pi^k$.

  On the other hand, if $\tau = \eta$ then $\eta \mid \pi^j$ implies $\eta^2 \mid \pi^j$ since $\theta(\pi^j) > 0$ (Proposition 3.7). Hence we obtain $(i, j) = (-2, k)$.

Hence the irreducible factors of ξ in $\mathbb{N}_0[\tau]$ are η, π, and $\eta^{-(1+\chi)}\pi^k$. It follows that the only atomic factorizations of ξ in $\mathbb{N}_0[\tau]$ are

$$\xi = (\eta)^{1+\chi}(\eta^{-(1+\chi)}\pi^k) = (\pi)^k,$$

so $\mathcal{L}(\xi) = \{2 + \chi, k\}$ as desired. □

Proposition 3.15. *The monoid* $\mathbb{N}_0[\tau]^\bullet$ *has a prime element.*

Proof. Let $\{1, \nu\}$ be a $\mathbb{Z}$-basis for $\mathcal{O}_K$ and suppose that $\tau = g + f\nu$ where $g, f \in \mathbb{Z}$ and $f \ne 0$. There are infinitely many rational primes which remain prime in $\mathcal{O}_K$ (this is true of asymptotically half of all rational primes when K is a quadratic number field—this is a consequence of quadratic reciprocity); fix such a prime $p > 0$ which does not divide f. An elementary argument shows that p is prime in $\mathbb{N}_0[\tau]^\bullet$; the details are left to the reader. □

We may now complete the proof of the main theorem. The fact that $\Delta(\mathbb{N}_0[\tau]^\bullet) = \mathbb{N}$ follows easily from Proposition 3.14. If $s \ge 1$ is rational, we can find j, k such that

$$\frac{j + k}{j + (2 + \chi)} = s.$$

So taking ξ as in the statement of Proposition 3.14, $\rho(p^j \xi) = s$ where p is some prime element of $\mathbb{N}_0[\tau]^\bullet$.

Theorems 2.3 and 3.1 raise a number of further questions about factorization in structures of this kind. For example, does every finite subset of $\{2, 3, \ldots\}$ occur as the factorization length set of some $f \in \mathbb{R}^+[X]$ or some $\alpha \in \mathbb{N}_0[\tau]$?

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Integer-valued polynomial in valued fields with an application to discrete dynamical systems

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Abstract. Integer-valued polynomials on subsets of discrete valuation domains are well studied. We undertake here a systematical study of integer-valued polynomials on subsets S of valued fields and of several connected notions: the polynomial closure of S, the Bhargava's factorial ideals of S and the v-orderings of S. A sequence of numbers is naturally associated to the subset S and a good description can be done in the case where S is regular (a generalization of the regular compact subsets of Y. Amice in local fields). Such a case arises naturally when we consider orbits under the action of an isometry.

Keywords. Integer-valued polynomial, generalized factorials, valued field, valuative capacity, discrete dynamical system.

AMS classification. 13F20, 13B65, 37B99.

1 Introduction

Integer-valued polynomials in a number field K is a natural notion introduced by Pólya [28] and Ostrowski [29]: they are polynomials f with coefficients in K such that $f(\mathcal{O}_K) \subseteq \mathcal{O}_K$ (where $\mathcal{O}_K$ denotes the ring of integers of K). The notion has been generalized, first in [7] by considering any integral domain D and the D-algebra

$$\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\} \tag{1.1}$$

(where K denotes the quotient field of D), and then, in [22] by considering any subset S of the integral domain D and the corresponding D-algebra

$$\mathrm{Int}(S, D) = \{f \in K[X] \mid f(S) \subseteq D\}, \tag{1.2}$$

that is called the ring of *integer-valued polynomials on S with respect to D*.

When D is Noetherian, the notion behaves well by localization: for every maximal ideal $\mathfrak{m}$ of D, one has the equality [8, I.2.7]

$$\mathrm{Int}(S, D)_{\mathfrak{m}} = \mathrm{Int}(S, D_{\mathfrak{m}}). \tag{1.3}$$

So that, when $D = \mathcal{O}_K$ is the ring of integers of a number field K, the study may be restricted to the case where S is a subset of the ring V of a discrete valuation v (with finite residue field). We then may use the notion of v-ordering introduced by Bhargava ([2] and [3]) that allows an algorithmic construction of bases of the V-module $\mathrm{Int}(S, V)$. Finally, the case of subsets of discrete valuation domains is well studied, although it remains some difficult questions. For instance, what happens when we replace the indeterminate X by several indeterminates (see Mulay [27] and Evrard [17])?

When the field K is no more a number field, but an infinite algebraic extension of $\mathbb{Q}$, the ring of integers $\mathcal{O}_K$ is no more a Dedekind domain, but a Prüfer domain. Then, we are not sure that things still work well under localization. We can find (see [11] or [8, VI.4.13]) characterizations of the equality

$$(\mathrm{Int}(\mathcal{O}_K))_{\mathfrak{m}} = \mathrm{Int}((\mathcal{O}_K)_{\mathfrak{m}}). \tag{1.4}$$

Without any condition on K, we only have the containment

$$(\mathrm{Int}(\mathcal{O}_K))_{\mathfrak{m}} \subseteq \mathrm{Int}((\mathcal{O}_K)_{\mathfrak{m}}), \tag{1.5}$$

and more generally, for every subset S of K, we have [8, I.2.4]

$$(\mathrm{Int}(S, \mathcal{O}_K))_{\mathfrak{m}} \subseteq \mathrm{Int}(S, (\mathcal{O}_K)_{\mathfrak{m}}). \tag{1.6}$$

In any case, it may be worth of interest to study the ring $\mathrm{Int}(S, V)$ formed by the integer-valued polynomials on a subset S of a (not necessarily discrete) rank-one valuation domain V.

In fact, it is known that many of the results concerning discrete valuation domains may be extended to rank-one valuation domains provided that the completion $\widehat{S}$ of S is assumed to be compact (see [9]) because, in that case we still have a p-adic Stone–Weierstrass theorem [10]. But, here, we wish to remove all restrictions on the subset S (while keeping the assumption that the valuation is rank one).

Classical definitions

Before beginning this study, let us first recall two general notions linked to integer-valued polynomials: the polynomial closure of a subset and the factorial ideals associated to a subset. For every integral domain D with quotient field K and every subset S of D, we may associate to the subset S, and to the corresponding D-algebra

$$\mathrm{Int}(S, D) = \{f \in K[X] \mid f(S) \subseteq D\} \tag{1.7}$$

of integer-valued polynomials on S with respect to D, the following notions:

Definition 1.1 (McQuillan [26]).

(i) A subset T of K is said to be *polynomially equivalent* to S if

$$\mathrm{Int}(T, D) = \mathrm{Int}(S, D). \tag{1.8}$$

(ii) The *polynomial closure* $\overline{S}$ of S is the largest subset of K which is polynomially equivalent to S, equivalently,

$$\overline{S} = \{t \in K \mid f(t) \in D \quad \forall f \in \mathrm{Int}(S, D)\}. \tag{1.9}$$

(iii) The subset S is said to be *polynomially closed* if

$$\overline{S} = S. \tag{1.10}$$

So that, to study the ring $\mathrm{Int}(S, D)$ we may replace S by its polynomial closure $\overline{S}$ or, when we do not know it, by any subset T which is polynomially equivalent to S. We now generalize to every subset S of an integral domain D the notion of factorial ideal introduced by Bhargava [4] for subsets of Dedekind domains.

Definition 1.2 ([4] and [9]). For every $n \in \mathbb{N}$, the *n-th factorial ideal* of the subset S with respect to the domain D is the inverse $n!_S^D$ (or simply $n!_S$) of the D-module generated by the leading coefficients of the polynomials of $\mathrm{Int}(S, D)$ of degree n, where the inverse of a sub-D-module N of K is $N^{-1} = \{x \in K \mid xN \subseteq D\}$.

Note that, in particular, $K^{-1} = (0)$ and $(0)^{-1} = K$. Bhargava [4] showed that these factorial ideals may have fine properties extending those of the classical factorials. For instance, when D is a Dedekind domain,

$$\forall n, m \in \mathbb{N} \quad n!_S^D \times m!_S^D \text{ divides } (n+m)!_S^D. \tag{1.11}$$

We fix now the hypotheses and notation for the whole paper.

2 Hypotheses, notation and v-orderings

2.1 Let K be a valued field,

that is, a field endowed with a rank-one valuation v. Then, the value group $\Gamma = v(K^*)$ is a subgroup of the additive group $\mathbb{R}$. We denote by V the corresponding valuation domain, by $\mathfrak{m}$ the maximal ideal and by k the residue field $V/\mathfrak{m}$.

As usual, we define an absolute value on K by letting

$$\forall\, x \in K^* \quad |x| = e^{-v(x)}. \tag{2.1}$$

For $x \in K$ and $\gamma \in \mathbb{R}$, we denote by $B(x, \gamma)$ the ball of center x and radius $e^{-\gamma}$, that is,

$$B(x, \gamma) = \{y \in K \mid v(x - y) \geq \gamma\}. \tag{2.2}$$

Remark 2.1. With respect to the polynomial closure, we may notice the following:

(i) Since every polynomial $f \in K[X]$ is a continuous function on K, the polynomial closure $\overline{S}$ of any subset S of K obviously contains the topological closure $\widetilde{S}$ of S in K:

$$\widetilde{S} \subseteq \overline{S}. \tag{2.3}$$

(ii) There are subsets S such that $\widetilde{S} \neq \overline{S}$ (Remarks 6.4 (ii) and 11.4 (2)).

(iii) In general, polynomially closed subsets are stable under intersection [8, IV.1.5], but not under finite union [8, IV.4.Exercise 2]. Nevertheless,we will see that, in a valued field K, a ball is polynomially closed and a finite union of balls is still polynomially closed (Proposition 8.2). (All the balls that we will consider are closed balls of the form $B(x, \gamma)$ unless the contrary is explicitly stated.)

2.2 Now we fix a subset S of K.

Since we are in a valued field K, the factorial ideals $n!_S$ of S are characterized by their valuations. Thus, we introduce the following arithmetical function:

Definition 2.2 ([9]). The *characteristic function* of S is the function w_S defined by

$$\forall n \in \mathbb{N} \quad w_S(n) = v(n!_S) \tag{2.4}$$

with the convention that $v((0)) = +\infty$ and $v(K) = -\infty$.

Obviously, $0!_S = V$, and then, $w_S(0) = 0$. In the case of valued fields, factorial ideals have still fine properties. For instance, (1.11) becomes

$$\forall n, m \in \mathbb{N} \quad w_S(n+m) \geq w_S(n) + w_S(m). \tag{2.5}$$

The following proposition is an obvious consequence of the previous inclusion (2.3):

Proposition 2.3. *Denoting by $\widetilde{S}$ the topological closure of S in K, we have*

$$\mathrm{Int}(\widetilde{S}, V) = \mathrm{Int}(S, V) \quad \textit{and hence, for every } n \geq 0,\ n!_{\widetilde{S}} = n!_S.$$

Denoting by $\widehat{V}$ and $\widehat{S}$ the completions of V and S with respect to the topology induced by v, we have

$$\mathrm{Int}(\widehat{S}, \widehat{V}) = \mathrm{Int}(S, \widehat{V}) = \mathrm{Int}(S, V)\widehat{V} \quad \textit{and, for every } n \geq 0,\ n!_{\widehat{S}} = n!_S \widehat{V}.$$

Equivalently,

$$\forall n \in \mathbb{N} \quad w_S(n) = w_{\tilde{S}}(n) = w_{\widehat{S}}(n). \tag{2.6}$$

We will see that most often the characteristic function w_S may be computed by means of the following notion of generalized v-ordering which extends the notion of v-ordering due to Bhargava [2].

2.3 v-orderings

Definition 2.4 ([9]). Let $N \in \mathbb{N} \cup \{+\infty\}$. A sequence $\{a_n\}_{n=0}^N$ of elements of S is called a *v-ordering* of S if, for $1 \leq n \leq N$, one has

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = \inf_{x \in S} v\left(\prod_{k=0}^{n-1}(x - a_k)\right). \tag{2.7}$$

The proof of [19, Proposition 4] may be extended to these generalized v-orderings, so that we have another characterization of the v-orderings of S:

Proposition 2.5. *The sequence $\{a_n\}_{n=0}^N$ of elements of S is a v-*ordering *of S if and only if*

$$\forall n \geq 1\ \forall x_0, \ldots, x_n \in S \quad \prod_{0 \leq i < j \leq n}(x_j - x_i) \ \textit{ is divisible by } \prod_{0 \leq i < j \leq n}(a_j - a_i). \tag{2.8}$$

Although these notions of integer-valued polynomial, factorial ideal, polynomial closure and v-ordering have already been partially studied, we wish to undertake here a systematical study of them. The following proposition shows strong links between them.

Proposition 2.6 ([2] and [9]). *Let $N \in \mathbb{N} \cup \{+\infty\}$ and let $\{a_n\}_{n=0}^N$ be a sequence of distinct elements of S. We associate to this sequence of elements a sequence $\{f_n\}_{n=0}^N$ of polynomials*

$$f_0(X) = 1 \quad \textit{and, for } 1 \leq n \leq N,\ f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k}. \tag{2.9}$$

The following assertions are equivalent:

(i) *The sequence $\{a_n\}_{n=0}^N$ is a v-ordering of S.*

(ii) *The sequence $\{a_n\}_{n=0}^N$ is a v-ordering of $\overline{S}$.*

(iii) *For each $n \leq N$, f_n belongs to* $\mathrm{Int}(S, V)$.

(iv) *The sequence $\{f_n\}_{n=0}^N$ is a basis of the V-module*

$$\mathrm{Int}_N(S, V) = \{f \in \mathrm{Int}(S, V) \mid \deg(f) \leq N\}. \tag{2.10}$$

(v) *For $1 \leq n \leq N$, one has*

$$n!_S = \prod_{k=0}^{n-1}(a_n - a_k)V, \textit{ that is, } w_S(n) = v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right). \tag{2.11}$$

This proposition shows in particular that, for $0 \leq n \leq N$, the real numbers $v\big(\prod_{k=0}^{n-1}(a_n - a_k)\big)$ do not depend on the v-ordering $\{a_n\}_{n=0}^N$. But, as shown in the following remark, there does not always exist v-orderings for a given subset S although there always exist integer-valued polynomials on S and factorials associated to S.

Remark 2.7. (i) There does not always exist v-orderings when the valuation is not discrete. For instance, assume that the valuation v is not discrete and the subset S is equal to the maximal ideal $\mathfrak{m}$ of V. Then, S does not admit any v-ordering since, for every $s, t \in \mathfrak{m}$, $v(s - t) > 0$ while $\inf_{t \in \mathfrak{m}} v(t - s) = 0$.

(ii) As a v-ordering of S is also a v-ordering of the polynomial closure $\overline{S}$ of S, the relevant question is more likely the existence of a v-ordering in $\overline{S}$. For instance, in the previous example, with v non discrete and $S = \mathfrak{m}$ (the maximal ideal of V), there is no v-ordering in $\mathfrak{m}$. Yet $\overline{S} = V$, since $\mathrm{Int}(\mathfrak{m}, V) = \mathrm{Int}(V, V) = V[X]$ (see [9, Remark 8] or Theorem 4.3 below). This larger subset $\overline{S}$ may admit v-orderings. Indeed, a sequence $\{a_n\}$ of elements of the non-discrete rank-one valuation domain V is a v-ordering of V if and only if the a_n's are in distinct classes modulo $\mathfrak{m}$, and such a sequence exists if and only if the residue field $k = V/\mathfrak{m}$ is infinite.

Corollary 2.8. *If the sequence $\{a_n\}_{n \in \mathbb{N}}$ is an infinite v-ordering of S, then the subset $T = \{a_n \mid n \in \mathbb{N}\}$ is polynomially equivalent to S.*

2.4 Notation

For $\gamma \in \mathbb{R}$ and $a, b \in K$, we say that

$$a \text{ and } b \text{ are } \textit{equivalent modulo } \gamma \text{ if } v(a-b) \geq \gamma.$$

Then, for our fixed subset S and for every $\gamma \in \mathbb{R}$, we denote by

- $S(a, \gamma)$ the equivalence class modulo γ of the element $a \in S$, that is,

$$S(a, \gamma) = S \cap B(a, \gamma), \tag{2.12}$$

- $S \bmod \gamma$ the set formed by the equivalence classes modulo γ of the elements of S,
- S_γ any set of representatives of $S \bmod \gamma$,
- q_γ the cardinality (finite or infinite) of $S \bmod \gamma$:

$$q_\gamma = q_\gamma(S) = \mathrm{Card}(S \bmod \gamma) = \mathrm{Card}(S_\gamma). \tag{2.13}$$

Since q_γ may be finite or infinite, it is natural to introduce the following:

$$\gamma_\infty = \gamma_\infty(S) = \sup\{\gamma \mid q_\gamma \text{ finite}\}. \tag{2.14}$$

Noticing that in the definition of a v-ordering what is important is not the valuation of the elements of S but the valuation of the differences of elements of S, we are led to consider another natural number:

$$\gamma_0 = \gamma_0(S) = \inf\{v(x-y) \mid x, y \in S, x \neq y\}. \tag{2.15}$$

Clearly,

$$-\infty \leq \gamma_0(S) \leq \gamma_\infty(S) \leq +\infty. \tag{2.16}$$

We will see that the fact that one of these three inequalities becomes an equality corresponds to three particular cases:

- $\gamma_0 = -\infty$ if and only if S is a non-fractional subset (see Section 3).
- $\gamma_0 = \gamma_\infty$ if and only if $\mathrm{Int}(S, V)$ is isomorphic to a generalized polynomial ring (see Section 4).
- $\gamma_\infty = +\infty$ if and only if the completion $\widehat{S}$ of S is compact, which corresponds to the well studied case (see Section 6).

2.5 The results

After studying these three particular cases, we shall assume that we have

$$-\infty < \gamma_0 < \gamma_\infty < +\infty,$$

hence in particular that S is a fractional subset. We consider two cases, whether q_{γ_∞} is finite or infinite. In Section 5, we associate to S a natural sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ of critical valuations. If q_{γ_∞} is finite, this sequence is finite, otherwise it is infinite. In Section 7, we prove an inequality concerning the characteristic function w_S:

$$\forall n \in \mathbb{N} \qquad \frac{w_S(n)}{n} \le \gamma_\infty.$$

In Sections 8 and 9, we establish some containments concerning the polynomial closure: in Section 8, when q_{γ_∞} is finite, we show that $\overline{S} \subseteq S + B(0, \gamma_\infty)$ and, in Section 9, we show that the polynomial closure contains not only the topological closure but also the pseudo-closure, which is a subset that we naturally associate to the pseudo-convergent sequences introduced by Ostrowski [29]. Then, in Section 10, we characterize the case where $S + B(0, \gamma_\infty) \subseteq \overline{S}$: we prove that this containment holds in particular when S is a regular subset, which is a generalization of the notion of regular compact subset introduced in 1964 by Y. Amice [1]. As an application, we show in Section 11 that, when S is any orbit under the action of an isometry, then S is a regular subset and this regular subset is either discrete or precompact. Finally, we end this paper by giving in Section 12 explicit examples. In a forthcoming paper [14], we will study such regular subsets and show that in this case the v-orderings have very strong properties.

3 The non-fractional case ($\gamma_0 = -\infty$)

The following lemma is obvious.

Lemma 3.1. *Let $a \in K^*$ and $b \in K$. Consider $T = aS + b = \{as + b \mid s \in S\}$. Then, the automorphism*

$$f(X) \in K[X] \mapsto f\left(\frac{X-b}{a}\right) \in K[X]$$

induces an isomorphism between the rings $\mathrm{Int}(S, V)$ *and* $\mathrm{Int}(T, V)$. *Obviously,*

$$n!_T = a^n n!_S, \quad \textit{equivalently}, \quad w_T(n) = nv(a) + w_S(n). \tag{3.1}$$

Moreover, for every $N \in \mathbb{N} \cup \{+\infty\}$, the sequence $\{a_n\}_{n=0}^N$ is a v-ordering of S if and only if the sequence $\{aa_n + b\}_{n=0}^N$ is a v-ordering of T.

Consequently, for our study, we may replace the set S by $aS + b$ for any $a \in K^*$ and $b \in K$. We are then led to consider whether S is a fractional subset of K or not. Recall that

Definition 3.2. The subset S of K is said to be a *fractional subset* of K if there exists some $d \in K^*$ such that $dS \subseteq V$.

Theorem 3.3. *The following assertions are equivalent:*

(i) *S is not a fractional subset,*

(ii) $\gamma_0 = -\infty$,

(iii) $\gamma_\infty = -\infty$,

(iv) $\mathrm{Int}(S, V) = V$,

(v) $n!_S = K$ *for* $n \geq 1$,

(vi) $w_S(n) = -\infty$ *for* $n \geq 1$,

(vii) *The polynomial closure* $\overline{S}$ *of* S *is equal to* K.

Proof. Obviously, on the one hand, assertions (i), (ii), and (iii) are equivalent and, on the other hand, assertions (iv), (v) and (vi) are equivalent. The equivalence between (i) and (iv) is known ([26] or [8, Corollary I.1.10]). The equivalence between (iv) and (vii) is obvious. □

Remark 3.4. Clearly, a non-fractional subset does not admit any v-ordering.

From now on, we will assume that S is a fractional subset. Then, Lemma 3.1 allows us to replace S by the subset $dS = \{ds \mid s \in S\}$ where d denotes any element of K such that $v(d) \geq -\gamma_0$, so that, we may assume that $S \subseteq V$. Moreover, replacing S by the subset $S - s_0 = \{s - s_0 \mid s \in S\}$ for some $s_0 \in S$, we may also assume that $0 \in S$.

4 The polynomial ring case ($-\infty < \gamma_0 = \gamma_\infty < +\infty$)

Notation. For $\gamma \in \mathbb{R}$ and $x \in K$, let

$$V[(X - x)/\gamma] = \left\{ \sum_{k=0}^{n} a_k (X - x)^k \in K[X] \mid v(a_k) \geq -k\gamma \right\}. \tag{4.1}$$

Obviously, if there exists $t \in K$ such that $v(t) = \gamma$, then the ring $V[(X - x)/\gamma]$ is a classical polynomial ring:

$$V[(X - x)/\gamma] = V\left[\frac{X - x}{t}\right]. \tag{4.2}$$

Lemma 4.1. *With the previous notation, for every* $a \in S$ *and* $\gamma \in \mathbb{R}$, *one has*

$$V[(X - a)/\gamma] \subseteq \mathrm{Int}(B(a, \gamma), V) \subseteq \mathrm{Int}(S(a, \gamma), V). \tag{4.3}$$

Proof. Let $f \in V[(X - a)/\gamma]$ and write

$$f(X) = \sum c_n (X - a)^n \quad \text{where} \quad v(c_n) \geq -n\gamma.$$

For every $b \in B(a, \gamma)$, one has

$$f(b) = \sum c_n (b - a)^n \quad \text{with} \quad v(c_n (b - a)^n) \geq 0,$$

thus $f(b) \in V$. □

To characterize the ring $\mathrm{Int}(S, V)$ we need a technical lemma that will be useful for other cases. Here, we denote by $\widehat{\Gamma}$ the completion of Γ in $\mathbb{R}$, that is, $\widehat{\Gamma} = \Gamma$ if v is discrete, and $\widehat{\Gamma} = \mathbb{R}$ if v is not discrete.

Lemma 4.2. *Assume that $\gamma \in \widehat{\Gamma}$, $\delta \in \mathbb{R}$ and $a \in S$ are such that $\gamma < \delta$ and $S(a, \gamma) \bmod \delta$ is infinite. Then, for every $f \in \mathrm{Int}(S, V)$, one has*

$$f \in V[(X-a)/\rho] \quad \textit{where } \rho = \gamma + \frac{n(n+1)}{2}(\delta - \gamma) \textit{ and } n = \deg(f). \tag{4.4}$$

Proof. We do not know whether $\gamma \in \Gamma$ but, for each integer s, there exists $z_s \in K$ with $\gamma - \frac{1}{s} \leq v(z_s) \leq \gamma$. Fix an integer s and let T be the following subset of V:

$$T = \left\{ \frac{x-a}{z_s} \mid x \in S(a, \gamma) \right\}.$$

Let $g \in \mathrm{Int}(T, V)$ be of degree n. As $S(a, \gamma) \bmod \delta$ is infinite, letting $\varepsilon_s = \delta - \gamma + \frac{1}{s}$, one can find $t_0, t_1, \ldots, t_n$ in T such that, for $i \neq j$, $v(t_i - t_j) < \varepsilon_s$. It then follows from Cramer's rule (see for instance [8, Proposition I.3.1]) that the valuation of each coefficient of g is greater or equal to

$$-v\Bigg(\prod_{0 \leq i < j \leq n} (t_j - t_i) \Bigg) > -\frac{n(n+1)}{2} \varepsilon_s.$$

Consequently, if $f(X) = \sum_{m=0}^{n} b_m (X-a)^m$ belongs to $\mathrm{Int}(S, V)$, then $g(X) = \sum_{m=0}^{n} b_m z_s^m (X-a)^m$ belongs to $\mathrm{Int}(T, V)$, and hence, one has

$$\forall m \geq 1 \quad v(b_m) > -m\gamma - \frac{n(n+1)}{2} \varepsilon_s.$$

Since s may tend to $+\infty$, we obtain the inequality

$$\forall m \geq 1 \quad v(b_m) \geq -m\gamma - \frac{n(n+1)}{2}(\delta - \gamma) \geq -m \left(\gamma + \frac{n(n+1)}{2}(\delta - \gamma) \right). \quad \square$$

Theorem 4.3. *If $\gamma_0 = \gamma_\infty > -\infty$, then*

$$\mathrm{Int}(S, V) = V[X/\gamma_0] \quad \textit{and} \quad \overline{S} = B(0, \gamma_0). \tag{4.5}$$

In particular,

$$w_S(n) = n\gamma_0. \tag{4.6}$$

Proof. We may apply the previous lemma. Clearly, $\gamma_0 \in \widehat{\Gamma}$ and $S = S(0, \gamma_0)$. By definition of γ_∞, for every $\delta > \gamma_\infty = \gamma_0$, q_δ is infinite, that is, $S(0, \gamma_0) \bmod \delta$ is infinite. So that, if $f \in \mathrm{Int}(S, V)$, then $f \in V[X/\rho]$ where ρ may tend to γ_0 when δ tends to γ_0. Finally, $f \in V[X/\gamma_0]$. By Lemma 4.1, as $S = S(0, \gamma_0)$, we thus have

$$V[X/\gamma_0] \subseteq \mathrm{Int}(B(0, \gamma_0), V) \subseteq \mathrm{Int}(S, V) \subseteq V[X/\gamma_0].$$

It follows that $\text{Int}(S, V) = V[X/\gamma_0]$ and also that $S \subseteq B(0, \gamma_0) \subseteq \overline{S}$. The following lemma shows that $B(0, \gamma_0)$ is polynomially closed and thus allows to conclude that $\overline{S} = B(0, \gamma_0)$. □

Lemma 4.4. *For every $x \in K$ and every $\gamma \in \mathbb{R}$, the ball $B(x, \gamma)$ is polynomially closed.*

Proof. If γ belongs to Γ, there exists $t \in K$ such that $v(t) = \gamma$, then the polynomial $f(X) = \frac{1}{t}(X - x)$ belongs to $\text{Int}(B(x, \gamma), V)$ and, for every $y \in K$, $f(y) \in V$ implies $v(y - x) \geq \gamma$, that is, $y \in B(x, \gamma)$.

If the valuation v is discrete, there exists $\delta \in \Gamma$ such that $B(x, \gamma)$ is equal to $B(x, \delta)$, which is polynomially closed. If the valuation v is not discrete, there exists an increasing sequence $\{\delta_n\}_{n \in \mathbb{N}}$ such that $\delta_n \in \Gamma$ for every n and $\lim_n \delta_n = \gamma$. Consequently, $B(x, \gamma) = \cap_{n \in \mathbb{N}} B(x, \delta_n)$. By the first argument, the balls $B(x, \delta_n)$ are polynomially closed and we know that an intersection of polynomially closed subsets is a polynomially closed subset (see [8, IV.1.5]). □

Theorem 4.3 says that, if $\gamma_0 = \gamma_\infty > -\infty$, then $\overline{S}$ is a ball. Conversely:

Proposition 4.5. *Assume that $S = B(x, \gamma)$. Then,*

(i) $\overline{S} = S$.

(ii) $\gamma_0 = \min\{\delta \in \Gamma \mid \delta \geq \gamma\}$ *(in fact, $\gamma_0 = \gamma$ if v is not discrete).*

(iii) *If v is discrete and $V/\mathfrak{m}$ is finite with cardinality q, then*

$$\gamma_\infty = +\infty \quad \textit{and} \quad w_S(n) = n\gamma_0 + w_V(n) = n\gamma_0 + \sum_{k \geq 1} \left[\frac{n}{q^k}\right].$$

(iv) *If either v is not discrete or $V/\mathfrak{m}$ is infinite, then*

$$\gamma_\infty = \gamma_0, \quad \text{Int}(S, V) = V[(X - x)/\gamma_0] \quad \textit{and} \quad w_S(n) = n\gamma_0.$$

(v) *S admits infinite v-orderings if and only if: either v is discrete, or $\gamma \in \Gamma$ and $V/\mathfrak{m}$ is infinite.*

Proof. (i) is Lemma 4.4.

(ii) is obvious.

Note that one can replace $B(x, \gamma)$ by $B(x, \gamma_0)$ and then, using Lemma 3.1 when $\gamma_0 \in \Gamma$, one can replace $B(x, \gamma_0)$ by $V = B(0, 0)$.

(iii) follows from [8, Theorem II.2.7].

(iv) The hypothesis implies $\gamma_\infty = \gamma_0$, and then, (iv) follows from Theorem 4.3.

(v) The existence of v-orderings is obvious in both cases. Conversely, assume that v is not discrete. If $\gamma \notin \Gamma$ then, for all $x, y \in S$, one has $v(x - y) > \gamma$ while $\inf_{x,y \in S} = \gamma$ so that, there is no element in S for the second term of a v-ordering. If $\gamma \in \Gamma$, we replace S by V. Then, it follows from (iv) that $w_V(n) = 0$. But, if $\text{Card}(V/\mathfrak{m}) = q$ then, for all $x_0, \ldots, x_q \in V$, $v(\prod_{0 \leq i < j \leq q}(x_j - x_i)) > 0$, so that, there is no element in V for the $(q + 1)$-th term of a v-ordering. □

5 The critical valuations of a fractional subset ($\gamma_0 < \gamma_\infty$)

Now, we may assume that $\gamma_0 < \gamma_\infty$, which implies in particular that S is a fractional subset. Moreover, since there exists $\gamma > \gamma_0$ such that $S \bmod \gamma$ is finite, there exist s_0 and $s_1 \in S$ such that $v(s_0 - s_1) = \gamma_0$. Then, if we replace S by $T = \left\{\frac{s-s_1}{s_0-s_1} \mid s \in S\right\}$, 0 and 1 belong to T and $\gamma_0(T) = 0$. So that:

From now on, we assume that

$$S \subseteq V,\ 0, 1 \in S,\ \gamma_0 = 0,\ q_0 = 1 \ \textit{and}\ \ 0 < \gamma_\infty \leq +\infty. \tag{5.1}$$

Moreover, we may choose $S_0 = \{0\}$.

We are interested in the study of the function

$$\gamma \in \mathbb{R} \mapsto q_\gamma \in \mathbb{N} \cup \{+\infty\}. \tag{5.2}$$

This is an increasing function and, by definition of γ_∞, $q_\gamma = +\infty$ for $\gamma > \gamma_\infty$. Moreover, for every $\gamma < \gamma_\infty$, q_γ is finite, thus $\sup\{v(a-b) \mid a, b \in S_\gamma, a \neq b\}$ is a maximum, and hence, is $< \gamma$. Consequently, there exists $\varepsilon > 0$ such that $q_\delta = q_\gamma$ for $\gamma - \varepsilon \leq \delta \leq \gamma$. The function $\gamma \mapsto q_\gamma$ is piecewise constant and left continuous. So that, we have the following proposition:

Proposition 5.1. *For every γ such that q_γ is finite, let*

$$\tilde{\gamma} = \sup\,\{\delta \mid q_\delta = q_\gamma\}. \tag{5.3}$$

These suprema are maxima and the $\tilde{\gamma}$'s may be written as elements of a strictly increasing sequence

$$\{\gamma_k\}_{0 \leq k \leq l} \ \ \textit{or}\ \ \{\gamma_k\}_{k \geq 0}.$$

The sequence $\{\gamma_k\}$ is finite if and only if q_{γ_∞} is finite, and then $\gamma_l = \gamma_\infty$. The sequence $\{\gamma_k\}$ is infinite if and only if q_{γ_∞} is infinite, and then

$$\lim_{k \to +\infty} \gamma_k = \gamma_\infty. \tag{5.4}$$

In other words, the γ_k's are characterized by

$$\gamma_0 = 0 \quad \textit{and, for}\ \ k \geq 1\text{:} \quad \gamma_{k-1} < \gamma \leq \gamma_k \ \Leftrightarrow\ q_\gamma = q_{\gamma_k}. \tag{5.5}$$

Note that, when q_{γ_∞} is infinite and γ_∞ is finite, then necessarily the valuation v is not discrete.

Definition 5.2. The sequence $\{\gamma_k\}$, finite or infinite introduced in Proposition 5.1, is called the *sequence of critical valuations* of S.

Remark 5.3. It follows from Proposition 5.1 that it is possible to choose the elements of the S_γ's, the sets of representatives of S modulo γ, for $\gamma < \gamma_\infty$, in such a way that

$$\gamma < \delta \ \Rightarrow\ S_\gamma \subseteq S_\delta. \tag{5.6}$$

Indeed, for each $k \geq 0$, we just have to choose the elements of $S_{\gamma_{k+1}}$ in such a way that $S_{\gamma_k} \subset S_{\gamma_{k+1}}$. We always assume that this condition is satisfied and also that $S_0 = \{0\}$. With such a choice, when $\gamma_\infty = \lim_{k\to\infty} \gamma_k$, we may also assume

$$\cup_{\gamma<\gamma_\infty} S_\gamma = \cup_{k\geq 0} S_{\gamma_k} \subseteq S_{\gamma_\infty}. \tag{5.7}$$

This last containment may be strict (see Example 5.4 (i) below).

Example 5.4. (i) If $K = \mathbb{Q}_p$ and $S = V = \mathbb{Z}_p$, then $\gamma_k = k$, $q_{\gamma_k} = p^k$, $S_{\gamma_k} = \{a \in \mathbb{N} \mid 0 \leq a < p^k\}$ and $\cup_{k\geq 0} S_{\gamma_k} = \mathbb{N}$ while $\gamma_\infty = +\infty$ and $S_{\gamma_\infty} = \mathbb{Z}_p = S$.

(ii) If $K = \mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$ and if

$$S = \left\{ \sum_{k=0}^{n} \varepsilon_k p^{1-\frac{1}{k+1}} \mid n \in \mathbb{N}, \varepsilon_k \in \{0,1\} \right\},$$

then $\gamma_k = 1 - \frac{1}{k+1}$, $q_{\gamma_k} = 2^k$, $\gamma_\infty = 1$ and $\cup_{k\geq 0} S_{\gamma_k} = S_{\gamma_\infty} = S$.

(iii) Consider the previous example and let

$$T = S \cup \{S + p^2\}.$$

Then,

$$\gamma_k(T) = \gamma_k(S),\ \gamma_\infty(T) = 1 \ \text{ and } \ \cup_{k\geq 0} T_{\gamma_k} = T_{\gamma_\infty} \neq T.$$

6 The precompact case ($\gamma_\infty = +\infty$)

The fact that $\gamma_\infty = +\infty$ is clearly equivalent to the fact that S is precompact, that is, the completion $\widehat{S}$ of S is compact (see for instance [10, Lemma 3.1]). We have to distinguish two cases whether S is finite or not.

6.1 *S* finite

This case is well described by McQuillan [25] for subsets of any integral domain (see also [8, Exercises IV.1, V.2, VI.20, VIII.25 and VIII.28] and [16]). For finite subsets of a valued field K, one can say a bit more. Obviously, there are v-orderings since at each step we just have to choose between a finite number of elements and, clearly, the first assertion of the following proposition is true.

Proposition 6.1. *Assume that S is finite with cardinality N.*

(i) *There exist infinite v-orderings $\{a_n\}_{n\in\mathbb{N}}$ of S. Moreover, if $\{a_n\}_{n\in\mathbb{N}}$ is a v-ordering of S, the a_n's for $n = 0$ and $n \geq N$ may be arbitrarily chosen, but necessarily, $\{a_n \mid 0 \leq n < N\} = S$.*

(ii) $\overline{S} = S = \widetilde{S}$.

(iii) $w_S(n) = +\infty \;\Leftrightarrow\; n \geq N$.

(iv) *Let*

$$f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k} \text{ for } 0 \leq n < N \text{ and } \varphi(X) = \prod_{n=0}^{N-1} (X - a_n) = \prod_{s \in S} (X - s).$$

Then,

$$\mathrm{Int}(S, V) = \oplus_{n=0}^{N-1} V f_n(X) \oplus K[X]\varphi(X). \tag{6.1}$$

Proof. Assertion (i) is obvious and assertion (ii) is also well known. Assertion (iii) results from assertion (i). Let us prove assertion (iv). Proposition 2.6 implies

$$\mathrm{Int}_{N-1}(S, V) = \sum_{n=0}^{N-1} V f_n(X). \tag{6.2}$$

Now, let $f \in \mathrm{Int}(S, V)$ and write $f = \varphi g + h$ where $g, h \in K[X]$ and $\deg(h) < N$. Then, $h = f - \varphi g \in \mathrm{Int}_{N-1}(S, V)$. □

Remark 6.2. The number of sequences such that $\{a_n\}_{n=0}^{N-1}$ is a v-ordering of S is at least N, since a_0 may be arbitrarily chosen in S, and at most $N!$. Note that the upper bound $N!$ is reached for instance when the elements of S are non-congruent modulo $\mathfrak{m}$. On the contrary, the lower bound N is never reached for $N > 2$.

6.2 S infinite

This is also a well studied case.

Proposition 6.3 ([9]). *Assume that S is infinite and $\widehat{S}$ is compact. Then:*

(i) *There always exist infinite v-orderings in S.*

(ii) *The polynomial closure $\overline{S}$ of S is equal to its topological closure $\widetilde{S}$ in K.*

Proof. The first assertion is [9, Lemma 17], and the second assertion is [9, Theorem 10]. □

Remark 6.4. (i) Note that $\gamma_\infty < +\infty$ implies that either the valuation v is not discrete or the residue field $k = V/\mathfrak{m}$ is infinite (else $\widehat{V}$ would be compact and $\widehat{S}$ as well).

(ii) When $\gamma_\infty < +\infty$, we may have $\widetilde{S} \neq \overline{S}$. For instance, let $t \neq 0$ be such that $v(t) > 0$ and let $S = \{t^{-k} \mid k \in \mathbb{N}\}$. Then, S is not a fractional subset, and hence its polynomial closure $\overline{S}$ is equal to K, while its topological closure $\widetilde{S}$ is equal to S. We will see more interesting cases with Remark 11.4.2.

(iii) In the precompact case ($\gamma_\infty = +\infty$), one has the equality $S = S_{\gamma_\infty}$. This last equality, that may be thought as a generalization of the precompact case, means when $\gamma_\infty < +\infty$ that every class modulo γ_∞ contains only one element: $\forall a \in S,\ S(a, \gamma_\infty) = \{a\}$ (S is uniformly discrete), and hence, $\widetilde{S} = S$. In particular, since $0 \in S$, for every nonzero element $a \in S$, $v(a) < \gamma_\infty$. Note also that, when $\gamma_\infty < +\infty$, $S = S_{\gamma_\infty}$ is equivalent to $S = \cup_k S_{\gamma_k}$ because q_{γ_∞} is infinite, and hence $\gamma_\infty = \lim_k \gamma_k$.

With respect to Proposition 6.3, note that the first assertion still holds: if $S = S_{\gamma_\infty}$, then S admits infinite v-orderings [12, Proposition 2.3]. On the other hand, the second assertion cannot be extended since we may have $S = S_{\gamma_\infty}$ and $\widetilde{S} \neq \overline{S}$ (see Section 12).

(iv) Another argument that leads to say that $S = S_{\gamma_\infty}$ generalizes the precompact case is the notion of pseudo-convergence introduced by Ostrowski [30, p. 368] and used by Kaplansky [24] in the study of immediate extensions of valued fields. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of K is said to be pseudo-convergent if

$$\forall\, i,\ j,\ k \quad [\, i < j < k \ \Rightarrow\ v(x_j - x_i) < v(x_k - x_j)\,]. \tag{6.3}$$

One may prove that, if $S_{\gamma_\infty} = S$, then from every infinite sequence of elements of S one can extract a pseudo-convergent subsequence (see [13, §1.12]).

7 On the characteristic function

In this section we assume that $\gamma_\infty < +\infty$.

Theorem 7.1. *For every subset S, one has*

$$\forall n \in \mathbb{N} \quad w_S(n) \le n\gamma_\infty. \tag{7.1}$$

Note that, when $\gamma_\infty = +\infty$, the previous theorem does not give any information on the function w_S, and that, when $\gamma_\infty = -\infty$, that is, when S is not a fractional subset, these inequalities are still true: they mean $w_S(0) = 0$ and, for $n \geq 1$, $w_S(n) = -\infty$, that is, $\mathrm{Int}(S, V) = V$.

Proof. Assume that $\gamma \in \Gamma$ and $\delta \in \mathbb{R}$ are such that $\gamma < \delta$, q_γ is finite and q_δ is infinite. Then, there necessarily exits an $a \in S$ such that $S(a, \gamma)$ mod δ is infinite. It follows from Lemma 4.2 that, for every n, the valuation of the leading coefficient of a polynomial $f \in \mathrm{Int}(S, V)$ of degree n is $\geq -n(\gamma + \frac{n(n+1)}{2}(\delta - \gamma))$. Consequently,

$$w_S(n) \le n\gamma + \frac{n^2(n+1)}{2}(\delta - \gamma).$$

Assume first that the sequence of critical valuations is finite. Then q_{γ_∞} is finite and, for every $\delta > \gamma_\infty$, q_δ is infinite. Then, the previous inequality, with $\gamma = \gamma_\infty$ and $\delta > \gamma_\infty$, becomes

$$\forall n \in \mathbb{N} \quad w_S(n) \le n\gamma_\infty + \frac{n^2(n+1)}{2}(\delta - \gamma_\infty).$$

These inequalities for all $\delta > \gamma_\infty$ imply that $w_S(n) \le n\gamma_\infty$.

Assume now that the sequence of critical valuations is infinite and that γ_∞ is finite. Then q_{γ_∞} is infinite and, for every $k \in \mathbb{N}$, q_{γ_k} is finite. It follows from the previous inequality, with $\gamma = \gamma_k$ and $\delta = \gamma_\infty$, that

$$\forall n \in \mathbb{N} \quad w_S(n) \le n\gamma_k + \frac{n^2(n+1)}{2}(\gamma_\infty - \gamma_k).$$

Since $\lim_{k\to\infty} \gamma_k = \gamma_\infty$, we may conclude that $w_S(n) \le n\gamma_\infty$. □

Recall that, by analogy with the Archimedean case (see for instance [21]), one defines the valuative capacity of S in the following way:

Definition 7.2 ([12, §4]). The *valuative capacity* of S with respect to v is the limit δ_S (finite or infinite) of the increasing sequence

$$\delta_S(n) = \frac{2}{n(n+1)} \inf_{x_0,\dots,x_n \in S} v\Bigg(\prod_{0\le i<j\le n} (x_j - x_i)\Bigg). \tag{7.2}$$

The link between the sequences $\{\delta_S(n)\}_{n\in\mathbb{N}}$ and $\{w_S(n)\}_{n\in\mathbb{N}}$ is given by the following formulas [12, Theorems 3.13 and 4.2]:

$$\sum_{k=1}^{n} w_S(n) = \frac{1}{2} n(n+1)\delta_S(n) \tag{7.3}$$

and

$$\lim_{n\to\infty} \frac{w_S(n)}{n} = \sup_{n\ge 1} \frac{w_S(n)}{n} = \delta_S. \tag{7.4}$$

Consequently, we always have the inequality

$$\delta_S \le \gamma_\infty(S). \tag{7.5}$$

8 On the polynomial closure

Proposition 8.1. *For every $\gamma < \gamma_\infty$, one has the containment*

$$\overline{S} \subseteq S + B(0,\gamma). \tag{8.1}$$

Moreover, if q_{γ_∞} is finite, one has also

$$\overline{S} \subseteq S + B(0,\gamma_\infty). \tag{8.2}$$

This is an easy consequence of the fact that a finite union of balls is polynomially closed (Proposition 8.2 below) since

$$S = \cup_{a\in S_\gamma} S(a,\gamma) \subseteq \cup_{a\in S_\gamma} B(a,\gamma) = S + B(0,\gamma). \tag{8.3}$$

Proposition 8.2. *Every finite union of balls is polynomially closed.*

This proposition is itself an easy consequence of the following lemma:

Lemma 8.3. *Let $a, t_1, \ldots, t_r$ be elements of K and let $\gamma, \gamma_1, \ldots, \gamma_r$ be positive real numbers such that the balls $B(a,\gamma), B(t_1,\gamma_1), \ldots, B(t_r,\gamma_r)$ are disjoint. Then, for every $\varepsilon > 0$, there exists $f \in K[X]$ such that*

$$\forall x \in \cup_{k=1}^{r} B(t_k,\gamma_k) \quad v(f(x)) \geq \varepsilon \quad \textit{and} \quad \forall x \in B(a,\gamma) \quad v(f(x)) = 0. \qquad (8.4)$$

Proof. We may assume that

$$\gamma > v(t_1 - a) \geq v(t_2 - a) \geq \cdots \geq v(t_r - a). \qquad (8.5)$$

Obviously, we have

$$\forall k \quad v(a - t_k) < \gamma_k.$$

Now consider

$$f(x) = \prod_{i=1}^{r} \left(\frac{x - t_i}{a - t_i} \right)^{m_i}$$

where the m_i's are integers that we are going to choose.

For $k \in \{1, \ldots, r\}$ and $x \in B(t_k, \gamma_k)$, one has

$$v\left(\frac{x - t_k}{a - t_k}\right) \geq \gamma_k - v(a - t_k) = \varepsilon_k > 0$$

and, for $j \in \{k, \ldots, r\}$ and $x \in B(t_k, \gamma_k)$, one has

$$v\left(\frac{x - t_j}{a - t_j}\right) = v\left(\frac{t_k - t_j}{a - t_j}\right) \geq 0.$$

Thus, for every $x \in B(t_k, \gamma_k)$,

$$v(f(x)) = \sum_{i=1}^{r} m_i v\left(\frac{x - t_i}{a - t_i}\right) \geq \sum_{i=1}^{k} m_i v\left(\frac{x - t_i}{a - t_i}\right) \geq m_k \varepsilon_k - \sum_{i=1}^{k-1} m_i v(a - t_i).$$

We may choose successively the integers $m_1, \ldots, m_k, \ldots, m_r$ such that

$$\forall k \in \{1, \ldots, r\} \quad m_k \varepsilon_k \geq \varepsilon + \sum_{i=1}^{k-1} m_i v(a - t_i).$$

With such a choice of the m_i's, for every $x \in \cup_{k=1}^{r} B(t_k, \gamma)$, one has $v(f(x)) \geq \varepsilon$. Of course, $f(a) = 1$. Moreover, if $x \in B(a, \gamma)$, then $v(x - t_k) = v(a - t_k)$ for every $k \in \{1, \ldots, r\}$, and hence, $v(f(x)) = 0$. □

Proof of Proposition 8.2. Assume that $S = \sqcup_{k=1}^{r} B(t_k, \gamma_k)$ where $\sqcup$ denotes a disjoint union. Let $a \in K \setminus S$ and $\delta \in \Gamma$, $\delta > 0$. Then, by Lemma 8.3, there exists $f \in K[X]$ such that $v(f(a)) = 0$ and, for every $x \in S$, $v(f(x)) \geq \delta$. Let $d \in V$ be such that $v(d) = \delta$. Then, the polynomial $\frac{1}{d} f(X)$ shows that $a \notin \overline{S}$. □

Theorem 8.4. *For every $\gamma < \gamma_\infty$, one has*

$$\overline{S} = \cup_{a \in S_\gamma} \left(\overline{S \cap B(a, \gamma)} \right). \tag{8.6}$$

In other words

$$\overline{\cup_{a \in S_\gamma} S(a, \gamma)} = \cup_{a \in S_\gamma} \overline{S(a, \gamma)}. \tag{8.7}$$

Proof. Obviously, $\cup_{a \in S_\gamma} \overline{S(a, \gamma)} \subseteq \overline{S}$. Let us prove the reverse containment. Let b be an element of $\overline{S}$. Then, by Proposition 8.2, b belongs to $\cup_{a \in S_\gamma} B(a, \gamma)$, thus there exits a_0 such that $b \in B(a_0, \gamma)$. Assume, by way of contradiction, that $b \notin \overline{S(a_0, \gamma)}$, then there exists $g \in K[X]$ such that $g(S(a_0, \gamma)) \subseteq V$ and $v(g(b)) < 0$. Since the values of a polynomial on a fractional subset is a fractional subset, there exists $\varepsilon > 0$ such that $-\varepsilon < \min\{v(g(x)) \mid x \in S\}$. It follows from Lemma 8.3 that there exists $f \in K[X]$ such that

$$\forall x \in \cup_{a \in S_\gamma, a \neq a_0} B(a, \gamma) \;\; v(f(x)) \geq \varepsilon \quad \text{and} \quad \forall x \in B(a_0, \gamma) \;\; v(f(x)) = 0.$$

Then, for $x \in \cup_{a \in S_\gamma, a \neq a_0} B(a, \gamma)$, one has $v(f(x)g(x)) \geq 0$ and, for $x \in S(a_0, \gamma)$, one has $v(f(x)g(x)) = v(g(x)) \geq 0$, while $v(f(b)g(b)) = v(g(b)) < 0$. Consequently, $fg \in \text{Int}(S, V)$ and $f(b)g(b) \notin V$, that is $b \notin \overline{S}$. We obtain a contradiction. □

If $q_{\gamma_\infty} < +\infty$, the previous proof still holds with $\gamma = \gamma_\infty$.

Corollary 8.5. *If q_{γ_∞} is finite, then $\overline{S} = \cup_{a \in S_{\gamma_\infty}} \overline{S(a, \gamma_\infty)}$.*

Now, we consider what happens with respect to v-orderings. We first recall:

Lemma 8.6 ([6], Lemma 3.4). *If $\{a_n\}_{n=0}^{N}$ is a v-ordering of S then, for every ball B, the (possibly empty) subsequence formed by the a_n's that belong to B is a v-ordering of $S \cap B$.*

Proposition 8.7. *Let γ be such that $\gamma < \gamma_\infty$. Then, S admits an infinite v-ordering if and only if, for every $b \in S_\gamma$, $S(b, \gamma)$ admits an infinite v-ordering.*

Proof. Assume that S admits an infinite v-ordering $\{a_n\}_{n \in \mathbb{N}}$. By Lemma 8.6, for every $b \in S_\gamma$, the subsequence formed by the a_n's that are in $B(b, \gamma)$ is a v-ordering of $S(b, \gamma)$. Let $T = \{a_n \mid n \in \mathbb{N}\}$ and, for every $b \in S_\gamma$, consider $T(b, \gamma) = T \cap B(b, \gamma)$. If $T(b, \gamma)$ is infinite, then $S(b, \gamma)$ admits an infinite v-ordering. Thus, assume that, for some $b \in S_\gamma$, $T(b, \gamma)$ is finite. By Corollary 2.8, $\overline{T} = \overline{S}$ and, by Theorem 8.4, $\overline{T(b, \gamma)} = \overline{S(b, \gamma)}$. Since $T(b, \gamma)$ is assumed to be finite, one has $\overline{T(b, \gamma)} = T(b, \gamma)$

(Proposition 6.1.ii). Consequently, $S(b,\gamma)$ is also finite, and hence, admits an infinite v-ordering (Proposition 6.1.i).

Conversely, assume that, for every $b \in S_\gamma$, $S(b,\gamma)$ admits an infinite v-ordering. We prove the existence of an infinite v-ordering of S by induction on n. Assume that $a_0,\ldots,a_{n-1}$ is a v-ordering of S. The question is: does there exist an element $a_n \in S$ which allows to reach the following infimum

$$\inf_{x\in S} v\left(\prod_{k=0}^{n-1}(x-a_k)\right) = \inf_{b\in S_\gamma}\ \inf_{x\in S(b,\gamma)} v\left(\prod_{k=0}^{n-1}(x-a_k)\right)?$$

It is then enough to prove that, for every $b \in S_\gamma$, the following infimum is a minimum:

$$\inf_{x\in S(b,\gamma)} v\left(\prod_{k=0}^{n-1}(x-a_k)\right) = v\left(\prod_{a_k\notin S(b,\gamma)}(b-a_k)\right) + \inf_{x\in S(b,\gamma)} v\left(\prod_{a_k\in S(b,\gamma)}(x-a_k)\right).$$

This last infimum is a minimum since the a_k's that belong to $S(b,\gamma)$ form a v-ordering of $S(b,\gamma)$ and, by hypothesis, $S(b,\gamma)$ admits infinite v-orderings. □

9 The pseudo-closure

When $\gamma_\infty = +\infty$, that is, when $\widehat{S}$ is compact, one has $\overline{S} = \widetilde{S}$ (Propositions 6.1 and 6.3). So that, we may assume that $\gamma_\infty < +\infty$, and hence, that either v is not discrete or $k = V/\mathfrak{m}$ is infinite.

We will generalize the fact that the topological closure $\widetilde{S}$ of S in K is contained in the polynomial closure $\overline{S}$ of S by considering the notion of pseudo-convergent sequence previously mentioned (see (6.3)).

Definition 9.1. (i) A sequence $\{x_n\}_{n\geq 0}$ of elements of K is *pseudo-convergent* if

$$\forall\, i,\ j,\ k \quad [\, i<j<k \ \Rightarrow\ v(x_j-x_i) < v(x_k-x_j)\,]. \tag{9.1}$$

(ii) An element x of K is *a pseudo-limit* of a sequence $\{x_n\}_{n\geq 0}$ if

$$\forall\, i,\ j \quad [\, i<j \ \Rightarrow\ v(x-x_i) < v(x-x_j)\,]. \tag{9.2}$$

(iii) The *pseudo-closure* of S in K is the union $\widetilde{\widetilde{S}}$ of S and of the set formed by the pseudo-limits in K of pseudo-convergent sequences of elements of S.

Suppose that x is a pseudo-limit of a sequence $\{x_n\}_{n\geq 0}$ and let

$$\delta = \lim_{n\to+\infty} v(x-x_n).$$

If $\delta = +\infty$, then the sequence $\{x_n\}$ is convergent with x as classical limit-point. In fact, clearly, one has $\widetilde{S} \subseteq \widetilde{\widetilde{S}}$. If $\delta < +\infty$, then the sequence $\{x_n\}$ is pseudo-convergent since, for $i<j<k$, one has

$$v(x_j-x_i) = v(x-x_i) < v(x-x_j) = v(x_k-x_j).$$

Note also that, in this case, every y such that $v(x-y) \geq \delta$ is also a pseudo-limit of the sequence $\{x_n\}$.

Theorem 9.2. *The polynomial closure $\overline{S}$ of S satisfies the following containments:*

$$\widetilde{\widetilde{S}} \subseteq \overline{S} \subseteq \cap_{k\geq 0}(S + B(0,\gamma_k)) \tag{9.3}$$

where $\widetilde{\widetilde{S}}$ denotes the pseudo-closure of S. Moreover, one has the following equalities:

$$\begin{aligned}\cap_{k\geq 0}(S + B(0,\gamma_k)) &= (S + B(0,\gamma_\infty)) \cup \widetilde{\widetilde{S}} = (S + B(0,\gamma_\infty)) \cup \overline{S}\\ &= \widetilde{\widetilde{S}} + B(0,\gamma_\infty).\end{aligned} \tag{9.4}$$

Proof. Let $x \in \widetilde{\widetilde{S}}$. Of course, if $x \in \widetilde{S}$, then $x \in \overline{S}$. Assume that $x \in \widetilde{\widetilde{S}} \setminus \widetilde{S}$, and then, that x is a pseudo-limit of a sequence $\{x_n\}_{n\geq 0}$ of elements of S. For every $n \geq 0$, let $\delta_n = v(x-x_n)$ and let $\delta = \lim_{n\to+\infty}\delta_n < +\infty$. We have $v(x_m - x_{m'}) = \delta_m$ for $m' > m > n$, with $\delta_n < \delta_m < \delta$, thus $S(x_n, \delta_n) \bmod \delta$ is infinite.

Consider now a polynomial $f(X) = \sum_{j=0}^{d} c_j X^j \in \mathrm{Int}(S,V)$ of degree d. It follows from Lemma 4.2 that $f \in V[(X-x_n)/\rho_n]$ where $\rho_n = \delta_n + \frac{d(d+1)}{2}(\delta - \delta_n)$, that is, $v(c_j) \geq -j\rho_n$ for every j. Consequently,

$$v(c_j(x-x_n)^j) \geq j(\delta_n - \rho_n) = -j\,\frac{d(d+1)}{2}(\delta - \delta_n),$$

and

$$v(f(x)) \geq -\frac{d^2(d+1)}{2}(\delta - \delta_n).$$

Since, $\lim \delta_n = \delta$, one may conclude that $v(f(x)) \geq 0$ and $x \in \overline{S}$. This is the first containment. The second containment is a straightforward consequence of Containment (8.1).

Since $S \subseteq \widetilde{\widetilde{S}} \subseteq \overline{S}$, it is obvious that the subset $(S + B(0,\gamma_\infty)) \cup \widetilde{\widetilde{S}}$ is contained in $(S + B(0,\gamma_\infty)) \cup \overline{S}$ and also in $\widetilde{\widetilde{S}} + B(0,\gamma_\infty)$. Moreover, it follows from (9.3) that $(S + B(0,\gamma_\infty)) \cup \overline{S}$ and $\widetilde{\widetilde{S}} + B(0,\gamma_\infty)$ are contained in $\cap_{k\geq 0}(S + B(0,\gamma_k))$. Thus, it remains to prove that $\cap_{k\geq 0}(S + B(0,\gamma_k)) \subseteq (S + B(0,\gamma_\infty)) \cup \widetilde{\widetilde{S}}$.

If the sequence $\{\gamma_k\}$ is finite, $\gamma_\infty = \gamma_l$ for some l and the result follows immediately. If the sequence $\{\gamma_k\}$ is infinite, let x be an element of $\cap_k(S + B(0,\gamma_k))$ which is not in $S + B(0,\gamma_\infty)$. We show that $x \in \widetilde{\widetilde{S}}$. Let $k_1 \geq 0$. As $x \in S + B(0,\gamma_{k_1})$, there is $x_1 \in S$ such that $v(x-x_1) \geq \gamma_{k_1}$. As $x \notin S + B(0,\gamma_\infty)$, $v(x-x_1) < \gamma_\infty$. There exists $k_2 > k_1$ such that $v(x-x_1) < \gamma_{k_2}$. As $x \in S + B(0,\gamma_{k_2})$, there is $x_2 \in S$ such that $v(x-x_2) \geq \gamma_{k_2}$. We then have

$$\gamma_{k_1} \leq v(x-x_1) < \gamma_{k_2} \leq v(x-x_2) < \gamma_\infty.$$

So that, we may construct two sequences $\{k_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 1}$ such that

$$\gamma_{k_n} \leq v(x-x_n) < \gamma_{k_{n+1}}.$$

Consequently, $\lim_n \gamma_{k_n} = \gamma_\infty$ and x is a pseudo-limit of the sequence $\{x_n\}$. □

Remark 9.3. (i) If $\gamma_\infty = +\infty$, we have the equalities

$$\widetilde{S} = \widetilde{\widetilde{S}} = \overline{S} = \cap_{k\geq 0}(S + B(0,\gamma_k)).$$

(ii) From Theorem 9.2, it follows that we always have

$$\widetilde{\widetilde{S}} + B(0,\gamma_\infty) = \overline{S} + B(0,\gamma_\infty) = \cap_{k\geq 0}(S + B(0,\gamma_k)).$$

If q_{γ_∞} is finite, we moreover have the equalities

$$S + B(0,\gamma_\infty) = \widetilde{S} + B(0,\gamma_\infty) = \widetilde{\widetilde{S}} + B(0,\gamma_\infty) = \overline{S} + B(0,\gamma_\infty).$$

If q_{γ_∞} is infinite, we only have a containment

$$S + B(0,\gamma_\infty) \subseteq \cap_{k\geq 0}(S + B(0,\gamma_k)).$$

This containment may be strict. For instance, if $\gamma_\infty = +\infty$, then $S + B(0,\gamma_\infty) = S$ and $\cap_{k\geq 0}(S + B(0,\gamma_k)) = \widetilde{S}$, but it may obviously be that $S \subsetneq \widetilde{S}$. The containment may also be strict in case $\gamma_\infty < +\infty$, according to the following example:

Assume that $v(K^*) = \mathbb{Q}$ and $v(2) = 0$. Let $S = \{0,1\} \cup \{2 + u_n\}_{n\geq 1}$ where $v(u_n) = 1 - \frac{1}{n}$. Then, $\gamma_0 = 0, \gamma_n = 1 - \frac{1}{n}, \gamma_\infty = 1$. Let $x = 2 + t$ with $v(t) \geq 1$. Then, x is the pseudo-limit of the sequence $\{2 + u_n\}$. Thus $x \in \widetilde{\widetilde{S}}$, and by (9.3), $x \in \cap_{k\geq 0}(S + B(0,\gamma_k))$. Yet $x \notin S + B(0,\gamma_\infty)$.

This fact still holds even if S is a very regular subset (see Remark 12.3 (ii)).

(iii) When $\gamma_\infty < +\infty$, the containments in (9.3) may be strict:

If $V/\mathfrak{m}$ is infinite and S is a set of representatives modulo $\mathfrak{m}$, then $\mathrm{Int}(S,V) = V[X]$ (from a Vandermonde argument). Thus $\widetilde{\widetilde{S}} = S \subsetneq \overline{S} = V$.

On the other hand, suppose that the valuation v is discrete and that $V/\mathfrak{m}$ is infinite, then set $S = \mathfrak{m} \cup (1 + \mathfrak{m}^2)$. From Proposition 8.2, S is polynomially closed, that is, $\overline{S} = S$. In this situation, $\gamma_0 = 0$ and $\gamma_\infty = 1$ (with $q_{\gamma_\infty} = 2$). Thus $S + B(0,\gamma_\infty) = \mathfrak{m} \cup (1 + \mathfrak{m})$, and hence, $S + B(0,\gamma_\infty) \not\subseteq \overline{S} = S$.

Let us now look at containments concerning the polynomial rings. Recall Lemma 4.1 that says that

$$\forall a \in S \;\forall \gamma \in \mathbb{R} \quad V[(X-a)/\gamma] \subseteq \mathrm{Int}(S(a,\gamma),V). \tag{9.5}$$

Consequently,

$$\cap_{a\in S_\gamma} V[(X-a)/\gamma] \subseteq \mathrm{Int}(S,V). \tag{9.6}$$

This leads us to recall and slightly generalize a notion introduced in [31]:

Definition 9.4. For every $\gamma \in \mathbb{R}$, the *Bhargava ring* with respect to S and γ is the following domain:

$$\mathrm{Int}_\gamma(S,V) = \cap_{a\in S} V[(X-a)/\gamma] = \cap_{a\in S_\gamma} V[(X-a)/\gamma]. \tag{9.7}$$

In the case where $\gamma \in \Gamma$, then $\gamma = v(t)$ for some $t \in K$ and then

$$\mathrm{Int}_\gamma(S, V) = \{f \in K[X] \mid \forall s \in S \quad f(tX + s) \in V[X]\}. \tag{9.8}$$

Yeramian [31] defines only $\mathrm{Int}_\gamma(S, V)$ when $S = V$ and $\gamma = v(t)$ and denotes it by $\mathbf{B}_t(V)$. As previously noticed

$$\forall \gamma \in \mathbb{R} \quad V[X] \subseteq \mathrm{Int}_\gamma(S, V) \subseteq \mathrm{Int}(S, V). \tag{9.9}$$

Obviously,

$$\gamma < \delta \quad \Rightarrow \quad \mathrm{Int}_\gamma(S, V) \subseteq \mathrm{Int}_\delta(S, V), \tag{9.10}$$

and if $\gamma_\infty = \lim_k \gamma_k < +\infty$, then

$$\cup_k \mathrm{Int}_{\gamma_k}(S, V) \subseteq \mathrm{Int}_{\gamma_\infty}(S, V). \tag{9.11}$$

We may also note that

Proposition 9.5. *If $\gamma_\infty < +\infty$, then*

$$\forall \gamma \in \mathbb{R} \quad \mathrm{Int}_\gamma(S, V) = \mathrm{Int}(S_\gamma + B(0, \gamma), V). \tag{9.12}$$

Proof. Since $\gamma_\infty < +\infty$, either v is not discrete or $k = V/\mathfrak{m}$ is infinite. Then, Proposition 4.5 says that

$$\forall a \in K \; \forall \gamma \in \mathbb{R} \quad \mathrm{Int}(B(a, \gamma), V) = V[(X - a)/\gamma]. \tag{9.13}$$

□

Thus, the containment $\overline{S} \subseteq \cap_{k \geq 0} (S + B(0, \gamma_k))$ corresponds to the containment

$$\cup_{k \geq 0} \mathrm{Int}_{\gamma_k}(S, V) \subseteq \mathrm{Int}(S, V). \tag{9.14}$$

10 When $\overline{S} = \cap_k (S + B(0, \gamma_k))$

In this section we still assume that $\gamma_\infty < +\infty$. We know that

$$\overline{S} \subseteq \cap_{k \geq 0} (S + B(0, \gamma_k)). \tag{10.1}$$

When does $\bar{S}$ equal $\cap_{k \geq 0} (S + B(0, \gamma_k))$?

Proposition 10.1. *The following assertions are equivalent:*

(i) $\overline{S} = \cap_{k \geq 0} (S + B(0, \gamma_k))$.

(ii) $S + B(0, \gamma_\infty) \subseteq \overline{S}$.

(iii) $\mathrm{Int}(S, V) = \mathrm{Int}_{\gamma_\infty}(S, V)$.

Proof. (i) $\leftrightarrow$ (ii) follows from the equality $\cap_k(S + B(0, \gamma_k)) = (S + B(0, \gamma_\infty)) \cup \overline{S}$ (Theorem 9.2).

(ii) $\leftrightarrow$ (iii) By Proposition 9.5, we have

$$\mathrm{Int}_{\gamma_\infty}(S, V) = \mathrm{Int}(S + B(0, \gamma_\infty), V)$$

and, clearly, we have

$$\mathrm{Int}(S + B(0, \gamma_\infty), V) \subseteq \mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V).$$

Thus, the containment $S + B(0, \gamma_\infty) \subseteq \overline{S}$ is equivalent to the equality $\mathrm{Int}_{\gamma_\infty}(S, V) = \mathrm{Int}(S, V)$. □

Now, we have to distinguish whether q_{γ_∞} is finite or not.

10.1 When q_{γ_∞} is finite

Recall that, when q_{γ_∞} is finite, one has $\cap_k \ (S + B(0, \gamma_k)) = S + B(0, \gamma_\infty)$. We begin with a lemma:

Lemma 10.2. *Assume that the sequence $\{\gamma_k\}$ of critical valuations of S is finite. If $a \in S$ is such that, for every $\delta > \gamma_\infty$, $S(a, \gamma_\infty)$ mod δ is infinite, then*

$$\mathrm{Int}(S, V) \subseteq V[(X - a)/\gamma_\infty] \quad \textit{and} \quad B(a, \gamma_\infty) \subseteq \overline{S}. \tag{10.2}$$

Proof. We apply Lemma 4.2 with $\gamma = \gamma_\infty$ and, when δ tends to γ_∞, ρ tends to γ_∞. □

This leads us to the following characterization:

Theorem 10.3. *Assume that q_{γ_∞} is finite. Then, the following three assertions are equivalent:*

$$\forall a \in S \ \ \forall \delta > \gamma_\infty \quad S(a, \gamma_\infty) \bmod \delta \textit{ is infinite}, \tag{10.3}$$

$$\overline{S} = S + B(0, \gamma_\infty), \tag{10.4}$$

$$\mathrm{Int}(S, V) = \mathrm{Int}_{\gamma_\infty}(S, V). \tag{10.5}$$

Proof. When assertion (10.3) holds, it follows from Lemma 10.2 that $B(a, \gamma_\infty) \subseteq \overline{S}$ for every $a \in S$, and hence, that $S + B(0, \gamma_\infty) \subseteq \overline{S}$. Since $\overline{S} \subseteq \cap_k(S + B(0, \gamma_k)) = S + B(0, \gamma_\infty)$, we have (10.4).

Now assume that (10.3) does not hold. Then there exist $a \in S$ and $\delta > \gamma_\infty$ such that $S(a, \gamma_\infty)$ mod δ is finite. Let $b_1, \ldots, b_s \in S$ be such that $S(a, \gamma_\infty) \subseteq \cup_{i=1}^s B(b_i, \delta)$. Then, $\overline{S(a, \gamma_\infty)} \subseteq \cup_{i=1}^s B(b_i, \delta)$. But $\cup_{i=1}^s B(b_i, \delta)$ cannot be equal to $B(a, \gamma_\infty)$ since either v is not discrete, or $V/\mathfrak{m}$ is infinite (because γ_∞ is finite). Consequently, by Corollary 8.5, we have $\overline{S} \cap B(a, \gamma_\infty) = \overline{S(a, \gamma_\infty)} \neq B(a, \gamma_\infty)$. Thus, assertion (10.4) does not hold.

The equivalence between (10.4) and (10.5) follows from Proposition 10.1. □

Remark 10.4. (i) In the case described by Theorem 10.3, $\overline{S}$ is a finite union of balls and, as already said, either the valuation v is not discrete, or the residue field $V/\mathfrak{m}$ is infinite. It follows from Propositions 8.7 and 4.5 that the subset $\overline{S}$ admits an infinite v-ordering if and only if $\gamma_\infty \in \Gamma$ and the residue field $V/\mathfrak{m}$ is infinite.

(ii) Recall that if S is a finite union of balls, the study of the characteristic function w_S is done in [6] in the case where the valuation v is discrete.

Let us now consider the case where q_{γ_∞} is infinite.

10.2 When q_{γ_∞} is infinite

The analog of Lemma 10.2 is

Lemma 10.5. *Assume that the sequence $\{\gamma_k\}$ of critical valuations of S is infinite. If there exists $a \in S$ such that, for every $\gamma < \gamma_\infty$, $S(a, \gamma)$ mod γ_∞ is infinite, then*

$$\mathrm{Int}(S, V) \subseteq V[(X - a)/\gamma_\infty] \quad \textit{and} \quad B(a, \gamma_\infty) \subseteq \overline{S}. \tag{10.6}$$

Proof. We apply Lemma 4.2 with $\gamma = \gamma_k$ and $\delta = \gamma_\infty$ and, when k tends to $+\infty$, ρ tends to γ_∞. □

This lemma leads to the following equivalences:

Theorem 10.6. *Assume that γ_∞ is finite and that q_{γ_∞} is infinite. Then, the following four assertions are equivalent:*

$$\overline{S} = \cap_{k \geq 0} (S + B(0, \gamma_k)), \tag{10.7}$$

$$S + B(0, \gamma_\infty) \subseteq \overline{S}, \tag{10.8}$$

$$\mathrm{Int}(S, V) = \mathrm{Int}_{\gamma_\infty}(S, V), \tag{10.9}$$

$$\forall a \in S \ \forall \gamma < \gamma_\infty \left\{ [\, S(a, \gamma_\infty) = S(a, \gamma)\,] \ \Rightarrow \ [\, \overline{S(a, \gamma_\infty)} = B(a, \gamma_\infty)\,] \right\}. \tag{10.10}$$

These equivalent assertions also hold when the following condition is satisfied:

$$\forall a \in S \ \ \forall \gamma < \gamma_\infty \quad S(a, \gamma) \text{ mod } \gamma_\infty \textit{ is infinite}. \tag{10.11}$$

Proof. The equivalences (10.7) ↔ (10.8) ↔ (10.9) are nothing else than Proposition 10.1. Let us prove that (10.8) → (10.10). Assume that $a \in S$ and $\gamma < \gamma_\infty$ are such that $S(a, \gamma_\infty) = S(a, \gamma)$ and that $S + B(0, \gamma_\infty) \subseteq \overline{S}$. Then, by Theorem 8.4,

$$B(a, \gamma_\infty) \subseteq \overline{S} \cap B(a, \gamma) = \overline{S(a, \gamma)} = \overline{S(a, \gamma_\infty)} \subseteq B(a, \gamma_\infty).$$

Thus, $\overline{S(a, \gamma_\infty)} = B(a, \gamma_\infty)$.

Conversely, suppose there exists $\gamma < \gamma_\infty$ such that $S(a, \gamma)$ mod γ_∞ is finite. Then, $S(a, \gamma)$ is a finite union of balls $\cup_j S(b_j, \gamma_\infty)$. Let $\delta = \min_{j \neq j'} v(b_j - b_{j'})$. Then,

$S(a,\delta) = S(a,\gamma_\infty)$. Assuming (10.10), we then have $\overline{S(a,\gamma_\infty)} = B(a,\gamma_\infty)$. On the other hand, if there is no such $\gamma < \gamma_\infty$ (with $S(a,\gamma)$ mod γ_∞ finite), it follows from Lemma 10.5 that $B(a,\gamma_\infty) \subseteq \overline{S}$. In both cases we then have $\cup_{a\in S} B(a,\gamma_\infty) \subseteq \overline{S}$.

Finally, Lemma 10.5 shows that (10.11) → (10.9). □

Note that assertion (10.11) means that, for every $a \in S$ and every $\gamma < \gamma_\infty$, the sequence formed by the cardinalities of $S(a,\gamma)$ mod γ_k is not a stationary sequence. The following example shows that condition (10.11) is not necessary in order to have (10.8).

Example 10.7. Let S be a subset satisfying condition (10.11), then it satisfies (10.8). Let $b \in K$ and $\delta < \gamma_\infty$ be such that $B(b,\delta) \cap S = \varnothing$ and let $T = S \cup B(b,\gamma_\infty)$. Then, obviously, $\gamma_\infty(T) = \gamma_\infty(S)$ and $\overline{T} = \overline{S} \cup B(b,\gamma_\infty)$. Consequently,

$$\begin{aligned} T + B(0,\gamma_\infty) &= (S \cup \{b\}) + B(0,\gamma_\infty) = (S + B(0,\gamma_\infty)) \cup B(b,\gamma_\infty) \\ &\subseteq \overline{S} \cup B(b,\gamma_\infty) = \overline{T}, \end{aligned}$$

while Card$(T(b,\delta)$ mod $\gamma_\infty) = 1$.

10.3 Regular subsets

The equivalent assertions of Theorems 10.3 and 10.6 are satisfied by regular subsets as in the following generalization of regular compact subsets introduced by Amice [1] in local fields and extended to precompact subsets of discrete valuation domains in [18]:

Definition 10.8. The fractional subset S of K is said to be a *regular subset* if, for every $\gamma < \delta$ such that q_γ is finite, Card$(S(x,\gamma)$ mod $\delta)$ does not depend on $x \in S$ in the following sense:

(i) if q_δ is finite, then every non-empty ball $S(x,\gamma)$ is the disjoint union of $\frac{q_\delta}{q_\gamma}$ balls $S(y,\delta)$,

(ii) if q_δ is infinite, then every non-empty ball $S(x,\gamma)$ is the disjoint union of infinitely many balls $S(y,\delta)$.

Condition (i) is equivalent to both following assertions:

$$\forall k \geq 0,\ q_{\gamma_{k+1}} = \alpha_k q_{\gamma_k} \text{ (where } \alpha_k \in \mathbb{N}),$$
$$\forall a \in S,\ \text{Card}\ \ S(a,\gamma_k) \bmod \gamma_{k+1}\ = \alpha_k.$$

If q_{γ_∞} is infinite, condition (ii) follows from condition (i). The next section shows that regular subsets appear naturally in discrete dynamical systems.

11 Orbits under the action of an isometry

Let φ be a map from S to S. Then, the pair (S,φ) may be considered as a discrete dynamical system. For every $x \in S$, we may consider the forward orbit $O_+^\varphi(x)$ of x under the action of φ:

$$O_+^\varphi(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}. \tag{11.1}$$

Proposition 11.1. *Let $\varphi : S \to S$ be an isometry. Fix $x \in S$ and let*

$$T = O_+^{\varphi}(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}.$$

Then, for every $\gamma \in \mathbb{R}$, denoting by $q_\gamma(T)$ the cardinality of $T \bmod \gamma$, we have

$$\forall n, m \in \mathbb{N} \quad [\, n \equiv m \pmod{q_\gamma(T)} \Leftrightarrow \varphi^n(x) \equiv \varphi^m(x) \pmod{\gamma}\,]. \tag{11.2}$$

In particular, if $q_\gamma(T)$ is finite, $x, \varphi(x), \dots, \varphi^{q_\gamma(T)-1}$ is a complete system of representatives of $T \bmod \gamma$, and, if $q_\gamma(T)$ is infinite, the $\varphi^k(x)$'s (for $k \in \mathbb{N}$) are non-congruent modulo γ.

Proof. If $q_\gamma(T)$ is finite, there exists $0 \le s < t$ such that $\varphi^s(x) \equiv \varphi^t(x) \pmod{\gamma}$. Conversely, assume that this is the case. Then, $\varphi^{t-s}(x) \equiv x \pmod{\gamma}$. Let $r > 0$ be the smallest integer such that $\varphi^r(x) \equiv x \pmod{\gamma}$. Then, $x, \varphi(x), \dots, \varphi^{r-1}(x)$ are non-congruent modulo γ. Moreover,

$$\forall h \in \mathbb{N} \quad \varphi^{(h+1)r}(x) = \varphi^{hr}(\varphi^r(x)) \equiv \varphi^{hr}(x) \pmod{\gamma},$$

and hence,

$$\forall h \in \mathbb{N} \quad \varphi^{hr}(x) \equiv x \pmod{\gamma}.$$

Now, for every $n \in \mathbb{N}$, let n_0 be such that

$$n \equiv n_0 \pmod{r} \quad \text{where} \quad 0 \le n_0 < r,$$

then

$$\varphi^{n-n_0}(x) \equiv x \pmod{\gamma}, \quad \text{that is,} \ \varphi^n(x) \equiv \varphi^{n_0}(x) \pmod{\gamma}.$$

Finally, the sequence $x, \varphi(x), \dots, \varphi^{r-1}(x)$ is a complete system of representatives of $T \bmod \gamma$, and $q_\gamma(T) = r$. In particular, $q_\gamma(T)$ is finite. It follows that $q_\gamma(T)$ is infinite if and only if the $\varphi^k(x)$'s are non-congruent modulo γ. □

Now we generalize a result obtained in discrete valuation domains with finite residue field (cf. [20, Théorème 7.1] or [15, Theorem 3.3]):

Theorem 11.2. *Let K be a valued field, S be an infinite fractional subset of K, and $\varphi : S \to S$ be an isometry. For every $x \in S$, the forward orbit*

$$O_+^{\varphi}(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}$$

is a regular subset.

Proof. Fix $x \in S$ and let $T = O_+^{\varphi}(x)$. Let $\gamma \in \mathbb{R}$ be such that $\mathrm{Card}(T \bmod \gamma) = q_\gamma(T) = r$ is finite and consider some $\delta > \gamma$.

Assume first that $\mathrm{Card}(T \bmod \delta)$ is infinite. Then, the $\varphi^k(x)$, for $k \in \mathbb{N}$, are non-congruent modulo δ. Consequently, in each class $T(\varphi^i(x), \gamma)$ $(0 \le i < r)$ of $S \bmod \gamma$, there are infinitely many elements that are non-congruent modulo δ, namely,

$$T(\varphi^i(x), \gamma) = \{\varphi^{i+kr}(x) \mid k \in \mathbb{N}\}.$$

Assume now that Card$(T \bmod \delta) = q_\delta(T) = s$ is finite. It follows from Proposition 11.1 and from $\delta > \gamma$ that

$$\begin{aligned} n \equiv m \pmod s \;&\Leftrightarrow\; \varphi^n(x) \equiv \varphi^m(x) \pmod \delta \\ &\Rightarrow\; \varphi^n(x) \equiv \varphi^m(x) \pmod \gamma \;\Leftrightarrow\; n \equiv m \pmod r. \end{aligned}$$

Thus, $r = q_\gamma(S)$ divides $s = q_\delta(T)$. Let $s = r\alpha$. Then,

$$T(x,\gamma) = \{\varphi^{rl}(x) \mid l \in \mathbb{N}\} = \cup_{k=0}^{\alpha-1}\{\varphi^{kr+sl} \mid l \in \mathbb{N}\} = \cup_{k=0}^{\alpha-1} T(\varphi^{kr}(x),\delta).$$

Thus, $T(x,\gamma)$ is the disjoint union of $\alpha = \frac{s}{r}$ balls of the form $T(y,\delta)$. Moreover, for $0 \le i < r$, one has $T(\varphi^i(x),\gamma) = \varphi^i(T(x,\gamma))$, so that $T(\varphi^i(x),\gamma)$ is also the disjoint union of α balls of the form $T(y,\delta)$. □

Proposition 11.3. *Let T be the forward orbit of an element x of a valued field K under the action of an isometry φ.*

(i) *If $\gamma_\infty(T) = +\infty$, then T is precompact.*

(ii) *If $\gamma_\infty(T) < +\infty$, then T is discrete.*

(iii) *If $q_{\gamma_\infty}(T) = +\infty$, then*

$$T = T_{\gamma_\infty} = \cup_{\gamma<\gamma_\infty} T_\gamma.$$

(iv) *If $q_{\gamma_\infty}(T) < +\infty$, then*

$$T = T_{\gamma_\infty} + T(x,\gamma_\infty)$$

where

$$T_{\gamma_\infty} = \{\varphi^k(x) \mid 0 \le k < q_{\gamma_\infty}\} \subseteq \{y \in K \mid v(x-y) < \gamma_\infty\}$$

and

$$T(x,\gamma_\infty) = \{\varphi^{rq_{\gamma_\infty}}(x) \mid r \in \mathbb{N}\} \subseteq \{y \in K \mid v(x-y) = \gamma_\infty\}.$$

Proof. It follows from Proposition 11.1 that, for every $\gamma < \gamma_\infty$, $\{\varphi^n(x) \mid 0 \le n < q_\gamma\}$ is a complete set of representatives of T modulo γ, that we may choose as T_γ. Clearly, for $\gamma < \delta < \gamma_\infty$, one has $T_\gamma \subseteq T_\delta$.

Assume that $q_{\gamma_\infty} = +\infty$. Then, $\gamma_k(T) \to \gamma_\infty(T)$, and hence, $T = \cup_{\gamma<\gamma_\infty} T_\gamma$. In particular, $T_{\gamma_\infty} = T$. This is assertion (iii).

Assume that $q_{\gamma_\infty} < +\infty$ (and hence, $\gamma_\infty < +\infty$) then, for every $\delta > \gamma_\infty$, one has $v(\varphi^n(x) - \varphi^m(x)) < \delta$ whatever $n \neq m$, and hence, $v(\varphi^n(x) - \varphi^m(x)) \le \gamma_\infty$. Consequently, $T(x,\gamma_\infty)$ which is a priori the intersection of T with the ball $B(x,\gamma_\infty)$, is here in fact, the intersection of T with the sphere $\{y \in K \mid v(x-y) = \gamma_\infty\}$. Thus, $T \subseteq T_{\gamma_\infty} + \{y \in K \mid v(y) = \gamma_\infty\}$. Clearly, the intersection of T with $B(x,\gamma_\infty)$ is the forward orbit of x under the action of $\varphi^{q_{\gamma_\infty}}$: $T(x,\gamma_\infty) = \{\varphi^{rq_{\gamma_\infty}}(x) \mid r \in \mathbb{N}\}$. This is assertion (iv).

Assertion (i) is obvious. Finally, assume that $\gamma_\infty < +\infty$. It follows from assertions (iii) and (iv) that, no matter whether q_{γ_∞} is finite or not, for all $x \neq y \in T$, $v(x-y) \leq \gamma_\infty$. Consequently, T is (uniformly) discrete: for each $t \in T$, $T \cap \{x \in K \mid v(x-t) > \gamma_\infty\} = \{t\}$. This is assertion (ii). □

Remark 11.4. Let T denote the orbit of an element x of K under the action of an isometry φ.

(1) If $q_{\gamma_\infty}(T)$ is finite, then $V/\mathfrak{m}$ is infinite. Indeed, the previous proof shows that, if q_{γ_∞} is finite, then

$$q_{\gamma_\infty} | n - m \;\Leftrightarrow\; v(\varphi^n(x) - \varphi^m(x)) = \gamma_\infty. \tag{11.3}$$

In particular, $\{y \in K \mid v(y) = \gamma_\infty\}$ is infinite, which is equivalent to the fact that the residue field $V/\mathfrak{m}$ is infinite.

(2) If $\gamma_\infty(T)$ is finite, then T is discrete, and hence, is equal to its topological closure $\widetilde{T}$ in V and also to its completion. If $q_{\gamma_\infty}(T)$ is infinite then, by Theorem 10.6 (assertion (10.11)), T is polynomially equivalent to $T + B(0, \gamma_\infty)$. Consequently, $\widetilde{T} \neq \overline{T}$.

Corollary 11.5. *Let S be an infinite fractional subset of K and let $\varphi : S \to S$ be an isometry. If the dynamical system (S, φ) is topologically transitive, that is, if there exists $x \in S$ such that $T = O_+^\varphi(x)$ is dense in S, then S is a regular subset. Moreover, either $\gamma_\infty < +\infty$, and then S is discrete with $S = S_{\gamma_\infty} = T$, or $\gamma_\infty = +\infty$, and then S is precompact with $S = S_{\gamma_\infty}$.*

Proof. By hypothesis, for every $\gamma \in \mathbb{R}$, one has $T \bmod \gamma = S \bmod \gamma$, and hence, for every γ such that $q_\gamma(S) < q_{\gamma_\infty}(S)$, $q_\gamma(S) = q_\gamma(T)$ and $(x, \varphi(x), \ldots, \varphi^{q_\gamma - 1}(x))$ is a complete system of representatives of $S \bmod \gamma$, that one may choose for S_γ. □

In summary, we obtain notably different results whether q_{γ_∞} is finite or infinite. The regular subsets such that q_{γ_∞} is infinite will be considered in a forthcoming paper [14] where we show that their v-orderings have very strong properties that may be used to describe the dynamics. In particular, extending results of [18], we prove that, for any infinite fractional subset S of K such that $S = S_{\gamma_\infty}$, the following assertions are equivalent:

(i) S is a regular subset.

(ii) There exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S such that

$$\forall \gamma \in \Gamma \quad [\, v(a_n - a_m) \geq \gamma \;\Leftrightarrow\; q_\gamma | (n-m) \,]. \tag{11.4}$$

(iii) The characteristic function w_S of S satisfies a generalized Legendre formula

$$w_S(n) = v(n!_S) = n\gamma_0 + \sum_{k=1}^{+\infty} \left[\frac{n}{q_{\gamma_k}(S)} \right] (\gamma_k - \gamma_{k-1}). \tag{11.5}$$

12 An example

The following example is a generalization of an example given in [12]. Let k be a field and let Γ be a subgroup of $\mathbb{R}$. Let $\Gamma_+ = \{\gamma \in \Gamma \mid \gamma \geq 0\}$ and consider the integral domain

$$A = k[\Gamma_+] = k[\{X^\gamma \mid \gamma \geq 0\}; X^\gamma X^\delta = X^{\gamma+\delta}] \tag{12.1}$$

endowed with the valuation v defined by

$$\forall k \in \mathbb{N}\ \forall a_k \in k\ \forall \delta_k \in \Gamma_+ \quad v\left(\sum_{k=0}^{n} a_k X^{\delta_k}\right) = \min\{\delta_k \mid a_k \neq 0\}. \tag{12.2}$$

Fix a strictly increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ of elements of Γ_+. For every $n \geq 0$, choose a finite subset C_n of k containing 0 with cardinality $\alpha_n > 1$. Now consider the following subset of A:

$$T = \{c_0 X^{r_0} + c_1 X^{r_1} + \cdots + c_l X^{r_l} \mid l \in \mathbb{N},\, c_h \in C_h, 0 \leq h \leq l\}. \tag{12.3}$$

Then,

$$q_{r_h} = q_{r_h}(T) = \mathrm{Card}(T \bmod r_h) \quad \text{satisfies} \quad q_{r_0} = 1 \text{ and } q_{r_{h+1}} = \alpha_h q_{r_h}. \tag{12.4}$$

Of course,

$$q_\gamma = q_{r_h} \quad \text{for} \quad r_{h-1} < \gamma \leq r_h. \tag{12.5}$$

In other words, the sequence of critical valuations of T, that is $\{\gamma_k\}_{k\in\mathbb{N}}$, is the sequence $\{r_k\}_{k\in\mathbb{N}}$. Consequently,

$$\gamma_\infty(T) = r_\infty = \sup\{\, r_n \mid n \in \mathbb{N}\,\}. \tag{12.6}$$

Thus, T is a discrete subspace if the sequence $\{r_n\}$ is bounded and T is precompact if the sequence is unbounded. In both cases, the subset T admits infinite v-orderings. We describe now such a v-ordering.

For every $n \geq 0$, the elements of C_n may be ordered in a finite sequence

$$\{c_{n,i}\}_{0\leq i<\alpha_n} \quad \text{with} \quad c_{n,0} = 0. \tag{12.7}$$

For every $n \geq 0$, denoting by $n \bmod \alpha$ the unique integer m such that $n \equiv m \pmod{\alpha}$ and $0 \leq m < \alpha$, we consider

$$n_0 = n \bmod \alpha_0, \quad n_1 = \frac{n - n_0}{\alpha_0} \bmod \alpha_1, \quad n_2 = \frac{n - n_0 - n_1\alpha_0}{\alpha_1} \bmod \alpha_2, \ \ldots \tag{12.8}$$

so that

$$n = n_0 + n_1\alpha_0 + n_2\alpha_0\alpha_1 + \cdots + n_l\alpha_0\alpha_1\cdots\alpha_{l-1} \tag{12.9}$$

or

$$n = n_0 + n_1 q_{r_1} + n_2 q_{r_2} + \cdots + n_l q_{r_l} \quad \text{with } 0 \leq n_h < \alpha_h. \tag{12.10}$$

Then, put

$$a_n = \sum_{h=0}^{l} c_{h,n_h} X^{r_h}. \tag{12.11}$$

Proposition 12.1. *The sequence* $\{a_n\}_{n\in\mathbb{N}}$ *defined by* (12.10) *and* (12.11) *is a* v*-ordering of the subset* T *defined by* (12.3). *Moreover, this* v*-ordering satisfies condition* (11.4) *and the following Legendre formula:*

$$w_T(n) = v(n!)_T = nr_0 + \sum_{h\geq 1} \left[\frac{n}{q_{r_h}}\right](r_h - r_{h-1})\,. \tag{12.12}$$

Proof. Denote by $\nu_T(n)$ the greatest integer k such that q_{r_k} divides n. Clearly,

$$\forall n, m \in \mathbb{N} \quad v(a_n - a_m) = r_{\nu_T(n-m)}. \tag{12.13}$$

One easily verifies that for $m \geq n$

$$\begin{aligned} v\left(\prod_{k=0}^{n-1}(a_m - a_k)\right) &= \sum_{k=0}^{n-1} r_{\nu_T(m-k)} = \sum_{l=1}^{m} r_{\nu_T(l)} - \sum_{l=1}^{m-n} r_{\nu_T(l)} \\ &= r_0 n + \sum_{k>0}\left(\left[\frac{m}{q_{r_k}}\right] - \left[\frac{m-n}{q_{r_k}}\right]\right)(r_k - r_{k-1}). \end{aligned}$$

In particular,

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = r_0 n + \sum_{k>0}\left[\frac{n}{q_{r_k}}\right](r_k - r_{k-1}).$$

Thus, the sequence $\{a_n\}$ is a v-ordering of T since

$$\left[\frac{m}{q_{r_k}}\right] - \left[\frac{m-n}{q_{r_k}}\right] \geq \left[\frac{n}{q_{r_k}}\right]. \qquad \square$$

Corollary 12.2. *The valuative capacity of* T *is equal to*

$$\delta_T = \sum_{k\geq 0} \frac{1}{\alpha_0\alpha_1\cdots\alpha_{k-1}}\; r_k \left(1 - \frac{1}{\alpha_k}\right). \tag{12.14}$$

In particular, if $\alpha_k = q$ *for every* k*, then*

$$\delta_T = \left(1 - \frac{1}{q}\right)\sum_{k\geq 0}\frac{r_k}{q^k}. \tag{12.15}$$

Since the condition on the sequence $\{r_k\}_{k\in\mathbb{N}}$ is just to be a strictly increasing sequence, it is easy to choose the r_k's in order to have δ_T either finite or infinite.

Proof. Recall that the definition of δ_T is given in Section 7. Note first that, if S is a regular subset such that $S = S_{\gamma_\infty}$, then the characteristic function w_S satisfies formula (11.5), and hence, the valuative capacity of S is given by

$$\delta_S = \lim_{n\to+\infty}\frac{w_S(n)}{n} = \gamma_0 + \sum_{k\geq 1}\frac{1}{q_{\gamma_k}}(\gamma_k - \gamma_{k-1}). \tag{12.16}$$

Replacing γ_k by r_k and q_{γ_k} by $\alpha_0\alpha_1\ldots\alpha_{k-1}$, we easily obtain formula (12.14). $\square$

Remark 12.3. The map $\varphi : T \to T$ defined by $\varphi(a_n) = a_{n+1}$ for $n \in \mathbb{N}$ is an isometry on T and $O_+^{\varphi}(0) = T \setminus \{0\}$.

(i) If $r_\infty = +\infty$, we consider the subset

$$\widehat{T} = \left\{ \sum_{l=0}^{\infty} d_n X^{r_n} \mid d_n \in C_n \right\} \tag{12.17}$$

and any subset S such that

$$T \subseteq S \subseteq \widehat{T}. \tag{12.18}$$

Then, S is precompact, $S_{\gamma_\infty} = T$ and $\overline{S} = \widehat{T} \cap K$. The subsets S and T are polynomially equivalent and the sequence $\{a_n\}$ is a v-ordering of S.

The previous map φ may be extended by continuity to $\widehat{T}$ and the dynamical system $(\widehat{T}, \varphi)$ is transitive since $O_+^{\varphi}(0) = T \setminus \{0\}$ is dense in T, and hence, in $\widehat{T}$.

(ii) If $r_\infty < +\infty$, we consider any subset S such that

$$T \subseteq S \subseteq \cap_k (T + B(0, r_k)). \tag{12.19}$$

Then, $S_{\gamma_\infty} = T$ and $\overline{S} = \cap_k (T + B(0, r_k))$. The subsets S and T are polynomially equivalent and the sequence $\{a_n\}$ is a v-ordering of S.

Note also that in this latter case ($r_\infty < \infty$), $\overline{T}$ may either be equal to $T + B(0, r_\infty)$ or not. Assume that the characteristic of K is $\neq 2$, $r_\infty \in \Gamma$, $C_0 = \{0, 1\}$ and $C_n = \{0, 1, 2\}$ for $n \geq 1$. Then, $2 + X^{r_\infty}$ belongs to $\cap_{k \geq 0}(T + B(0, r_k))$ while $2 + X^{r_\infty}$ does not belong to $T + B(0, r_\infty)$ (in fact, this is quite the example given in Remark 9.3). On the other hand, if all the C_n's are equal, then

$$\overline{T} = T + B(0, r_\infty) = \cap_{k \geq 0}(T + B(0, r_k)).$$

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Almost clean rings and arithmetical rings

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Abstract. It is shown that a commutative Bézout ring R with compact minimal prime spectrum is an elementary divisor ring if and only if so is R/L for each minimal prime ideal L. This result is obtained by using the quotient space pSpec R of the prime spectrum of the ring R modulo the equivalence generated by the inclusion. When every prime ideal contains only one minimal prime, for instance if R is arithmetical, pSpec R is Hausdorff and there is a bijection between this quotient space and the minimal prime spectrum Min R, which is a homeomorphism if and only if Min R is compact. If x is a closed point of pSpec R, there is a pure ideal $A(x)$ such that $x = V(A(x))$. If R is almost clean, i.e., each element is the sum of a regular element with an idempotent, it is shown that pSpec R is totally disconnected and, $\forall x \in$ pSpec R, $R/A(x)$ is almost clean; the converse holds if every principal ideal is finitely presented. Some questions posed by Facchini and Faith at the second International Fez Conference on Commutative Ring Theory in 1995, are also investigated. If R is a commutative ring for which the ring $Q(R/A)$ of quotients of R/A is an IF-ring for each proper ideal A, it is proved that R_P is a strongly discrete valuation ring for each maximal ideal P and R/A is semicoherent for each proper ideal A.

Keywords. Totally disconnected space, Bézout ring, Hermite ring, elementary divisor ring, IF-ring, valuation ring, clean ring, almost clean ring.

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1 Introduction

In this paper we consider the following two questions:

Question 1.1. Is every Bézout domain an elementary divisor ring?

Question 1.2. More generally, is every Bézout semihereditary ring an elementary divisor ring?

The first question was posed by M. Henriksen in [17] in 1955, and the second by M. D. Larsen, W. J Lewis and T. S. Shores in [20] in 1974.

In Section 3 we prove that these two questions are equivalent but they are still unsolved.

To show this equivalence, we use the quotient space pSpec R of Spec R modulo the equivalence generated by the inclusion, where R is a commutative ring. When R is a *Gelfand ring*, i.e., each prime ideal is contained in only one maximal, pSpec R is Hausdorff and homeomorphic to Max R (Proposition 2.1). On the other hand, if each prime ideal contains a unique minimal prime, then pSpec R is Hausdorff and there is a continuous bijection from Min R into pSpec R which is a homeomorphism if and only if Min R is compact. There is also a continuous surjection τ_R : pSpec $R \to$ Spec B(R), where $\mathrm{B}(R)$ is the Boolean ring associated to R, and τ_R is a homeomorphism if and only if pSpec R is totally disconnected. In this case, it is possible to get some interesting algebraic results by using Lemma 2.9.

A ring R is said to be *clean* (respectively *almost clean* (see [24])) if each element of R is the sum of an idempotent with a unit (respectively a regular element). In Section 4 we show that the total disconnectedness of pSpec R is necessary if the ring R is almost clean. Recall that a ring R is clean if and only if R is Gelfand and Max R totally disconnected: see [25, Theorem 1.7], [22, Corollary 2.7] or [9, Theorem I.1]. Almost clean rings were introduced by McGovern in [24] and studied by several authors: Ahn and Anderson [1], Burgess and Raphael [3] and [4], Varadarajan [30]. If Q is the quotient ring of R and if each prime ideal of R contains a unique minimal prime, we show that pSpec R and pSpec Q are homeomorphic and $\mathrm{B}(R) = \mathrm{B}(Q)$; moreover, if R is arithmetical, then R is almost clean if Q is clean, and the converse holds if Q is coherent.

In Section 5 we give partial answers to some questions posed by Facchini and Faith at the second International Fez Conference on Commutative Ring Theory in 1995 [12]. If R is fractionally IF, it is shown that R/A is semicoherent for each ideal A and R_P is a strongly discrete valuation ring for each maximal ideal P. We give an example of a finitely fractionally self FP-injective ring which is not arithmetical; recall that Facchini and Faith proved that each fractionally self FP-injective ring is arithmetical. It is also proven that any ring which is either clean, coherent and arithmetical or semihereditary is finitely fractionally IF. However, there exist examples of clean coherent arithmetical rings with a non-compact minimal prime spectrum; recall that the author proved that Min R/A is compact for any ideal A of a fractionally self FP-injective ring R.

All rings in this paper are associative and commutative with unity, and all modules are unital.We denote respectively Spec R, Max R and Min R, the space of prime ideals, maximal ideals and minimal prime ideals of R, with the Zariski topology. If A a subset of R, then we denote $V(A) = \{P \in \mathrm{Spec}\, R \mid A \subseteq P\}$ and $D(A) = \mathrm{Spec}\, R \setminus V(A)$.

2 A quotient space of the prime spectrum of a ring

If R is a ring, we consider on Spec R the equivalence relation $\mathcal{R}$ defined by $L\mathcal{R}L'$ if there exists a finite sequence of prime ideals $(L_k)_{1\leq k\leq n}$ such that $L = L_1$, $L' = L_n$ and $\forall k,\ 1 \leq k \leq (n-1)$, either $L_k \subseteq L_{k+1}$ or $L_k \supseteq L_{k+1}$. We denote by pSpec R the quotient space of Spec R modulo $\mathcal{R}$ and by λ_R : Spec $R \to$ pSpec R the natural map. The quasi-compactness of Spec R implies the one of pSpec R, but generally pSpec R is not T_1: see [21, Propositions 6.2 and 6.3]. However:

Proposition 2.1. *The following conditions are equivalent for a ring R:*

(i) *The restriction of λ_R to* Max R *is a homeomorphism;*

(ii) *the restriction of λ_R to* Max R *is injective;*

(iii) *R is Gelfand.*

In this case pSpec R *is Hausdorff.*

Proof. It is obvious that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). If a prime ideal is contained in two maximal ideals P_1 and P_2 we get that $\lambda_R(P_1) = \lambda_R(P_2)$.

(iii) $\Rightarrow$ (i). If L is a prime ideal we denote by $\mu(L)$ the unique maximal ideal containing L. It is easy to verify that $\mu(L) = \mu(L')$ if $L\mathcal{R}L'$. So, μ induces a map $\bar{\mu}$: pSpec $R \to$ Max R. We easily show that $\bar{\mu}^{-1} = \lambda_R|_{\text{Max}\,R}$. By [10, Theorem 1.2] μ is continuous and Max R is Hausdorff. Hence $\bar{\mu}$ is a homeomorphism and pSpec R is Hausdorff. □

Proposition 2.2 ([9, Proposition IV.1]). *Let R be a ring such that each prime ideal contains only one minimal prime. Then* pSpec R *is Hausdorff and* $\lambda_R|_{\text{Min}\,R}$ *is bijective. Moreover* $\lambda_R|_{\text{Min}\,R}$ *is a homeomorphism if and only if* Min R *is compact.*

Proposition 2.3. *Let $\varphi : R \to T$ be a ring homomorphism. Then φ induces a continuous map* ${}^b\varphi$: pSpec $T \to$ pSpec R *such that* $\lambda_R \circ {}^a\varphi = {}^b\varphi \circ \lambda_T$, *where* ${}^a\varphi$: Spec $T \to$ Spec R *is the continuous map induced by* φ.

Proof. If L and L' are prime ideals of T such that $L \subseteq L'$ then ${}^a\varphi(L) \subseteq {}^a\varphi(L')$. Hence, if $x \in$ pSpec T, we can put ${}^b\varphi(x) = \lambda_R({}^a\varphi(L))$ where $L \in x$. Since λ_R and ${}^a\varphi$ are continuous, so is ${}^b\varphi$. □

An exact sequence $0 \to F \to E \to G \to 0$ is *pure* if it remains exact when tensoring it with any R-module. Then, we say that F is a *pure* submodule of E. By [13, Proposition 8.6] F is a pure submodule of E if every finite system of equations

$$\sum_{i=1}^{n} r_{j,i}x_i = y_j \in F \qquad (1 \le j \le p),$$

with coefficients $r_{j,i} \in R$ and unknowns $x_1, \dots, x_n$, has a solution in F whenever it is solvable in E. The following proposition is well known.

Proposition 2.4. *Let A be an ideal of a ring R. The following conditions are equivalent:*

(i) *A is a pure ideal of R;*

(ii) *for each finite family $(a_i)_{1\le i\le n}$ of elements of A there exists $t \in A$ such that $a_i = a_i t$, $\forall i$, $1 \le i \le n$;*

(iii) *for all $a \in A$ there exists $b \in A$ such that $a = ab$;*

(iv) *R/A is a flat R-module.*

Moreover, if A is finitely generated, then A is pure if and only if it is generated by an idempotent.

Proof. (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Let B be an ideal of R. We must prove that $A \cap B = AB$. If $a \in A \cap B$, there exists $t \in A$ such that $a = at$. Hence $a \in AB$.

(iv) ⇒ (iii). If $a \in A$, then $Ra = A \cap Ra = Aa$ by (iv).

(i) ⇒ (iii). If $a \in A$, 1 is solution of the equation $ax = a$. So, this equation has a solution in A.

(iii) ⇒ (ii). Let $a_1, \dots, a_n$ be elements of A. We proceed by induction on n. There exists $t \in A$ such that $a_n = ta_n$. By induction hypothesis there exists $s \in A$ such that $a_i - ta_i = s(a_i - ta_i)$, $\forall i,\ 1 \le i \le (n-1)$. Now, it is easy to check that $(s + t - st)a_i = a_i,\ \forall i,\ 1 \le i \le n$.

(ii) ⇒ (i). We consider the following system of equations:

$$\sum_{i=1}^{n} r_{j,i} x_i = a_j \in A,\ 1 \le j \le p.$$

Assume that $(c_1, \dots, c_n)$ is a solution of this system in R. There exists $s \in A$ such that $a_j = sa_j,\ \forall j,\ 1 \le j \le p$. So, $(sc_1, \dots, sc_n)$ is a solution of this system in A. □

We set 0_P the kernel of the natural map $R \to R_P$ where $P \in \operatorname{Spec} R$.

Lemma 2.5. *Let R be a ring and let C a closed subset of* Spec R*. Then C is the inverse image of a closed subset of* pSpec R *by λ_R if and only if $C = V(A)$ where A is a pure ideal. Moreover, in this case, $A = \cap_{P \in C} 0_P$.*

Proof. Let A be a pure ideal, and let P and L be prime ideals such that $A \subseteq P$ and $L \subseteq P$. Since A is pure, for each $a \in A$ there exists $b \in A$ such that $a = ab$. Then $(1-b)a = 0$ and $(1-b) \notin P$, whence $(1-b) \notin L$ and $a \in L$. So, $L \in V(A)$ and $V(A)$ is the inverse image of a closed subset of pSpec R by λ_R.

Let $C = V(B)$ where $B = \cap_{L \in C} L$. Suppose that C is the inverse image of a closed subset of pSpec R by λ_R. We put $A = \cap_{P \in C} 0_P$. Let $b \in B$ and $P \in C$. Then C contains each minimal prime ideal contained in P. So, the image of b, by the natural map $R \to R_P$, belongs to the nilradical of R_P. It follows that there exist $0 \neq n_P \in \mathbb{N}$ and $s_P \in R \setminus P$ such that $s_P b^{n_P} = 0$. Hence, $\forall L \in D(s_P) \cap C,\ b^{n_P} \in 0_L$. A finite family $(D(s_{P_j}))_{1 \le j \le m}$ covers C. Let $n = \max\{n_{P_1}, \dots, n_{P_m}\}$. Then $b^n \in 0_L,\ \forall L \in C$, whence $b^n \in A$. We deduce that $C = V(A)$. Now, we have $A_P = 0$ if $P \in V(A)$ and $A_P = R_P$ if $P \in D(A)$. Hence A is a pure ideal. □

Corollary 2.6. *For any ring R the following assertions hold:*

(i) *A subset U of* pSpec R *is open and closed if and only if there exists an idempotent $e \in R$ such that $\lambda_R^{\leftarrow}(U) = D(e)$;*

(ii) *R is indecomposable if and only if* pSpec R *is connected.*

Proof. A subset U of pSpec R is open and closed if and only if is so $\lambda_R^{\leftarrow}(U)$ and it is well known that a subset U' of Spec R is open and closed if and only if $U' = D(e)$ for some idempotent $e \in R$. The second assertion is an immediate consequence of the first. □

If $\{x\}$ is closed in pSpec R we denote by $A(x)$ the pure ideal of R for which $x = \mathrm{V}(A(x))$. A topological space is called *totally disconnected* if each of its connected components contains only one point. Every Hausdorff topological space X with a base of clopen neighbourhoods is totally disconnected and the converse holds if X is compact (see [16, Theorem 16.17]).

Proposition 2.7. *Let R be a ring. Then the following conditions are equivalent:*

(i) pSpec R *is totally disconnected;*

(ii) *for each $x \in$ pSpec R, $\{x\}$ is closed and $A(x)$ is generated by idempotents.*

Proof. (ii) $\Rightarrow$ (i). Let $x, y \in$ pSpec R, $x \neq y$. Then $V(A(x)) \cap V(A(y)) = \varnothing$. So, $A(x) + A(y) = R$, whence $\exists a \in A(x)$ such that $(1-a) \in A(y)$. There exists an idempotent $e \in A(x)$ such that $a = ae$. So, $(1-e)(1-a) = (1-e) \in A(y)$. We easily deduce that $x \subseteq D(1-e)$ and $y \subseteq D(e)$.

(i) $\Rightarrow$ (ii). Let $x \in$ pSpec R and $a \in A(x)$. There exists $b \in A(x)$ such that $a = ab$. So $(1-b)a = 0$. Clearly $x \subseteq D(1-b)$. Since pSpec R is Hausdorff and Spec R is quasi-compact, $\lambda_R^{\rightarrow}(V(1-b))$ is closed. Therefore $U =$ pSpec $R \setminus \lambda_R^{\rightarrow}(V(1-b))$ is open and contains x. The condition pSpec R is totally disconnected implies that there exists an idempotent e such that $x \subseteq D(e) \subseteq \lambda_R^{\leftarrow}(U) \subseteq D(1-b)$. If follows that $e \in R(1-b)$. So $ea = 0$ and consequently $a = a(1-e)$. From $x \subseteq D(e)$ and $e(1-e) = 0$ we deduce that $(1-e) \in A(x)$ by Lemma 2.5. □

For any ring R, $\mathrm{B}(R)$ is the set of idempotents of R. For any $e, e' \in \mathrm{B}(R)$ we put $e \oplus e' = e + e' - ee'$ and $e \odot e' = ee'$. With these operations $\mathrm{B}(R)$ is a Boolean ring. The space Spec $\mathrm{B}(R)$ is denoted by $\mathrm{X}(R)$. Then $\mathrm{X}(R)$ is Hausdorff, compact and totally disconnected. If $x \in \mathrm{X}(R)$ the *stalk* of R at x is the quotient of R by the ideal generated by the idempotents contained in x.

Proposition 2.8. *Let R be a ring. The following assertions hold:*

(i) *There exists a surjective continuous map τ_R :* pSpec $R \to \mathrm{X}(R)$*;*

(ii) pSpec R *is totally disconnected if and only if τ_R is a homeomorphism. In this case, for each $x \in$* pSpec R*, $R/A(x)$ is the stalk of R at $\tau_R(x)$.*

Proof. (i). If L and L' are prime ideals of R, $L \subseteq L'$, then $L \cap \mathrm{B}(R) = L' \cap \mathrm{B}(R)$ since each prime ideal of $\mathrm{B}(R)$ is maximal. So, τ_R is well defined. It is easy to check that for any idempotent $e \in R$, $\tau_R^{\leftarrow}(D(e)) = \lambda_R^{\rightarrow}(D(e))$. Hence τ_R is continuous. For each $x \in \mathrm{X}(R)$, if L is a maximal ideal containing all elements of x, then $x = \tau_R(\lambda_R(L))$, whence τ_R is surjective.

(ii). It is obvious that pSpec R is totally disconnected if τ_R is a homeomorphism. Conversely, since pSpec R is compact and $\mathrm{X}(R)$ is Hausdorff it is enough to show that τ_R is injective. Let $x, x' \in$ pSpec R, $x \neq x'$. There exists an idempotent e such that $x \in \lambda_R^{\rightarrow}(D(e))$ and $x' \in \lambda_R^{\rightarrow}(D(1-e))$. It follows that $e \notin \tau_R(x)$ and $e \in \tau_R(x')$. Hence $\tau_R(x) \neq \tau_R(x')$. The last assertion is a consequence of Proposition 2.7 and Lemma 2.5. □

The following lemma will be useful to show some important results of this paper.

Lemma 2.9. *Let R be a ring such that* pSpec R *is totally disconnected. Then any R-algebra S (which is not necessarily commutative) satisfies the following condition: let $f_1, \dots, f_k$ be polynomials over S in noncommuting variables $x_1, \dots, x_m, y_1, \dots, y_n$. Let $a_1, \dots, a_m \in S$. Assume that, $\forall x \in$* pSpec R *there exist $b_1, \dots, b_n \in S$ such that:*

$$f_i(a_1, \dots, a_m, b_1, \dots, b_n) \in A(x)S, \quad \forall i,\ 1 \le i \le k.$$

Then there exist $d_1, \dots, d_n \in S$ such that:

$$f_i(a_1, \dots, a_m, d_1, \dots, d_n) = 0, \quad \forall i,\ 1 \le i \le k.$$

Proof. Let $x \in \operatorname{pSpec} R$ and let $b_{x,1}, \dots, b_{x,n} \in S$ such that

$$f_i(a_1, \dots, a_m, b_{x,1}, \dots, b_{x,n}) \in A(x)S, \quad \forall i,\ 1 \le i \le k.$$

Then there exists a finitely generated ideal $A \subseteq A(x)$ such that

$$f_i(a_1, \dots, a_m, b_{x,1}, \dots, b_{x,n}) \in AS, \quad \forall i,\ 1 \le i \le k.$$

There exists an idempotent e_x such that $A \subseteq R(1 - e_x) \subseteq A(x)$. Hence

$$e_x f_i(a_1, \dots, a_m, b_{x,1}, \dots, b_{x,n}) = 0, \quad \forall i,\ 1 \le i \le k.$$

A finite family $(\lambda_R^{\rightarrow}(D(e_{x_j})))_{1 \le j \le p}$ covers pSpec R. We may assume that $(e_{x_j})_{1 \le j \le p}$ is a family of orthogonal idempotents. We put $d_\ell = e_{x_1} b_{x_1,\ell} + \cdots + e_{x_p} b_{x_p,\ell}$, $\forall \ell,\ 1 \le \ell \le n$. Then $f_i(a_1, \dots, a_m, d_1, \dots, d_n) = 0, \forall i, 1 \le i \le k$. □

We denote by gen M the minimal number of generators of a finitely generated R-module M. The following proposition is an example of an algebraic result that can be proven by using Lemma 2.9. Recall that the *trivial extension* $R \ltimes M$ of R by M is defined by: $R \ltimes M = \{\begin{pmatrix} r & x \\ 0 & r \end{pmatrix} \mid r \in R,\ x \in M\}$. It is convenient to identify $R \ltimes M$ with the R-module $R \oplus M$ endowed with the following multiplication: $(r, x)(s, y) = (rs, ry + sx)$, where $r, s \in R$ and $x, y \in M$.

Proposition 2.10. *Let R be a ring such that* pSpec R *is totally disconnected. Let M be a finitely generated R-module and F a finitely presented R-module. Then:*

(i) *If, $\forall x \in$* pSpec R*, $M/A(x)M$ is a homomorphic image of $F/A(x)F$, then M is a homomorphic image of F;*

(ii) gen M = sup{gen$(M/A(x)M) \mid x \in$ pSpec R}*;*

(iii) *if M is finitely presented and, $\forall x \in$* pSpec R*, $M/A(x)M \cong F/A(x)F$, then $M \cong F$.*

Proof. (i). Let $\{m_1, \dots, m_p\}$ be a spanning set of M. Let $\{f_1, \dots, f_n\}$ be a spanning set of F with the following relations: $\forall \ell, 1 \le \ell \le n'$, $\sum_{i=1}^{n} c_{l,i} f_i = 0$. We put

$S = R \ltimes M$ the trivial extension of R by M. We consider the following system $\mathcal{E}_1$ of polynomial equations in variables $X_{j,i},\ Y_i, Z_{i,j},\ 1 \le j \le k,\ 1 \le i \le n$:

$$\sum_{i=1}^{n} X_{j,i} Y_i = (0, m_j), \quad \forall j,\ 1 \le j \le k; \quad Y_i = \sum_{j=1}^{p} Z_{i,j}(0, m_j), \quad \forall i,\ 1 \le i \le n;$$

$$\sum_{i=1}^{n} (c_{l,i}, 0) Y_i = 0, \quad \forall \ell,\ 1 \le \ell \le n'.$$

Thus $\mathcal{E}_1$ has a solution modulo $A(x)S$ for each $x \in \operatorname{pSpec} R$. By Lemma 2.9 $\mathcal{E}_1$ has a solution $x_{j,i},\ y_i, z_{i,j},\ 1 \le j \le p,\ 1 \le i \le n$ in S. It is easy to check that $y_i = (0, m'_i),\ \forall i,\ 1 \le i \le n$, and if $x_{j,i} = (r_{j,i}, x'_{j,i}),\ \forall j, i,\ 1 \le j \le p,\ 1 \le i \le n$, then $m_j = \sum_{i=1}^{n} r_{j,i} m'_i,\ \forall j, 1 \le j \le p$. We have also: $\forall \ell, 1 \le \ell \le n'$, $\sum_{i=1}^{n} c_{l,i} m'_i = 0$. Hence we get an epimorphism $\phi : F \to M$ defined by $\phi(f_i) = m'_i,\ \forall i, 1 \le i \le n$.

(ii) is an easy consequence of (i).

(iii). Let the notations be as in (i). We assume that $m_1, \dots, m_p$ verify the following relations: $\forall k,\ 1 \le k \le p'$, $\sum_{j=1}^{p} d_{k,j} m_j = 0$.

Observe that $F \cong M$ if M has a spanning set $\{m'_1, \dots, m'_n\}$ with the relations: $\forall \ell,\ 1 \le \ell \le n'$, $\sum_{i=1}^{n} c_{l,i} m' i = 0$. In this case there exist $r_{j,i} \in R$ such that $m_j = \sum_{i=1}^{n} r_{j,i} m'_i,\ \forall j,\ 1 \le j \le p$. Thus

$$\sum_{j=1}^{p} d_{k,j} m_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{p} d_{k,j} r_{j,i} \right) m'_i = 0, \quad \forall k,\ 1 \le k \le p'.$$

It follows that there exist $w_{k,l} \in R$ such that:

$$\begin{aligned}\sum_{j=1}^{p} d_{k,j} m_j &= \sum_{\ell=1}^{n'} w_{k,l} \left(\sum_{i=1}^{n} c_{\ell,i} m'_i \right) \\ &= \sum_{i=1}^{n} \left(\sum_{\ell=1}^{n'} w_{k,l} c_{l,i} \right) m' i = 0, \quad \forall k,\ 1 \le k \le p'.\end{aligned}$$

We deduce that

$$\sum_{j=1}^{p} d_{k,j} r_{j,i} = \sum_{\ell=1}^{n'} w_{k,l} c_{l,i}, \quad \forall k,\ 1 \le k \le p', \quad \forall i,\ 1 \le i \le n. \tag{2.1}$$

Conversely, if there exists an epimorphism $\phi : F \to M$ defined by $\phi(f_i) = m'_i$, then ϕ is bijective if each relation (*rel*) $\sum_{i=1}^{n} a_i m'_i = 0$ is a linear combination of the relations $\sum_{i=1}^{n} c_{\ell,i} m'_i = 0$. Since m'_i is a linear combination of $m_1, \dots, m_p$, from the relation (*rel*) we get a relation (*rel*1) which is a linear combination of the relations $\sum_{j=i}^{p} d_{k,j} m_j = 0$. By using the equalities 2.1, we get that (*rel*) is a linear combination of the relations $\sum_{i=1}^{n} c_{\ell,i} m'_i = 0$.

Let $\mathcal{E}_2$ be the system of polynomial equations in variables $X_{j,i}$, $W_{k,\ell}$, $1 \leq j \leq p$, $1 \leq i \leq n$, $1 \leq k \leq p'$, $1 \leq \ell \leq n'$,

$$\sum_{j=1}^{p}(d_{k,j},0)X_{j,i} = \sum_{\ell=1}^{n'} W_{k,l}(c_{\ell,i},0), \quad \forall k,\ 1 \leq k \leq p', \quad \forall i,\ 1 \leq i \leq n.$$

We put $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. As in (i) and by using the above observation we show that $\mathcal{E}$ has a solution. We define $\phi : F \to M$ as in (i), and by using the fact that $\mathcal{E}_2$ has a solution, we prove that ϕ is injective by using the above observation. □

3 Hermite rings and elementary divisor rings

An R-module is called *uniserial* if the set of its submodules is totally ordered by inclusion. A ring R is a *valuation ring* if it is a uniserial R-module. We say that R is *arithmetical* if R_L is a valuation ring for each maximal ideal L. A ring is a *Bézout ring* if every finitely generated ideal is principal. A ring R is *Hermite* if R satisfies the following property : for every $(a, b) \in R^2$, there exist d, a', b' in R such that $a = da'$, $b = db'$ and $Ra' + Rb' = R$. We say that R is an *elementary divisor ring* if for every matrix A, with entries in R, there exist a diagonal matrix D and invertible matrices P and Q, with entries in R, such that $PAQ = D$. Then we have the following implications:

elementary divisor ring ⇒ Hermite ring ⇒ Bézout ring ⇒ arithmetical ring;

but these implications are not reversible: see [14] or [5].

Theorem 3.1. *Let R be a ring such that* pSpec R *is totally disconnected. Assume that $R/A(x)$ is Bézout for each $x \in$* pSpec R*. Then R is Hermite.*

Proof. By Proposition 2.10 R is Bézout. It follows that each prime ideal contains a unique minimal prime, whence $R/A(x)$ has a unique minimal prime ideal. By [17, Theorem 2], a ring with a unique minimal ideal is Hermite if and only if it is Bézout. So, $R/A(x)$ is Hermite. Now, let $a,\ b \in R$. Consider the following polynomial equations: $a = XZ$, $b = YZ$ and $1 = SX + TY$. For each $x \in$ pSpec R, these equations have a solution modulo $A(x)$. By Lemma 2.9 they have a solution in R. □

Recall that a ring R is *(semi)hereditary* if each (finitely generated) ideal is projective. If F is a submodule of a module E and x an element of E, then the ideal $\{r \in R \mid rx \in F\}$ is denoted by $(F : x)$.

The following was already proved, see [8, Theorem III.3] and [20, Theorem 2.4].

Corollary 3.2. *Let R be a Bézout ring. Then the following assertions hold:*

(i) *R is Hermite if* Min R *is compact;*

(ii) *R is Hermite if it is semihereditary.*

Proof. (i). Since each prime ideal contains a unique minimal prime, $\lambda_R|_{\mathrm{Min}\,R}$ is bijective. Moreover, by [9, Proposition IV.1] pSpec R is Hausdorff. It follows that $\lambda_R|_{\mathrm{Min}\,R}$ is a homeomorphism because Min R is compact. We can apply the previous theorem because Min R is always totally disconnected ([18, Corollary 2.4]):

$$\forall a \in R, \quad D(a) \cap \mathrm{Min}\,R = V((N : a)) \cap \mathrm{Min}\,R$$

where N is the nilradical of R.

(ii) is an immediate consequence of (i) because Min R is compact if R is semihereditary by [26, Proposition 10]. □

The following example shows that pSpec R is not generally totally disconnected, even if R is arithmetical.

Example 3.3. Consider [31, Example 6.2 (due to Jensen)] defined in the following way: let $\mathcal{I}$ be a family of pairwise disjoint intervals of the real line with rational endpoints, such that between any two intervals of $\mathcal{I}$ there is at least another interval of $\mathcal{I}$; let R be the ring of continuous maps $\mathbb{R} \to \mathbb{R}$ which are rational constant by interval except on finitely many intervals of $\mathcal{I}$ on which it is given by a rational polynomial. It is easy to check that R is a reduced indecomposable ring. It is also Bézout (left as an exercise!). Let $[a, b] \in \mathcal{I}$ and $f \in R$ defined by $f(x) = (x-a)(b-x)$ if $a \leq x \leq b$ and $f(x) = 0$ elsewhere. Then $(0 : f)$ is not finitely generated, whence R is not semihereditary. So, pSpec R is an infinite set and a compact connected topological space.

Theorem 3.4. *Let R be a ring such that* pSpec R *is totally disconnected. Assume that $R/A(x)$ is Bézout for each $x \in$* pSpec R*. Then R is an elementary divisor ring if and only if so is R/L, for each minimal prime ideal L.*

Proof. Only "if" requires a proof. By Theorem 3.1 R is Hermite. Let $x \in$ pSpec R and $R' = R/A(x)$. Then R' has a unique minimal prime ideal. Let L be the minimal prime ideal of R such that $L/A(x)$ is the minimal prime of R'. Thus $L/A(x)$ is contained in the Jacobson radical $\mathcal{J}(R')$ of R'. So, $R'/\mathcal{J}(R')$ is an elementary divisor ring since it is a homomorphic image of R/L. By [17, Theorem 3] a Hermite ring S is an elementary divisor ring if and only if so is $S/\mathcal{J}(S)$. Hence $R/A(x)$ is an elementary divisor ring. Let $a, b, c \in R$ such that $Ra + Rb + Rc = R$. We consider the polynomial equation in variables X, Y, S, T: $aSX + bTX + cTY = 1$. By [15, Theorem 6], this equation has a solution modulo $A(x)$, $\forall x \in$ pSpec R. So, by Lemma 2.9 there is a solution in R. We conclude by [15, Theorem 6]. □

With a similar proof as in Corollary 3.2, we get Corollary 3.5. The second condition shows that the two questions 1.1 and 1.2 have the same answer.

Corollary 3.5. *The following assertions hold:*

(i) *Let R be a Bézout ring with compact minimal prime spectrum. Then R is an elementary divisor ring if and only if so is R/L, for each minimal prime ideal L.*

(ii) *Let R be a semihereditary ring. Then R is an elementary divisor ring if and only if so is R/L, for each minimal prime ideal L.*

(iii) *Let R be a hereditary ring. Then R is an elementary divisor ring if and only if R/L is Bézout for each minimal prime ideal L.*

The third assertion can be also deduced from [27, Corollary].

4 Almost clean rings

In [24, Proposition 15], McGovern proved that each element of a ring R is the product of an idempotent with a regular element if and only if R is a *PP-ring*, i.e., each principal ideal is projective, and he showed that each PP-ring is almost clean ([24, Proposition 16]). The aim of this section is to study almost clean rings.

In the sequel, if R is a ring, $\mathfrak{R}(R)$ is its set of regular elements of R and

$$\Phi_R = \{L \in \operatorname{Spec} R \mid L \cap \mathfrak{R}(R) = \varnothing\}.$$

By [2, Corollaire 2, p. 92] each zero-divisor is contained in an element of Φ_R. So, the following proposition is obvious.

Proposition 4.1. *Let R be a ring. The following conditions are equivalent:*

(i) *For each $a \in R$, either a or $(a-1)$ is regular.*

(ii) *R is almost clean and indecomposable.*

(iii) $\forall L, L' \in \Phi_R,\ L + L' \neq R.$

Corollary 4.2. *Let R be an arithmetical ring, Q its ring of fractions and N its nilradical. Then:*

(i) *R is almost clean and indecomposable if and only if Q is a valuation ring;*

(ii) *R/A is almost clean and indecomposable for each ideal $A \subseteq N$ if and only if N is prime and uniserial.*

Proof. (i). Assume that R is almost clean and indecomposable. Then if $L, L' \in \Phi_R$ then there exists a maximal ideal P such that $L + L' \subseteq P$. Since R_P is a valuation ring, either $L \subseteq L'$ or $L' \subseteq L$. By [2, Corollaire, p. 129] Φ_R is homeomorphic to $\operatorname{Spec} Q$. It follows that Q is local. Conversely, Φ_R contains a unique maximal element.

(ii). First, assume that R/A is almost clean and indecomposable for each ideal $A \subseteq N$. Since Q is a valuation ring then N is prime. By way of contradiction suppose $\exists a, b \in N$ such that neither divides the other. We may assume that $Ra \cap Rb = 0$. Let A and B be maximal submodules of Ra and Rb respectively. We may replace R by $R/(A+B)$ and assume that Ra and Rb are simple modules. Let L and P be their respective annihilators. Since R_L is a valuation ring and $R_L a \neq 0$, we have $R_L b = 0$. So, $L \neq P$. It follows that $\exists c \in L$ such that $(1-c) \in P$. Neither c nor $(1-c)$ is regular. This contradicts that R is almost clean.

Conversely, suppose that N is prime and uniserial. Then, if A is an ideal contained in N, N/A is also uniserial. So, the ring of fractions of R/A is a valuation ring: see [29, p. 218, between the definition of a torch ring and Theorem B]. □

Following Vámos [29], we say that R is a *torch ring* if the following conditions are satisfied:

(i) R is an arithmetical ring with at least two maximal ideals;

(ii) R has a unique minimal prime ideal N which is a nonzero uniserial module.

We follow T. S. Shores and R. Wiegand [28], by defining a *canonical form* for an R-module E to be a decomposition $E \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n$, where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \neq R$, and by calling a ring R a *CF-ring* if every direct sum of finitely many cyclic modules has a canonical form.

Corollary 4.3. *Each CF-ring is almost clean.*

Proof. By [28, Theorem 3.12] every CF-ring is arithmetical and a finite product of indecomposable CF-rings. If R is indecomposable then R is either a domain, or a local ring, or a torch ring. By Corollary 4.2 R is almost clean. □

By Proposition 2.8, there is some similarity between [4, Theorem 2.4] and the following theorem.

Theorem 4.4. *Let R be a ring. Consider the following conditions:*

(i) *R is almost clean;*

(ii) pSpec R *is totally disconnected and* $\forall r \in R$, $\forall x \in$ pSpec R, $\exists s_x \in \mathfrak{R}(R)$ *such that either* $r \equiv s_x$ modulo $A(x)$ *or* $(r-1) \equiv s_x$ modulo $A(x)$*;*

(iii) pSpec R *is totally disconnected and for each* $x \in$ pSpec R, $R/A(x)$ *is almost clean.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *and the three conditions are equivalent if every principal ideal of R is finitely presented.*

Proof. (i) $\Rightarrow$ (ii). Let x and y be two distinct points of pSpec R. Let P and P' be two minimal prime ideals of R such that $P \in x$ and $P' \in y$. There is no maximal ideal containing P and P'. So, $P + P' = R$ and there exist $a \in P$ and $a' \in P'$ such that $a + a' = 1$. We have $a = s + e$ where s is regular and e idempotent. Since $s \notin P$ we get that $e \notin P$. It follows that for each $L \in x$, $(1-e) \in L$ and $e \notin L$. So, $x \subseteq D(e)$. We have $a' = -s + (1-e)$. In the same way we get $y \subseteq D(1-e)$. Therefore x and y belong to disjoint clopen neighbourhoods. Hence pSpec R is totally disconnected. Now, let $r \in R$ and $z \in$ pSpec R. We have $r = s + e$ where s is regular and e idempotent. If $z \subseteq V(e)$ then $e \in A(z)$ and $r \equiv s$ modulo $A(z)$; and, if $z \subseteq V(1-e)$ then $(1-e) \in A(z)$ and $(r-1) \equiv s$ modulo $A(z)$.

(ii) $\Rightarrow$ (i). Let $a \in R$. Let Q be the ring of fractions of R and let $S = R \ltimes Q$ be the trivial extension of R by Q. We consider the following polynomial equations in S:

$E^2 = E$, $E + X = (a, 0)$ and $XY = (0, 1)$. Let $x \in \operatorname{pSpec} R$. If $a \equiv s_x$ modulo $A(x)$ where s_x is a regular element of R, then $E = (0, 0)$, $X = (s_x, 0)$, $Y = (0, 1/s_x)$ is a solution of these polynomial equations modulo $A(x)S$; if $(a - 1) \equiv s_x$ modulo $A(x)$, we take $E = (1, 0)$. So, by Lemma 2.9, these equations have a solution in S: $E = (e, q)$, $X = (s, u)$, $Y = (t, v)$. From $E^2 = E$ we deduce that $e^2 = e$ and $(2e - 1)q = 0$. So, $q = 0$ since $(2e - 1)$ is a unit. From $E + X = (a, 0)$ we deduce that $u = 0$, and from $XY = (0, 1)$ we deduce that $sv = 1$. Hence s is a regular element of R and $a = e + s$. We conclude that R is almost clean.

(ii) $\Rightarrow$ (iii). Clearly, if $r \in R$ then, $\forall x \in \operatorname{pSpec} R$ either r or $(r - 1)$ is regular modulo $A(x)$. So, $R/A(x)$ is almost clean $\forall x \in \operatorname{pSpec} R$.

(iii) $\Rightarrow$ (ii). We assume that each principal ideal is finitely presented. Let $x \in \operatorname{pSpec} R$. It remains to show that any regular element a modulo $A(x)$ is congruent to a regular element of R modulo $A(x)$. Since $(0 : a)$ is finitely generated and $A(x)$ is generated by idempotents, there exists an idempotent $e \in A(x)$ such that $(0 : a) \subseteq Re$. Now, it is easy to check that $a(1 - e) + e$ is a regular element. □

The following examples show that the conditions (i) and (iii) are not generally equivalent.

Example 4.5 ([4, Examples 2.2 and 2.9.(i)]). are non-almost clean arithmetical rings (the second is reduced) with almost clean stalks. These are defined in the following way: let D be a principal ideal domain and $\forall n \in \mathbb{N}$, let R_n be a quotient of D by a non-zero proper ideal I_n; let R be the set of elements $r = (r_n)_{n \in \mathbb{N}}$ of $\Pi_{n \in \mathbb{N}} R_n$ which satisfy $\exists m_r \in \mathbb{N}$ and $\exists d_r \in D$ such that $\forall n \geq m_r$, $r_n = d_r + I_n$. We put $e_m = (\delta_{m,n})_{n \in \mathbb{N}}$, $\forall m \in \mathbb{N}$. It is easy to check that the points of $\operatorname{pSpec} R$ are: $x_\infty = V(\bigoplus_{n \in \mathbb{N}} R_n)$ and $\forall n \in \mathbb{N}$, $x_n = V(1 - e_n)$. Since $x_\infty \subseteq D(1 - e_m)$ and $x_m \subseteq D(e_m)$, $\forall m \in \mathbb{N}$, $\operatorname{pSpec} R$ is totally disconnected. So, (by using Proposition 2.8) these examples satisfy condition (iii) of Theorem 4.4.

The following example shows that the condition "each principal ideal is finitely presented" is not necessary if R is almost clean.

Example 4.6. We consider [31, Example 1.3b]. Let $R = \mathbb{Z} \oplus S$ where $S = (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$. The multiplication is defined by $(m, x)(n, y) = (mn, nx + my + xy)$, where $m, n \in \mathbb{Z}$ and $x, y \in S$. It is known that R is a reduced arithmetical ring which is not semi-hereditary. Let $p \in \mathbb{Z}$ and $s \in S$. Then $(2p - 1, s) = (2p - 1, 0) + (0, s)$ and $(2p, s) = (2p - 1, 0) + (1, s)$. It is easy to check that $(2p - 1, 0)$ is regular and, $(0, s)$ and $(1, s)$ are idempotents. Hence R is almost clean. But $(0 : (2, 0)) = 0 \times S$ which is not finitely generated.

Proposition 4.7. *Let R be a ring and Q its ring of fractions. Assume that each prime ideal of R contains only one minimal prime. Then:*

(i) $\operatorname{pSpec} Q$ *and* $\operatorname{pSpec} R$ *are homeomorphic;*

(ii) *each idempotent of Q belongs to R;*

(iii) *if Q is clean then R is almost clean.*

Proof. (i). If $\varphi : R \to Q$ is the natural map then $\operatorname{Min} R \subseteq \Phi_R = \operatorname{Im} {}^a\varphi$. It follows that each prime ideal of Q contains only one minimal prime and ${}^b\varphi$ is bijective. Moreover, since pSpec Q and pSpec R are compact, ${}^b\varphi$ is a homeomorphism.

(ii). Let e an idempotent of Q. Then $\lambda_Q^{\rightarrow}(D(e))$ is a clopen subset of pSpec Q, whence its image by ${}^b\varphi$ is a clopen subset of pSpec R and consequently it is of the form $\lambda_R^{\rightarrow}(D(e'))$ where e' is an idempotent of R. But the inverse image of $D(e') \subseteq \operatorname{Spec} R$ by ${}^a\varphi$ is $D(e') \subseteq \operatorname{Spec} Q$. So, $e = e' \in R$.

(iii). Assume that Q is clean. Let $r \in R$. Then $r = q + e$ where q is a unit of Q and e an idempotent. Since $e \in R$, q is a regular element of R. □

Corollary 4.8. *Let R be an almost clean arithmetical ring and Q its ring of fractions. If Q is coherent then Q is clean. In this case Q is an elementary divisor ring.*

Proof. Recall that an arithmetical ring is coherent if and only if each principal ideal is finitely presented because the intersection of any two finitely generated ideals is finitely generated by [28, Corollary 1.11]. By Proposition 4.7 we may assume that pSpec $Q =$ pSpec R. This space is totally disconnected by Theorem 4.4. Let $x \in$ pSpec Q and let $A(x)$ be the pure ideal of R such that $x = V(A(x))$. Then $x = V(QA(x))$. If s is a regular element of R then $s + A(x)$ is a regular element of $R/A(x)$. So, $R/A(x)$ and $Q/QA(x)$ have the same ring of fractions which is a valuation ring by Corollary 4.2. Hence $Q/QA(x)$ is almost clean, $\forall x \in$ pSpec Q. By Theorem 4.4 Q is almost clean too. We conclude that Q is clean since each regular element is a unit.

The last assertion is a consequence of [9, Theorem I.1 and Corollary II.2]. □

We don't know if the assumption "Q is coherent" can be omitted. The following example shows that the conclusion of the previous corollary doesn't hold if R is not arithmetical, even if R has a unique minimal prime ideal.

Example 4.9. Let K be a field, $D = K[x, y]_{(x,y)}$ where x, y are indeterminates, $E = D/Dx \oplus D/Dy$ and R the trivial extension of D by E. Since R contains a unique minimal prime ideal, Q is indecomposable. We put $r = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and $s = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$. Clearly r and s are zerodivisors but $r + s$ is regular. It follows that $\frac{r}{r+s}$ and $\frac{s}{r+s}$ are two zerodivisors of Q whose sum is 1. So, Q is not almost clean.

5 Fractionally IF-rings

Let $\mathcal{P}$ be a ring property. We say that a ring R is *(finitely) fractionally $\mathcal{P}$* if the classical ring of quotients $Q(R/A)$ of R/A satisfies $\mathcal{P}$ for each (finitely generated) ideal A. In [12] Facchini and Faith studied fractionally self FP-injective rings. They proved that these rings are arithmetical ([12, Theorem 1]) and they gave some examples ([12, Theorem 6]). In the first part of this section we investigate fractionally IF-rings and partially answer a question posed by Facchini and Faith in [12, question Q1, p. 301].

Some preliminary results are needed. As in [23] a ring R is said to be *semicoherent* if $\operatorname{Hom}_R(E, F)$ is a submodule of a flat R-module for any pair of injective R-modules

E, F. An R-module E is *FP-injective* if $\mathrm{Ext}^1_R(F, E) = 0$ for any finitely presented R-module F, and R is *self FP-injective* if R is FP-injective as R-module. We recall that a module E is FP-injective if and only if it is a pure submodule of every overmodule. If each injective R-module is flat we say that R is an *IF-ring*. By [6, Theorem 2], R is an IF-ring if and only if it is coherent and self FP-injective.

Proposition 5.1. *Let R be a self FP-injective ring. Then R is coherent if and only if it is semicoherent.*

Proof. If R is coherent then $\mathrm{Hom}_R(E, F)$ is flat for any pair of injective modules E, F by [13, Theorem XIII.6.4(b)]; so, R is semicoherent. Conversely, let E be the injective hull of R. Since R is a pure submodule of E, then, for each injective R-module F, the following sequence is exact:

$$0 \to \mathrm{Hom}_R(F \otimes_R E/R, F) \to \mathrm{Hom}_R(F \otimes_R E, F) \to \mathrm{Hom}_R(F \otimes_R R, F) \to 0.$$

By using the natural isomorphisms $\mathrm{Hom}_R(F \otimes_R B, F) \cong \mathrm{Hom}_R(F, \mathrm{Hom}_R(B, F))$ and $F \cong \mathrm{Hom}_R(R, F)$ we get the following exact sequence:

$$0 \to \mathrm{Hom}_R(F, \mathrm{Hom}_R(E/R, F)) \to \mathrm{Hom}_R(F, \mathrm{Hom}_R(E, F)) \to \mathrm{Hom}_R(F, F) \to 0.$$

So, the identity map on F is the image of an element of $\mathrm{Hom}_R(F, \mathrm{Hom}_R(E, F))$. Consequently the following sequence splits:

$$0 \to \mathrm{Hom}_R(E/R, F) \to \mathrm{Hom}_R(E, F) \to F \to 0.$$

It follows that F is a direct summand of a flat module. So, R is an IF-ring. □

Corollary 5.2. *Let R be a ring. Assume that its ring of quotients Q is self FP-injective. Then R is semicoherent if and only if Q is coherent.*

Proof. If R is semicoherent, then so is Q by [23, Proposition 1.2]. From Proposition 5.1 we deduce that Q is coherent. Conversely, let E and F be injective R-modules. It is easy to check that the multiplication by a regular element of R in $\mathrm{Hom}_R(E, F)$ is injective. So, $\mathrm{Hom}_R(E, F)$ is a submodule of the injective hull of $Q \otimes_R \mathrm{Hom}_R(E, F)$ which is flat over Q and R because Q is an IF-ring. □

Corollary 5.3. *Let R be a valuation ring and A an ideal. Then R/A is semicoherent if and only if A is either prime or the inverse image of a proper principal ideal of $R_{A^\sharp}$ by the natural map $R \to R_{A^\sharp}$, where $A^\sharp = \{r \in R \mid rA \subset A\}$.*

Proof. Assume that A is not prime and let $A' = AR_{A^\sharp}$. Then $R_{A^\sharp}/A'$ is the ring of quotients of R/A. So, by [7, Théorème 2.8], $R_{A^\sharp}/A'$ is self FP-injective because each non-unit is a zero-divisor. By [8, Corollary II.14] it is coherent if and only if A' is principal. So, we conclude by Corollary 5.2. □

A valuation ring R is called *strongly discrete* if there is no non-zero idempotent prime ideal.

Corollary 5.4. *Let R be a valuation ring. Then R/A is semicoherent for each ideal A if and only if R is strongly discrete.*

Proof. Assume that R is strongly discrete. Each ideal A is of the form $A = aL$, where L is a prime ideal and $a \in R$. Clearly $L = A^\sharp$. Then, $L^2 \neq L$ implies that AR_L is principal over R_L. Since A is the inverse image of AR_L by the natural map $R \to R_L$, R/A is semicoherent by Corollary 5.3.

Conversely, let L be non-zero prime ideal, let $A = aLR_L$, where $0 \neq a \in R_L$ and let A' be the inverse image of A by the natural map $R \to R_L$. Clearly $L = (A')^\sharp$. Since R/A' is semicoherent, A is principal over R_L by Corollary 5.3. It follows that L is principal over R_L. So, $L \neq L^2$. □

Now, we can prove one of the main results of this section.

Theorem 5.5. *Let R be a fractionally IF-ring. Then, R/A is semicoherent for each ideal A and R_P is a strongly discrete valuation ring for each maximal ideal P.*

Proof. By [12, Theorem 1] R is arithmetical because it is fractionally self FP-injective. Let P be a maximal ideal and let A be an ideal of R_P. If B is the kernel of the following composition of natural maps $R \to R_P \to R_P/A$, then $Q(R_P/A) = Q(R/B)$ is an IF-ring. We conclude by Corollaries 5.3 and 5.4. □

It is obvious that each von Neumann regular ring is a fractionally IF-ring. Moreover:

Proposition 5.6. *Let R be an arithmetical ring which is locally strongly discrete. Then R is a fractionally IF-ring in the following cases:*

(i) *R is fractionally semilocal;*

(ii) *R is semilocal;*

(iii) *R is a Prüfer domain of finite character, i.e., each non-zero element is contained in but a finite number of maximal ideals.*

Proof. (i). We may assume that $R = Q(R)$. By [12, Lemma 7], for each maximal ideal P, $R_P = Q(R_P)$. It follows that R_P is self FP-injective. Moreover it is coherent by Corollary 5.4 and Proposition 5.1. Since $\Pi_{P \in \mathrm{Max}\, R} R_P$ is a faithfully flat R-module and an IF-ring, we deduce that R is IF too.

(ii) follows from (i) by [12, Lemma 5].

(iii) follows from (ii) since R/A is semilocal for each non-zero ideal A. □

Question 5.7. What are the locally strongly discrete Prüfer domains which are fractionally IF?

The following example shows that an arithmetical ring which is locally artinian is not necessarily fractionally IF.

Example 5.8. Let K be a field, $V = K[X]/(X^2)$ where X is an indeterminate and let x be the image of X in V. For each $p \in \mathbb{N}$ we put $R_{2p} = V$ and $R_{2p+1} = V/xV \cong K$. Let $S = \prod_{n\in\mathbb{N}} R_n$, $J = \bigoplus_{n\in\mathbb{N}} R_n$ and let R be the unitary V-subalgebra of S generated by J. For each $n \in \mathbb{N}$ we set $\mathbf{e}_n = (\delta_{n,p})_{p\in\mathbb{N}}$ and we denote by $\mathbf{1}$ the identity element of R. Let P be a maximal ideal of R:

- Either $J \subseteq P$; in this case $P = P_\infty = J + xR$ and $R_{P_\infty} = R/J \cong V$;
- or $\exists n \in \mathbb{N}$ such that $\mathbf{e}_n \notin P$; in this case $P = P_n = R(\mathbf{1} - \mathbf{e}_n) + Rx\mathbf{e}_n$, $R_{P_n} = R/R(\mathbf{1} - \mathbf{e}_n) \cong V$ if n is even and $R_{P_n} \cong K$ if n is odd.

Then, for each maximal ideal P, R_P is an artinian valuation ring. Now it is easy to check that $(0 : x\mathbf{1}) = Rx\mathbf{1} + \bigoplus_{n\in\mathbb{N}} R\mathbf{e}_{2n+1}$. So, R is not coherent.

The following proposition is a short answer to another question posed by Facchini and Faith in [12, question Q3, p. 301].

Proposition 5.9. *There exists a non-arithmetical zero-Krull-dimensional ring R which is finitely fractionally self FP-injective.*

Proof. Let V be the artinian valuation ring of Example 5.8, $S = V^{\mathbb{N}}$ and $J = V^{(\mathbb{N})}$. Let $y = (y_n)_{n\in\mathbb{N}}$, $z = (z_n)_{n\in\mathbb{N}} \in S$ such that, $\forall p \in \mathbb{N}$, $y_{2p} = z_{2p+1} = x$ and $y_{2p+1} = z_{2p} = 0$, and let R be the unitary V-subalgebra of S generated by y, z and J. The idempotents $(\mathbf{e}_n)_{n\in\mathbb{N}}$ are defined as in Example 5.8. Let $P \in \mathrm{Max}\, R$:

- Either $J \subseteq P$; in this case $P = P_\infty = J + yR + zR$ and $R_{P_\infty} = R/J \cong K[Y, Z]/(Y, Z)^2$;
- or $\exists n \in \mathbb{N}$ such that $\mathbf{e}_n \notin P$; in this case $P = P_n = R(\mathbf{1} - \mathbf{e}_n) + Rx\mathbf{e}_n$, $R_{P_n} = R/R(\mathbf{1} - \mathbf{e}_n) \cong V$.

Clearly, R_{P_∞} is not a valuation ring. So, R is not arithmetical. First, we show that R is a pure submodule of S. It is sufficient to prove that R_P is a pure submodule of S_P for each maximal ideal P. It is obvious that $R_{P_n} \cong S_{P_n} \cong \mathbf{e}_n S \cong V$. It remains to be shown that R_{P_∞} is a pure submodule of $S_{P_\infty} \cong S/J$. We consider the following equations:

$$\forall i,\ 1 \leq i \leq p, \quad \sum_{1\leq j\leq m} r_{i,j} x_j \equiv s_i \text{ modulo } J,$$

where $r_{i,j},\ s_i \in R$, $\forall i,\ 1 \leq i \leq p$, $\forall j,\ 1 \leq j \leq m$. When these equations have a solution in S, we must prove they have a solution in R too. This can be done by using the basis $\{\mathbf{1}, y, z, \mathbf{e}_n, x\mathbf{e}_n \mid n \in \mathbb{N}\}$ of R over K. Consequently R is pure in S. Now, let A be a finitely generated ideal of R. Then R/A is a pure submodule of S/SA. We have $S/SA \cong \prod_{n\in\mathbb{N}} (R/A)_{P_n}$. For each $n \in \mathbb{N}$, $(R/A)_{P_n}$ is self injective, whence it is an injective (R/A)-module. We deduce that S/SA is injective over R/A. Hence R/A is self FP-injective. □

Finally, for the second question posed by Facchini and Faith in [12, question Q2, p. 301], we shall prove Theorem 5.11. The following lemma is needed.

Lemma 5.10. *Let R be a clean ring such that $R = Q(R)$ and $(0 : a)$ is finitely generated for each $a \in R$. Then, for each maximal ideal P, $R_P = Q(R_P)$.*

Proof. Let P be a maximal ideal of R. By way of contradiction, suppose that R_P contains a regular element which is not a unit. So, $\exists a \in P$ such that $(0_P : a) = 0_P$ (since R is clean, $R_P = R/0_P$ by [9, Proposition III.1]). It follows that $(0 : a) \subseteq 0_P$. Since $(0 : a)$ is finitely generated and 0_P is generated by idempotents, there exists an idempotent $e \in 0_P$ such that $(0 : a) \subseteq Re$. Now, it is easy to check that $a(1-e)+e$ is a regular element contained in P. This contradicts that $R = Q(R)$. □

Theorem 5.11. *The following assertions hold:*

(i) *Let R be an almost clean coherent arithmetical ring. Assume that $R/A(x)$ is either torch or local or a domain $\forall x \in$ pSpec R. Then R is finitely fractionally IF;*

(ii) *each clean coherent arithmetical ring is finitely fractionally IF;*

(iii) *each semihereditary ring is finitely fractionally IF;*

(iv) *let R be a zero-Krull-dimensional ring or a one-Krull-dimensional domain. Then R is finitely fractionally IF if and only if R is coherent and arithmetical.*

Proof. (i). Let A be a finitely generated ideal of an almost clean coherent arithmetical ring R and let N be the nilradical of R. Since each prime ideal contains only one minimal prime, by [9, Lemme IV.2] $D((N : A))$ is the inverse image by λ_R of an open subset U of pSpec R. For each $x \in U$ there exists an idempotent $e_x \in (N : A)$ such that $x \subseteq D(e_x) \subseteq D((N : A))$. Hence $(N : A) = \Sigma_{x\in U}(Re_x + N)$. Since R is coherent $(0 : A)$ is finitely generated. It follows that there exists an idempotent $e \in (N : A)$ such that $(0 : A) \subseteq (Re + N)$. We have $R/A \cong (R(1-e)/A(1-e)) \times (Re/Ae)$. Since $(0 :_{R(1-e)} A(1-e)) \subseteq N(1-e)$, $(R(1-e)/A(1-e))$ is IF by [8, Proposition II.15]. On the other hand, Re is an almost coherent arithmetical ring and Ae is a finitely generated ideal contained in Ne. Let $T = (Re/Ae)$ and let $\phi : R \to T$ be the natural epimorphism. Then pSpec T is totally disconnected because it is homeomorphic to pSpec(Re). Let $x \in$ pSpec T. Then $T/A(x)$ is the quotient of $R/A({}^b\phi(x))$ modulo an ideal contained in the minimal prime of $R/A({}^b\phi(x))$. By Corollary 4.2 $T/A(x)$ is almost clean. We deduce that T is coherent and almost clean by Theorem 4.4. By Corollary 4.8 $T' = Q(T)$ is clean. By Lemma 5.10 $T'_P = Q(T'_P)$ for each maximal ideal P of T'. We deduce that T'_P is IF because it is a valuation ring. So, since T' is locally IF, it is IF too. Hence $Q(R/A)$ is IF.

(ii) and (iii). If R is either clean, coherent and arithmetical or semihereditary, then R satisfies the conditions of (i). Hence R is finitely fractionally IF.

(iv). Observe that $Q(R/A) = R/A$ for each non-zero proper ideal A.

First assume that R is arithmetical and coherent. We deduce that R is finitely fractionally IF from (ii) and (iii).

Conversely, let P be a maximal ideal of R and A a finitely generated ideal of R_P. There exists a finitely generated ideal B of R such that $A = B_P$. So, $R_P/A \cong (R/B)_P$. Since R/B is IF, so is R_P/A by [7, Proposition 1.2]. Hence we may assume that R is local and we must prove that R is a valuation ring. If not, there exist $a, b \in R$ such that $a \notin Rb$ and $b \notin Ra$. The coherence of $R/(ab)$ implies that $Ra \cap Rb$ is finitely generated. It follows that $R/(Ra \cap Rb)$ is IF. We may assume that $Ra \cap Rb = 0$. By [19, Corollary 2.5] $A = (0 : (0 : A))$ for each finitely generated ideal A. We deduce that $0 = Ra \cap Rb = (0 : (0 : a) + (0 : b))$. Then $(0 : a) + (0 : b)$ is a faithful finitely generated proper ideal. By [19, Corollary 2.5] this is not possible. Hence R is a valuation ring. □

If R is fractionally self FP-injective, then, by [8, Theorem III.1] Min R/A is compact for each proper ideal A. The following example shows that this is not true if R is finitely fractionally IF, even if R is a coherent clean arithmetical ring.

Example 5.12. Let D be a valuation domain. Assume that its maximal ideal P' is the only non-zero prime and it is not finitely generated. Let $0 \neq d \in P'$. We put $V = D/dD$, $P = P'/dD$ and $R = V^{\mathbb{N}}$. It is easy to check that R is clean, Bézout and coherent. So, by (ii) of Theorem 5.11 R is finitely fractionally IF. Since P is not finitely generated, $\forall n \in \mathbb{N}, \exists b_n \in P$ such that $b_n^n \neq 0$. We set $b = (b_n)_{n \in \mathbb{N}}$. Let N be the nilradical of R. If there exists $c = (c_n)_{n \in \mathbb{N}} \in R$ such that $(b - bcb) \in N$, then $\exists m \in \mathbb{N}$ such that $b^m(1 - cb)^m = 0$. If $n \in \mathbb{N},\ n \geq m$ we get that $b_n^n(1 - c_n b_n)^n = 0$. Clearly there is a contradiction. So, R/N is not von Neumann regular. Now, let $c \in R$ such that $(N : c) = N$. We shall prove that c is a unit. By way of contradiction, suppose $\exists k \in \mathbb{N}$ such that $c_k \in P$. We put $e_k = (\delta_{k,n})_{n \in \mathbb{N}}$. Then $ce_k = c_k e_k \in N$. It follows that $e_k \in N$, which is absurd. So, $\forall k \in \mathbb{N},\ c_k \notin P$. Therefore c is a unit and R/N is equal to its quotient ring. By [11, Theorem 5] a reduced arithmetic ring S is semihereditary if and only if $Q(S)$ is von Neumann regular. By [26, Proposition 10] a reduced arithmetic ring S is semihereditary if and only if Min S is compact. Consequently Min R/N is not compact . Hence Min R is not compact too. (If P' is finitely generated by p and $R = \prod_{n \in \mathbb{N}} D/p^{n+1}D$, then R is clean, arithmetical, coherent and finitely fractionally IF, but Min R is not compact. We do the same proof by taking $b_n = p + p^{n+1}D,\ \forall n \in \mathbb{N}$.)

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Amalgamated algebras along an ideal

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Abstract. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. In this paper, we initiate a systematic study of a new ring construction called the "amalgamation of A with B along J with respect to f". This construction finds its roots in a paper by J. L. Dorroh appeared in 1932 and provides a general frame for studying the amalgamated duplication of a ring along an ideal, introduced and studied by D'Anna and Fontana in 2007, and other classical constructions such as the $A + XB[X]$ and $A + XB[\![X]\!]$ constructions, the CPI-extensions of Boisen and Sheldon, the $D + M$ constructions and the Nagata's idealization.

Keywords. Nagata's idealization, pullback, $D + M$ construction, amalgamated duplication.

AMS classification. 13B99, 13E05, 13F20, 14A05.

1 Introduction

Let A and B be commutative rings with unity, let J ba an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can define the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A,\ j \in J\}$$

called *the amalgamation of A with B along J with respect to f*. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied in [6] and [7]). Moreover, other classical constructions (such as the $A + XB[X]$ construction, the $D + M$ construction and the Nagata's idealization) can be studied as particular cases of the amalgamation.

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by D. D. Anderson in [1] and motivated by a classical construction due to Dorroh [8], concerning the embedding of a ring without identity in a ring with identity.

The level of generality that we have chosen is due to the fact that the amalgamation can be studied in the frame of pullback constructions. This point of view allows us to provide easily an ample description of the properties of $A \bowtie^f J$, in connection with the properties of A, J and f.

In this paper, we begin a study of the basic properties of $A \bowtie^f J$. In particular, in Section 2, we present all the constructions cited above as particular cases of the amalgamation. Moreover, we show that the CPI extensions (in the sense of Boisen and Sheldon [3]) are related to amalgamations of a special type and we compare Nagata's idealization with the amalgamation. In Section 3, we consider the iteration of the amalgamation process, giving some geometrical applications of it.

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In the last two sections, we show that the amalgamation can be realized as a pullback and we characterize those pullbacks that arise from an amalgamation (Proposition 4.7). Finally we apply these results to study the basic algebraic properties of the amalgamation, with particular attention to the finiteness conditions.

2 The genesis

Let A be a commutative ring with identity and let $\mathcal{R}$ be a ring without identity which is an A-module. Following the construction described by D. D. Anderson in [1], we can define a multiplicative structure in the A-module $A \oplus \mathcal{R}$, by setting $(a,x)(a',x') := (aa', ax' + a'x + xx')$, for all $a, a' \in A$ and $x, x' \in \mathcal{R}$. We denote by $A\dot{\oplus}\mathcal{R}$ the direct sum $A \oplus \mathcal{R}$ endowed also with the multiplication defined above.

The following properties are easy to check.

Lemma 2.1 ([1, Theorem 2.1]). *With the notation introduced above, we have:*

(1) *$A\dot{\oplus}\mathcal{R}$ is a ring with identity $(1,0)$, which has an A-algebra structure induced by the canonical ring embedding $\iota_A : A \hookrightarrow A\dot{\oplus}\mathcal{R}$, defined by $a \mapsto (a,0)$ for all $a \in A$.*

(2) *If we identify $\mathcal{R}$ with its canonical image $(0) \times \mathcal{R}$ under the canonical embedding $\iota_{\mathcal{R}} : \mathcal{R} \hookrightarrow A\dot{\oplus}\mathcal{R}$, defined by $x \mapsto (0,x)$, for all $x \in \mathcal{R}$, then $\mathcal{R}$ becomes an ideal in $A\dot{\oplus}\mathcal{R}$.*

(3) *If we identify A with $A \times (0)$ (respectively, $\mathcal{R}$ with $(0) \times \mathcal{R}$) inside $A\dot{\oplus}\mathcal{R}$, then the ring $A\dot{\oplus}\mathcal{R}$ is an A-module generated by $(1,0)$ and $\mathcal{R}$, i.e., $A(1,0) + \mathcal{R} = A\dot{\oplus}\mathcal{R}$. Moreover, if $p_A : A\dot{\oplus}\mathcal{R} \twoheadrightarrow A$ is the canonical projection (defined by $(a,x) \mapsto a$ for all $a \in A$ and $x \in \mathcal{R}$), then*

$$0 \to \mathcal{R} \xrightarrow{\iota_{\mathcal{R}}} A\dot{\oplus}\mathcal{R} \xrightarrow{p_A} A \to 0$$

is a splitting exact sequence of A-modules. □

Remark 2.2. (1) The previous construction takes its roots in the classical construction, introduced by Dorroh [8] in 1932, for embedding a ring (with or without identity, possibly without regular elements) in a ring with identity (see also Jacobson [14, page 155]). For completeness, we recall Dorroh's construction starting with a case which is not the motivating one, but that leads naturally to the relevant one (Case 2).

Case 1. Let R be a commutative ring (with or without identity) and let $\mathrm{Tot}(R)$ be its total ring of fractions, i.e., $\mathrm{Tot}(R) := N^{-1}R$, where N is the set of regular elements of R. If we assume that R has a regular element r, then it is easy to see that $R \subseteq \mathrm{Tot}(R)$, and $\mathrm{Tot}(R)$ has identity $1 := \frac{r}{r}$, even if R does not. In this situation we can consider $R[1] := \{x + m \cdot 1 \mid x \in R, m \in \mathbb{Z}\}$. Obviously, if R has an identity, then $R = R[1]$; otherwise, we have that $R[1]$ is a commutative ring with identity, which contains properly R and it is the smallest subring of $\mathrm{Tot}(R)$ containing R and 1. It is easy to see that:

(a) R and $R[1]$ have the same characteristic,

(b) R is an ideal of $R[1]$, and

(c) if $R \subsetneq R[1]$, then the quotient-ring $R[1]/R$ is canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$, where n (≥ 0) is the characteristic of $R[1]$ (or, equivalently, of R).

Case 2. Let R be a commutative ring (with or without identity) and, possibly, without regular elements. In this situation, we possibly have $R = \mathrm{Tot}(R)$, so we cannot perform the previous construction. Following Dorroh's ideas, we can consider in any case R as a $\mathbb{Z}$-module and, with the notation introduced at the beginning of this section, we can construct the ring $\mathbb{Z}\dot{\oplus}R$, that we denote by $\mathtt{Dh}(R)$ in Dorroh's honour. Note that $\mathtt{Dh}(R)$ is a commutative ring with identity $1_{\mathtt{Dh}(R)} := (1,0)$. If we identify, as usual, R with its canonical image in $\mathtt{Dh}(R)$, then R is an ideal of $\mathtt{Dh}(R)$ and $\mathtt{Dh}(R)$ has a kind of minimal property over R, since $\mathtt{Dh}(R) = \mathbb{Z}(1,0) + R$. Moreover, the quotient-ring $\mathtt{Dh}(R)/R$ is naturally isomorphic to $\mathbb{Z}$.

On the bad side, note that if R has an identity 1_R, then the canonical embedding of R into $\mathtt{Dh}(R)$ (defined by $x \mapsto (0,x)$ for all $x \in R$) does not preserve the identity, since $(0, 1_R) \neq 1_{\mathtt{Dh}(R)}$. Moreover, in any case (whenever R is a ring with or without identity), the canonical embedding $R \hookrightarrow \mathtt{Dh}(R)$ may not preserve the characteristic.

In order to overcome this difficult, in 1935, Dorroh [9] gave a variation of the previous construction. More precisely, if R has positive characteristic n, then R can be considered as a $\mathbb{Z}/n\mathbb{Z}$-module, so $\mathtt{Dh}_n(R) := (\mathbb{Z}/n\mathbb{Z})\dot{\oplus}R$ is a ring with identity, having characteristic n. Moreover, as above, $\mathtt{Dh}_n(R) = (\mathbb{Z}/n\mathbb{Z})\,(1,0) + R$ and $\mathtt{Dh}_n(R)/R$ is canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

(2) Note that a general version of the Dorroh's construction (previous Case 2) was considered in 1974 by Shores [18, Definition 6.3] for constructing examples of local commutative rings with arbitrarily large Loewy length. We are indebted to L. Salce for pointing out to us that the amalgamated duplication of a ring along an ideal [6] can also be viewed as a special case of Shores construction (cf. also [17]). Moreover, before Shores, Corner in 1969 [4], for studying endomorphisms rings of Abelian groups, considered a similar construction called "split extension of a ring by an ideal".

A natural situation in which we can apply the previous general construction (Lemma 2.1) is the following. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Note that f induces on J a natural structure of A-module by setting $a \cdot j := f(a)j$, for all $a \in A$ and $j \in J$. Then, we can consider $A\dot{\oplus}J$.

The following properties, except (2) that is easy to verify, follow from Lemma 2.1.

Lemma 2.3. *With the notation introduced above, we have:*

(1) $A\dot{\oplus}J$ *is a ring.*

(2) *The map* $f^{\bowtie} : A\dot{\oplus}J \to A \times B$, *defined by* $(a, j) \mapsto (a, f(a) + j)$ *for all* $a \in A$ *and* $j \in J$, *is an injective ring homomorphism.*

(3) *The map* $\iota_A : A \to A\dot{\oplus}J$ *(respectively,* $\iota_J : J \to A\dot{\oplus}J$*), defined by* $a \mapsto (a, 0)$ *for all* $a \in A$ *(respectively, by* $j \mapsto (0, j)$ *for all* $j \in J$*), is an injective ring homomorphism (respectively, an injective* A*-module homomorphism). If we identify* A *with* $\iota_A(A)$ *(respectively,* J *with* $\iota_J(J)$*), then the ring* $A\dot{\oplus}J$ *coincides with* $A + J$*.*

(4) *Let* $p_A : A\dot{\oplus}J \to A$ *be the canonical projection (defined by* $(a, j) \mapsto a$ *for all* $a \in A$ *and* $j \in J$*), then the following is a split exact sequence of* A*-modules:*

$$0 \to J \xrightarrow{\iota_J} A\dot{\oplus}J \xrightarrow{p_A} A \to 0\,. \qquad \square$$

We set

$$A \bowtie^f J := f^{\bowtie}(A\dot{\oplus}J), \quad \Gamma(f) := \{(a, f(a)) \mid a \in A\}.$$

Clearly, $\Gamma(f) \subseteq A \bowtie^f J$ and they are subrings of $A \times B$. The motivation for replacing $A\dot{\oplus}J$ with its canonical image $A \bowtie^f J$ inside $A \times B$ (under $f^{\bowtie}$) is related to the fact that the multiplicative structure defined in $A\dot{\oplus}J$, which looks somewhat "artificial", becomes the restriction to $A \bowtie^f J$ of the natural multiplication defined componentwise in the direct product $A \times B$. The ring $A \bowtie^f J$ will be called *the amalgamation of* A *with* B *along* J*, with respect to* $f : A \to B$*.*

Example 2.4. *The amalgamated duplication of a ring.*
A particular case of the construction introduced above is the amalgamated duplication of a ring [6]. Let A be a commutative ring with unity, and let E be an A-submodule of the total ring of fractions $\mathrm{Tot}(A)$ of A such that $E \cdot E \subseteq E$. In this case, E is an ideal in the subring $B := (E : E)$ $(:= \{z \in \mathrm{Tot}(A) \mid zE \subseteq E\})$ of $\mathrm{Tot}(A)$. If $\iota : A \to B$ is the natural embedding, then $A \bowtie^{\iota} E$ coincides with $A \bowtie E$, the amalgamated duplication of A along E, as defined in [6]. A particular and relevant case is when $E := I$ is an ideal in A. In this case, we can take $B := A$, we can consider the identity map $\mathrm{id} := \mathrm{id}_A : A \to A$ and we have that $A \bowtie I$, the amalgamated duplication of A along the ideal I, coincides with $A \bowtie^{\mathrm{id}} I$, that we will call also *the simple amalgamation of* A *along* I (instead of the amalgamation of A along I, with respect to id_A).

Example 2.5. *The constructions* $A + \boldsymbol{X}B[\boldsymbol{X}]$ *and* $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$*.*
Let $A \subset B$ be an extension of commutative rings and $\boldsymbol{X} := \{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over B. In the polynomial ring $B[\mathbf{X}]$, we can consider the following subring

$$A + \boldsymbol{X}B[\boldsymbol{X}] := \{h \in B[\boldsymbol{X}] \mid h(\mathbf{0}) \in A\}\,,$$

where $\mathbf{0}$ is the n-tuple whose components are 0. This is a particular case of the general construction introduced above. In fact, if $\sigma' : A \hookrightarrow B[\boldsymbol{X}]$ is the natural embedding and $J' := \boldsymbol{X}B[\boldsymbol{X}]$, then it is easy to check that $A \bowtie^{\sigma'} J'$ is isomorphic to $A + \boldsymbol{X}B[\boldsymbol{X}]$ (see also the following Proposition 5.1(3)).

Similarly, the subring $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!] := \{h \in B[\![\boldsymbol{X}]\!] \mid h(\mathbf{0}) \in A\}$ of the ring of power series $B[\![\boldsymbol{X}]\!]$ is isomorphic to $A \bowtie^{\sigma''} J''$, where $\sigma'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]$ is the natural embedding and $J'' := \boldsymbol{X}B[\![\boldsymbol{X}]\!]$.

Example 2.6. *The $D + M$ construction.*
Let M be a maximal ideal of a ring (usually, an integral domain) T and let D be a subring of T such that $M \cap D = (0)$. The ring $D + M := \{x + m \mid x \in D,\ m \in M\}$ is canonically isomorphic to $D \bowtie^{\iota} M$, where $\iota : D \hookrightarrow T$ is the natural embedding.

More generally, let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a subset of the set of the maximal ideals of T, such that $M_\lambda \cap D = (0)$ for some $\lambda \in \Lambda$, and set $J := \bigcap_{\lambda \in \Lambda} M_\lambda$. The ring $D + J := \{x + j \mid x \in D,\ j \in J\}$ is canonically isomorphic to $D \bowtie^{\iota} J$. In particular, if $D := K$ is a field contained in T and $J := \mathrm{Jac}(T)$ is the Jacobson ideal of (the K-algebra) T, then $K + \mathrm{Jac}(T)$ is canonically isomorphic to $K \bowtie^{\iota} \mathrm{Jac}(T)$, where $\iota : K \hookrightarrow T$ is the natural embedding.

Example 2.7. *The CPI-extensions (in the sense of Boisen–Sheldon* [3]*).*
Let A be a ring and P be a prime ideal of A. Let $\boldsymbol{k}(P)$ be the residue field of the localization A_P and denote by ψ_P (or simply, by ψ) the canonical surjective ring homomorphism $A_P \longrightarrow \boldsymbol{k}(P)$. It is well known that $\boldsymbol{k}(P)$ is canonically isomorphic to the quotient field of A/P, so we can identify A/P with its canonical image into $\boldsymbol{k}(P)$. Then the subring $\boldsymbol{C}(A, P) := \psi^{-1}(A/P)$ of A_P is called the *CPI-extension of A with respect to P*. It is immediately seen that, if we denote by λ_P (or, simply, by λ) the localization homomorphism $A \longrightarrow A_P$, then $\boldsymbol{C}(A, P)$ coincides with the ring $\lambda(A) + PA_P$. On the other hand, if $J := PA_P$, we can consider $A \bowtie^{\lambda} J$ and we have the canonical projection $A \bowtie^{\lambda} J \to \lambda(A) + PA_P$, defined by $(a, \lambda(a) + j) \mapsto \lambda(a) + j$, where $a \in A$ and $j \in PA_P$. It follows that $\boldsymbol{C}(A, P)$ is canonically isomorphic to $(A \bowtie^{\lambda} PA_P)/(P \times \{0\})$ (Proposition 5.1(3)).

More generally, let I be an ideal of A and let S_I be the set of the elements $s \in A$ such that $s + I$ is a regular element of A/I. Obviously, S_I is a multiplicative subset of A and if $\overline{S_I}$ is its canonical projection onto A/I, then $\mathrm{Tot}(A/I) = (\overline{S_I})^{-1}(A/I)$. Let $\varphi_I : S^{-1}A \longrightarrow \mathrm{Tot}(A/I)$ be the canonical surjective ring homomorphism defined by $\varphi_I(as^{-1}) := (a + I)(s + I)^{-1}$, for all $a \in A$ and $s \in S$. Then, the subring $\boldsymbol{C}(A, I) := \varphi_I^{-1}(A/I)$ of $S_I^{-1}A$ is called the *CPI-extension of A with respect to I*. If $\lambda_I : A \longrightarrow S_I^{-1}A$ is the localization homomorphism, then it is easy to see that $\boldsymbol{C}(A, I)$ coincides with the ring $\lambda_I(A) + S_I^{-1}I$. It will follow by Proposition 5.1(3) that, if we consider the ideal $J := S_I^{-1}I$ of $S_I^{-1}A$, then $\boldsymbol{C}(A, I)$ is canonically isomorphic to $(A \bowtie^{\lambda_I} J)/(\lambda_I^{-1}(J) \times \{0\})$.

Remark 2.8. *Nagata's idealization.*
Let A be a commutative ring and $\mathcal{M}$ a A-module. We recall that, in 1955, Nagata introduced the ring extension of A called *the idealization of $\mathcal{M}$ in A,* denoted here by $A \ltimes \mathcal{M}$, as the A-module $A \oplus \mathcal{M}$ endowed with a multiplicative structure defined by

$$(a, x)(a', x') := (aa', ax' + a'x)\,, \quad \text{for all } a, a' \in A \text{ and } x, x' \in \mathcal{M}$$

(cf. [15], Nagata's book [16, page 2], and Huckaba's book [13, Chapter VI, Section 25]). The idealization $A \ltimes \mathcal{M}$ is a ring, such that the canonical embedding $\iota_A : A \hookrightarrow A \ltimes \mathcal{M}$ (defined by $a \mapsto (a, 0)$, for all $a \in A$) induces a subring $A^{\ltimes}$ $(:= \iota_A(A))$ of $A \ltimes \mathcal{M}$ isomorphic to A and the embedding $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow A \ltimes \mathcal{M}$ (defined by $x \mapsto (0, x)$, for all

$x \in \mathcal{M}$) determines an ideal $\mathcal{M}^{\ltimes}$ ($:= \iota_{\mathcal{M}}(\mathcal{M})$) in $A \ltimes \mathcal{M}$ (isomorphic, as an A-module, to $\mathcal{M}$), which is nilpotent of index 2 (i.e. $\mathcal{M}^{\ltimes} \cdot \mathcal{M}^{\ltimes} = 0$).

For the sake of simplicity, we will identify $\mathcal{M}$ with $\mathcal{M}^{\ltimes}$ and A with $A^{\ltimes}$. If $p_A : A \ltimes \mathcal{M} \to A$ is the canonical projection (defined by $(a, x) \mapsto a$, for all $a \in A$ and $x \in \mathcal{M}$), then

$$0 \to \mathcal{M} \xrightarrow{\iota_{\mathcal{M}}} A \ltimes \mathcal{M} \xrightarrow{p_A} A \to 0$$

is a spitting exact sequence of A-modules. (Note that the idealization $A \ltimes \mathcal{M}$ is also called in [11] *the trivial extension of A by $\mathcal{M}$*.)

We can apply the construction of Lemma 2.1 by taking $\mathcal{R} := \mathcal{M}$, where $\mathcal{M}$ is an A-module, and considering $\mathcal{M}$ as a (commutative) ring without identity, endowed with a trivial multiplication (defined by $x \cdot y := 0$ for all $x, y \in \mathcal{M}$). In this way, we have that the Nagata's idealization is a particular case of the construction considered in Lemma 2.1. Therefore, the Nagata's idealization can be interpreted as a particular case of the general amalgamation construction. Let $B := A \ltimes \mathcal{M}$ and ι $(= \iota_A) : A \hookrightarrow B$ be the canonical embedding. After identifying $\mathcal{M}$ with $\mathcal{M}^{\ltimes}$, $\mathcal{M}$ becomes an ideal of B. It is now straightforward that $A \ltimes \mathcal{M}$ coincides with the amalgamation $A \bowtie^{\iota} \mathcal{M}$.

Although this, the Nagata's idealization and the constructions of the type $A \bowtie^f J$ can be very different from an algebraic point of view. In fact, for example, if $\mathcal{M}$ is a nonzero A-module, the ring $A \ltimes \mathcal{M}$ is always not reduced (the element $(0, x)$ is nilpotent, for all $x \in \mathcal{M}$), but the amalgamation $A \bowtie^f J$ can be an integral domain (see Example 2.6 and Proposition 5.2).

3 Iteration of the construction $A \bowtie^f J$

In the following all rings will always be commutative with identity, and every ring homomorphism will send 1 to 1.

If A is a ring and I is an ideal of A, we can consider the amalgamated duplication of the ring A along its ideal I (= the simple amalgamation of A along I), i.e., $A \bowtie I := \{(a, a + i) \mid a \in A,\ i \in I\}$ (Example 2.4). For the sake of simplicity, set $A' := A \bowtie I$. It is immediately seen that $I' := \{0\} \times I$ is an ideal of A', and thus we can consider again the simple amalgamation of A' along I', i.e., the ring $A'' := A' \bowtie I'$ $(= (A \bowtie I) \bowtie (\{0\} \times I))$. It is easy to check that the ring A'' may not be considered as a simple amalgamation of A along one of its ideals. However, we can show that A'' can be interpreted as an amalgamation of algebras, giving in this way an answer to a problem posed by B. Olberding in 2006 at Padova's Conference in honour of L. Salce.

We start by showing that it is possible to iterate the amalgamation of algebras and the result is still an amalgamation of algebras.

More precisely, let $f : A \to B$ be a ring homomorphism and J an ideal of B. Since $J'^f := \{0\} \times J$ is an ideal of the ring $A'^f := A \bowtie^f J$, we can consider the simple amalgamation of A'^f along J'^f, i.e., $A''^f := A'^f \bowtie J'^f$ (which coincides with $A'^f \bowtie^{\mathrm{id}} J'^f$, where $\mathrm{id} := \mathrm{id}_{A'^f}$ is the identity mapping of A'^f). On the other hand, we can consider the mapping $f^{(2)} : A \to B^{(2)} := B \times B$, defined by $a \mapsto (f(a), f(a))$ for all $a \in A$. Since $J^{(2)} := J \times J$ is an ideal of the ring $B^{(2)}$, we can consider the amalgamation $A \bowtie^{f^{(2)}} J^{(2)}$. Then, the mapping $A''^f \to A \bowtie^{f^{(2)}} J^{(2)}$, defined by $((a, f(a)+j_1), (a, f(a)+j_1)+(0, j_2)) \mapsto (a, (f(a), f(a)) + (j_1, j_1+j_2))$ for all $a \in A$ and $j_1, j_2 \in J$, is a ring isomorphism, having as inverse map the map $A \bowtie^{f^{(2)}} J^{(2)} \to A''^f$, defined by $(a, (f(a)+j_1, f(a)+j_2)) \mapsto ((a, f(a)+j_1), (a, f(a)+j_1)+(0, j_2-j_1))$ for all $a \in A$ and $j_1, j_2 \in J$. We will denote by $A \bowtie^{2,f} J$ or, simply, $A^{(2,f)}$ (if no confusion can arise) the ring $A \bowtie^{f^{(2)}} J^{(2)}$, that we will call the *2-amalgamation of the A-algebra B along J (with respect to f)*.

For $n \geq 2$, we define the *n-amalgamation of the A-algebra B along J (with respect to f)* by setting

$$A \bowtie^{n,f} J := A^{(n,f)} := A \bowtie^{f^{(n)}} J^{(n)},$$

where $f^{(n)} : A \to B^{(n)} := B \times B \times \cdots \times B$ (n-times) is the diagonal homomorphism associated to f and $J^{(n)} := J \times J \times \cdots \times J$ (n-times). Therefore,

$$A \bowtie^{n,f} J \\ = \{(a, (f(a), f(a), \ldots, f(a)) + (j_1, j_2, \ldots, j_n)) \mid a \in A,\ j_1, j_2, \ldots, j_n \in J\}.$$

Proposition 3.1. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then $A \bowtie^{n,f} J$ is canonically isomorphic to the simple amalgamation $A^{(n-1,f)} \bowtie J^{(n-1,f)}$ $(= A^{(n-1,f)} \bowtie^{\mathrm{id}} J^{(n-1,f)})$, where $J^{(n-1,f)}$ is the canonical isomorphic image of J inside $A^{(n-1,f)}$ and* $\mathrm{id} := \mathrm{id}_{A^{(n-1,f)}}$ *is the identity mapping of $A^{(n-1,f)}$.*

Proof. The proof can be given by induction on $n \geq 2$. For the sake of simplicity, we only consider here the inductive step from $n = 2$ to $n+1$ $(= 3)$. It is straightforward that the mapping $A \bowtie^{3,f} J \to A''^f \bowtie J''^f$, defined by $(a, (f(a), f(a), f(a)) + (j_1, j_2, j_3)) \mapsto (a'', a'' + j'')$, where $a'' := ((a, f(a)+j_1), (a, f(a)+j_1)+(0, j_2 - j_1)) \in A''^f$ and $j'' := ((0,0), (0, j_3 - j_2)) \in J''^f$, for all $a \in A$ and $j_1, j_2, j_3 \in J$ establishes a canonical ring isomorphism. □

In particular, let A be a ring and I an ideal of A, the simple amalgamation of $A' := A \bowtie I$ along $I' := \{0\} \times I$, that is $A'' := A' \bowtie I'$, is canonically isomorphic to the 2-amalgamation $A \bowtie^{2,\mathrm{id}} I = \{(a, (a,a) + (i_1, i_2)) \mid a \in A,\ i_1, i_2 \in I\}$.

Example 3.2. We can apply the previous (iterated) construction to curve singularities. Let A be the ring of an algebroid curve with h branches (i.e., A is a one-dimensional reduced ring of the form $K[\![X_1, X_2, \ldots, X_r]\!]/\bigcap_{i=1}^{h} P_i$, where K is an algebraically closed field, $X_1, X_2, \ldots, X_r$ are indeterminates over K and P_i is an height $r-1$ prime ideal of $K[\![X_1, X_2, \ldots, X_r]\!]$, for $1 \leq i \leq r$). If I is a regular and proper ideal of A, then, with an argument similar to that used in the proof of [5, Theorem 14] (where the case of a simple amalgamation of the ring of the given algebroid curve is investigated), it can be shown that $A \bowtie^n I$ is the ring of an algebroid curve with $(n+1)h$ branches; moreover, for each branch of A, there are exactly $n+1$ branches of $A \bowtie^n I$ isomorphic to it.

4 Pullback constructions

Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B. In the remaining part of the paper, we intend to investigate the algebraic properties of the ring $A \bowtie^f J$, in relation with those of A, B, J and f. One important tool we can use for this purpose is the fact that the ring $A \bowtie^f J$ can be represented as a pullback (see next Proposition 4.2). On the other hand, we will provide a characterization of those pullbacks that give rise to amalgamated algebras (see next Proposition 4.7). After proving these facts, we will make some pertinent remarks useful for the subsequent investigation on amalgamated algebras.

Definition 4.1. We recall that, if $\alpha : A \to C,\ \beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β.

The fact that D is a pullback can also be described by saying that the triplet (D, p_A, p_B) is a solution of the universal problem of rendering commutative the diagram built on α and β,

$$\begin{array}{ccc} D & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

where p_A (respectively, p_B) is the restriction to $\alpha \times_C \beta$ of the projection of $A \times B$ onto A (respectively, B).

Proposition 4.2. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If $\pi : B \to B/J$ is the canonical projection and $\breve{f} := \pi \circ f$, then $A \bowtie^f J = \breve{f} \times_{B/J} \pi$.*

Proof. The statement follows easily from the definitions. □

Remark 4.3. Notice that we have many other ways to describe the ring $A \bowtie^f J$ as a pullback. In fact, if $C := A \times B/J$ and $u : A \to C$, $v : A \times B \to C$ are the canonical ring homomorphisms defined by $u(a) := (a, f(a) + J)$, $v((a, b)) := (a, b + J)$, for every $(a, b) \in A \times B$, it is straightforward to show that $A \bowtie^f J$ is canonically isomorphic to $u \times_C v$. On the other hand, if $I := f^{-1}(J)$, $\breve{u} : A/I \to A/I \times B/J$ and $\breve{v} : A \times B \to A/I \times B/J$ are the natural ring homomorphisms induced by u and v, respectively, then $A \bowtie^f J$ is also canonically isomorphic to the pullback of $\breve{u}$ and $\breve{v}$.

The next goal is to show that the rings of the form $A \bowtie^f J$, for some ring homomorphism $f : A \to B$ and some ideal J of B, determine a distinguished subclass of the class of all fiber products.

Proposition 4.4. *Let A, B, C, α, β be as in Definition 4.1, and let $f : A \to B$ be a ring homomorphism. Then the following conditions are equivalent:*

(i) *There exist an ideal J of B such that $A \bowtie^f J$ is the fiber product of α and β.*

(ii) *α is the composition $\beta \circ f$.*

If the previous conditions hold, then $J = \mathrm{Ker}(\beta)$.

Proof. Assume that condition (i) holds, and let a be an element of A. Then $(a, f(a)) \in A \bowtie^f J$ and, by assumption, we have $\alpha(a) = \beta(f(a))$. This prove condition (ii).

Conversely, assume that $\alpha = \beta \circ f$. We want to show that the ring $A \bowtie^f \mathrm{Ker}(\beta)$ is the fiber product of α and β. The inclusion $A \bowtie^f \mathrm{Ker}(\beta) \subseteq \alpha \times_C \beta$ is clear. On the other hand, let $(a, b) \in \alpha \times_C \beta$. By assumption, we have $\beta(b) = \alpha(a) = \beta(f(a))$. This shows that $b - f(a) \in \mathrm{Ker}(\beta)$, and thus $(a, b) = (a, f(a) + k)$, for some $k \in \mathrm{Ker}(\beta)$. Then $A \bowtie^f \mathrm{Ker}(\beta) = \alpha \times_C \beta$ and condition (i) is true.

The last statement of the proposition is straightforward. □

In the previous proposition we assume the existence of the ring homomorphism f. The next step is to give a condition for the existence of f. We start by recalling that a ring homomorphism $r : B \to A$ is called *a ring retraction* if there exists a ring homomorphism $\iota : A \to B$, such that $r \circ \iota = \mathrm{id}_A$. In this situation, ι is necessarily injective, r is necessarily surjective, and A is called a *retract of B*.

Example 4.5. If $r : B \to A$ is a ring retraction and $\iota : A \hookrightarrow B$ is a ring embedding such that $r \circ \iota = \mathrm{id}_A$, then B is naturally isomorphic to $A \bowtie^\iota \mathrm{Ker}(r)$. This is a consequence of the facts, easy to verify, that $B = \iota(A) + \mathrm{Ker}(r)$ and that $\iota^{-1}(\mathrm{Ker}(r)) = \{0\}$ (for more details see Proposition 5.1(3)).

Remark 4.6. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then A is a retract of $A \bowtie^f J$. More precisely, $\pi_A : A \bowtie^f J \to A$, $(a, f(a), j) \mapsto a$, is a retraction, since the map $\iota : A \to A \bowtie^f J$, $a \mapsto (a, f(a))$, is a ring embedding such that $\pi_A \circ \iota = \mathrm{id}_A$.

Proposition 4.7. *Let $A, B, C, \alpha, \beta, p_A, p_B$ be as in Definition 4.1. Then, the following conditions are equivalent:*

(i) *$p_A : \alpha \times_C \beta \to A$ is a ring retraction.*

(ii) *There exist an ideal J of B and a ring homomorphism $f : A \to B$ such that $\alpha \times_C \beta = A \bowtie^f J$.*

Proof. Set $D := \alpha \times_C \beta$. Assume that condition (i) holds and let $\iota : A \hookrightarrow D$ be a ring embedding such that $p_A \circ \iota = \mathrm{id}_A$. If we consider the ring homomorphism $f := p_B \circ \iota : A \to B$, then, by using the definition of a pullback, we have $\beta \circ f = \beta \circ p_B \circ \iota = \alpha \circ p_A \circ \iota = \alpha \circ \mathrm{id}_A = \alpha$. Then, condition (ii) follows by applying Proposition 4.4. Conversely, let $f : A \to B$ be a ring homomorphism such that $D = A \bowtie^f J$, for some ideal J of B. By Remark 4.6, the projection of $A \bowtie^f J$ onto A is a ring retraction. □

Remark 4.8. Let $f, g : A \to B$ be two ring homomorphisms and J be an ideal of B. It can happen that $A \bowtie^f J = A \bowtie^g J$, with $f \neq g$. In fact, it is easily seen that $A \bowtie^f J = A \bowtie^g J$ if and only if $f(a) - g(a) \in J$, for every $a \in A$.

For example, let $f, g : A[X] \to A[X]$ be the ring homomorphisms defined by $f(X) := X^2$, $f(a) := a$, $g(X) := X^3$, $g(a) := a$, for all $a \in A$, and set $J := XA[X]$. Then $f \neq g$, but $A[X] \bowtie^f J = A[X] \bowtie^g J$, since $f(p) - g(p) \in J$, for all $p \in A[X]$.

The next goal is to give some sufficient conditions for a pullback to be reduced. Given a ring A, we denote by $\mathrm{Nilp}(A)$ the ideal of all nilpotent elements of A.

Proposition 4.9. *With the notation of Definition 4.1, we have:*

(1) *If D $(= \alpha \times_C \beta)$ is reduced, then*

$$\mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha) = \{0\} \quad \textit{and} \quad \mathrm{Nilp}(B) \cap \mathrm{Ker}(\beta) = \{0\}.$$

(2) *If at least one of the following conditions holds*

  (a) *A is reduced and* $\mathrm{Nilp}(B) \cap \mathrm{Ker}(\beta) = \{0\}$,

  (b) *B is reduced and* $\mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha) = \{0\}$,

  then D is reduced.

Proof. (1) Assume D reduced. By symmetry, it suffices to show that $\mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha) = \{0\}$. If $a \in \mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha)$, then $(a, 0)$ is a nilpotent element of D, and thus $a = 0$.

(2) By the symmetry of conditions (a) and (b), it is enough to show that, if condition (a) holds, then D is reduced. Let (a, b) be a nilpotent element of D. Then $a = 0$, since $a \in \mathrm{Nilp}(A)$ and A is reduced. Thus we have $(a, b) = (0, b) \in \mathrm{Nilp}(D)$, hence $b \in \mathrm{Nilp}(B) \cap \mathrm{Ker}(\beta) = \{0\}$. □

We study next the Noetherianity of a ring arising from a pullback construction as in Definition 4.1.

Proposition 4.10. *With the notation of Definition 4.1, the following conditions are equivalent:*

(i) *D $(= \alpha \times_C \beta)$ is a Noetherian ring.*

(ii) $\mathrm{Ker}(\beta)$ *is a Noetherian D-module (with the D-module structure naturally induced by p_B) and $p_A(D)$ is a Noetherian ring.*

Proof. It is easy to see that $\mathrm{Ker}(p_A) = \{0\} \times \mathrm{Ker}(\beta)$. Thus, we have the following short exact sequence of D-modules

$$0 \longrightarrow \mathrm{Ker}(\beta) \xrightarrow{i} D \xrightarrow{p_A} p_A(D) \longrightarrow 0,$$

where i is the natural D-module embedding (defined by $x \mapsto (0, x)$ for all $x \in \mathrm{Ker}(\beta)$). By [2, Proposition (6.3)], D is a Noetherian ring if and only if $\mathrm{Ker}(\beta)$ and $p_A(D)$ are Noetherian as D-modules. The statement now follows immediately, since the D-submodules of $p_A(D)$ are exactly the ideals of the ring $p_A(D)$. □

Remark 4.11. Note that, in Proposition 4.10, we did not require β to be surjective. However, if β is surjective, then p_A is also surjective and so $p_A(D) = A$. Therefore, in this case, D is a Noetherian ring if and only if A is a Noetherian ring and $\mathrm{Ker}(\beta)$ is a Noetherian D-module.

5 The ring $A \bowtie^f J$: some basic algebraic properties

We start with some straightforward consequences of the definition of amalgamated algebra $A \bowtie^f J$.

Proposition 5.1. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B and let $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\}$ be as in Section 2.*

(1) *Let $\iota := \iota_{A,f,J} : A \to A \bowtie^f J$ be the natural the ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in A$. Then ι is an embedding, making $A \bowtie^f J$ a ring extension of A (with $\iota(A) = \Gamma(f)$ ($:= \{(a, f(a)) \mid a \in A\}$ subring of $A \bowtie^f J$).*

(2) *Let I be an ideal of A and set $I \bowtie^f J := \{(i, f(i) + j) \mid i \in I, j \in J\}$. Then $I \bowtie^f J$ is an ideal of $A \bowtie^f J$, the composition of canonical homomorphisms $A \overset{\iota}{\hookrightarrow} A \bowtie^f J \twoheadrightarrow A \bowtie^f J / I \bowtie^f J$ is a surjective ring homomorphism and its kernel coincides with I.*

Hence, we have the following canonical isomorphism:

$$\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}.$$

(3) *Let $p_A : A \bowtie^f J \to A$ and $p_B : A \bowtie^f J \to B$ be the natural projections of $A \bowtie^f J \subseteq A \times B$ into A and B, respectively. Then p_A is surjective and* $\mathrm{Ker}(p_A) = \{0\} \times J$.

Moreover, $p_B(A \bowtie^f J) = f(A) + J$ and $\mathrm{Ker}(p_B) = f^{-1}(J) \times \{0\}$. *Hence, the following canonical isomorphisms hold:*

$$\frac{A \bowtie^f J}{(\{0\} \times J)} \cong A \quad \text{and} \quad \frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J.$$

(4) *Let $\gamma : A \bowtie^f J \to (f(A) + J)/J$ be the natural ring homomorphism, defined by $(a, f(a) + j) \mapsto f(a) + J$. Then γ is surjective and* $\mathrm{Ker}(\gamma) = f^{-1}(J) \times J$. *Thus, there exists a natural isomorphism*

$$\frac{A \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{f(A) + J}{J}.$$

In particular, when f is surjective we have

$$\frac{A \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{B}{J}. \qquad \square$$

The ring $B_\diamond := f(A) + J$ (which is a subring of B) has an important role in the construction $A \bowtie^f J$. For instance, if $f^{-1}(J) = \{0\}$, we have $A \bowtie^f J \cong B_\diamond$ (Proposition 5.1(3)). Moreover, in general, J is an ideal also in $B_\diamond$ and, if we denote by $f_\diamond : A \to B_\diamond$ the ring homomorphism induced from f, then $A \bowtie^{f_\diamond} J = A \bowtie^f J$. The next result shows one more aspect of the essential role of the ring $B_\diamond$ for the construction $A \bowtie^f J$.

Proposition 5.2. *With the notation of Proposition 5.1, assume $J \neq \{0\}$. Then, the following conditions are equivalent:*

(i) *$A \bowtie^f J$ is an integral domain.*

(ii) *$f(A) + J$ is an integral domain and $f^{-1}(J) = \{0\}$.*

In particular, if B is an integral domain and $f^{-1}(J) = \{0\}$, then $A \bowtie^f J$ is an integral domain.

Proof. (ii) $\Rightarrow$ (i) is obvious, since $f^{-1}(J) = \{0\}$ implies that $A \bowtie^f J \cong f(A) + J$ (Proposition 5.1(3)).

Assume that condition (i) holds. If there exists an element $a \in A \setminus \{0\}$ such that $f(a) \in J$, then $(a, 0) \in (A \bowtie^f J) \setminus \{(0, 0)\}$. Hence, if j is a nonzero element of J, we have $(a, 0)(0, j) = (0, 0)$, a contradiction. Thus $f^{-1}(J) = \{0\}$. In this case, as observed above, $A \bowtie^f J \cong f(A) + J$ (Proposition 5.1(3)), so $f(A) + J$ is an integral domain. □

Remark 5.3. (1) Note that, if $A \bowtie^f J$ is an integral domain, then A is also an integral domain, by Proposition 5.1(1).

(2) Let $B = A$, $f = \mathrm{id}_A$ and $J = I$ be an ideal of A. In this situation, $A \bowtie^{\mathrm{id}_A} I$ (the simple amalgamation of A along I) coincides with the amalgamated duplication of A along I (Example 2.4) and it is never an integral domain, unless $I = \{0\}$ and A is an integral domain.

Now, we characterize when the amalgamated algebra $A \bowtie^f J$ is a reduced ring.

Proposition 5.4. *We preserve the notation of Proposition 5.1. The following conditions are equivalent:*

(i) *$A \bowtie^f J$ is a reduced ring.*

(ii) *A is a reduced ring and* $\mathrm{Nilp}(B) \cap J = \{0\}$.

In particular, if A and B are reduced, then $A \bowtie^f J$ is reduced; conversely, if J is a radical ideal of B and $A \bowtie^f J$ is reduced, then B (and A) is reduced.

Proof. From Proposition 4.9(2a) we deduce easily that (ii) $\Rightarrow$ (i), after noting that, with the notation of Proposition 4.2, in this case $\mathrm{Ker}(\pi) = J$.

(i) $\Rightarrow$ (ii) By Proposition 4.9(1) and the previous equality, it is enough to show that if $A \bowtie^f J$ is reduced, then A is reduced. This is trivial because, if $a \in \mathrm{Nilp}(A)$, then $(a, f(a)) \in \mathrm{Nilp}(A \bowtie^f J)$.

Finally, the first part of the last statement is straightforward. As for the second part, we have $\{0\} = \mathrm{Nilp}(B) \cap J = \mathrm{Nilp}(B)$ (since J is radical, and so $J \supseteq \mathrm{Nilp}(B)$). Hence B is reduced. □

Remark 5.5. (1) Note that, from the previous result, when $B = A$, $f = \mathrm{id}_A$ ($= \mathrm{id}$) and $J = I$ is an ideal of A, we reobtain easily that $A \bowtie I$ ($= A \bowtie^{\mathrm{id}} I$) is a reduced ring if and only if A is a reduced ring [7, Proposition 2.1].

(2) The previous proposition implies that the property of being reduced for $A \bowtie^f J$ is independent of the nature of f.

(3) If A and $f(A)+J$ are reduced rings, then $A \bowtie^f J$ is a reduced ring, by Proposition 5.4. But the converse is not true in general. As a matter of fact, let $A := \mathbb{Z}$, $B := \mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})$, $f : A \to B$ be the ring homomorphism such that $f(n) = (n, [n]_4)$, for every $n \in \mathbb{Z}$ (where $[n]_4$ denotes the class of n modulo 4). If we set $J := \mathbb{Z} \times \{[0]_4\}$, then $J \cap \mathrm{Nilp}(B) = \{0\}$, and thus $A \bowtie^f J$ is a reduced ring, but $(0, [2]_4) = (2, [2]_4) + (-2, [0]_4)$ is a nonzero nilpotent element of $f(\mathbb{Z}) + J$.

The next proposition provides an answer to the question of when $A \bowtie^f J$ is a Noetherian ring.

Proposition 5.6. *With the notation of Proposition 5.1, the following conditions are equivalent:*

(i) *$A \bowtie^f J$ is a Noetherian ring.*

(ii) *A and $f(A) + J$ are Noetherian rings.*

Proof. (ii) ⇒ (i). Recall that $A \bowtie^f J$ is the fiber product of the ring homomorphism $\breve{f} : A \to B/J$ (defined by $a \mapsto f(a) + J$) and of the canonical projection $\pi : B \to B/J$. Since the projection $p_A : A \bowtie^f J \to A$ is surjective (Proposition 5.1(3)) and A is a Noetherian ring, by Proposition 4.10, it suffices to show that J ($= \mathrm{Ker}(\pi)$), with the structure of $A \bowtie^f J$-module induced by p_B, is Noetherian. But this fact is easy, since every $A \bowtie^f J$-submodule of J is an ideal of the Noetherian ring $f(A) + J$.

(i) ⇒ (ii) is a straightforward consequence of Proposition 5.1(3). □

Note that, from the previous result, when $B = A$, $f = \mathrm{id}_A$ ($= \mathrm{id}$) and $J = I$ is an ideal of A, we reobtain easily that $A \bowtie I$ ($= A \bowtie^{\mathrm{id}} I$) is a Noetherian ring if and only if A is a Noetherian ring [6, Corollary 2.11].

However, the previous proposition has a moderate interest because the Noetherianity of $A \bowtie^f J$ is not directly related to the data (i.e., A, B, f and J), but to the ring $B_\diamond = f(A) + J$ which is canonically isomorphic $A \bowtie^f J$, if $f^{-1}(J) = \{0\}$ (Proposition 5.1(3)). Therefore, in order to obtain more useful criteria for the Noetherianity of $A \bowtie^f J$, we specialize Proposition 5.6 in some relevant cases.

Proposition 5.7. *With the notation of Proposition 5.1, assume that at least one of the following conditions holds:*

(a) *J is a finitely generated A-module (with the structure naturally induced by f).*

(b) *J is a Noetherian A-module (with the structure naturally induced by f).*

(c) *$f(A) + J$ is Noetherian as A-module (with the structure naturally induced by f).*

(d) *f is a finite homomorphism.*

Then $A \bowtie^f J$ is Noetherian if and only if A is Noetherian. In particular, if A is a Noetherian ring and B is a Noetherian A-module (e.g., if f is a finite homomorphism [2, Proposition 6.5]*), then $A \bowtie^f J$ is a Noetherian ring for all ideals J of B.*

Proof. Clearly, without any extra assumption, if $A \bowtie^f J$ is a Noetherian ring, then A is a Noetherian ring, since it is isomorphic to $A \bowtie^f J/(\{0\} \times J)$ (Proposition 5.1(3)).

Conversely, assume that A is a Noetherian ring. In this case, it is straightforward to verify that conditions (a), (b), and (c) are equivalent [2, Propositions 6.2, 6.3, and 6.5]. Moreover (d) implies (a), since J is an A-submodule of B, and B is a Noetherian A-module under condition (d) [2, Proposition 6.5].

Using the previous observations, it is enough to show that $A \bowtie^f J$ is Noetherian if A is Noetherian and condition (c) holds. If $f(A) + J$ is Noetherian as an A-module, then $f(A) + J$ is a Noetherian ring (every ideal of $f(A) + J$ is an A-submodule of $f(A) + J$). The conclusion follows from Proposition 5.6((ii) $\Rightarrow$ (i)).

The last statement is a consequence of the first part and of the fact that, if B is a Noetherian A-module, then (a) holds [2, Proposition 6.2]. □

Proposition 5.8. *We preserve the notation of Propositions 5.1 and 4.2. If B is a Noetherian ring and the ring homomorphism $\breve{f} : A \to B/J$ is finite, then $A \bowtie^f J$ is a Noetherian ring if and only if A is a Noetherian ring.*

Proof. If $A \bowtie^f J$ is Noetherian we already know that A is Noetherian. Hence, we only need to show that if A and B are Noetherian rings and $\breve{f}$ is finite then $A \bowtie^f J$ is Noetherian. But this fact follows immediately from [10, Proposition 1.8]. □

As a consequence of the previous proposition, we can characterize when rings of the form $A + XB[X]$ and $A + XB[\![X]\!]$ are Noetherian. Note that S. Hizem and A. Benhissi [12] have already given a characterization of the Noetherianity of the power series rings of the form $A + XB[\![X]\!]$. The next corollary provides a simple proof of Hizem and Benhissi's theorem and shows that a similar characterization holds for the polynomial case (in several indeterminates). At the Fez Conference in June 2008, S. Hizem has announced to have proven a similar result in the polynomial ring case with a totally different approach.

Corollary 5.9. *Let $A \subseteq B$ be a ring extension and $\boldsymbol{X} := \{X_1, \dots, X_n\}$ a finite set of indeterminates over B. Then the following conditions are equivalent:*

(i) *$A + \boldsymbol{X}B[\boldsymbol{X}]$ is a Noetherian ring.*

(ii) *$A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is a Noetherian ring.*

(iii) *A is a Noetherian ring and $A \subseteq B$ is a finite ring extension.*

Proof. (iii) $\Rightarrow$ (i, ii). With the notations of Example 2.5, recall that $A + \boldsymbol{X}B[\boldsymbol{X}]$ is isomorphic to $A \bowtie^{\sigma'} \boldsymbol{X}B[\boldsymbol{X}]$ (and $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is isomorphic to $A \bowtie^{\sigma''} \boldsymbol{X}B[\![\boldsymbol{X}]\!]$). Since we have the following canonical isomorphisms

$$\frac{B[\boldsymbol{X}]}{\boldsymbol{X}B[\boldsymbol{X}]} \cong B \cong \frac{B[\![\boldsymbol{X}]\!]}{\boldsymbol{X}B[\![\boldsymbol{X}]\!]},$$

in the present situation, the homomorphism $\breve{\sigma}' : A \hookrightarrow B[\boldsymbol{X}]/\boldsymbol{X}B[\boldsymbol{X}]$ (or, $\breve{\sigma}'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]/\boldsymbol{X}B[\![\boldsymbol{X}]\!]$) is finite. Hence, statements (i) and (ii) follow easily from Proposition 5.8.

(i) (or, (ii)) $\Rightarrow$ (iii). Assume that $A+\boldsymbol{X}B[\boldsymbol{X}]$ (or, $A+\boldsymbol{X}B[\![\boldsymbol{X}]\!]$) is a Noetherian ring. By Proposition 5.6, or by the isomorphism $(A + \boldsymbol{X}B[\boldsymbol{X}])/\boldsymbol{X}B[\boldsymbol{X}] \cong A$ (respectively $(A + \boldsymbol{X}B[\![\boldsymbol{X}]\!])/\boldsymbol{X}B[\![\boldsymbol{X}]\!] \cong A$), we deduce that A is also a Noetherian ring. Moreover, by assumption, the ideal I of $A + \boldsymbol{X}B[\boldsymbol{X}]$ (respectively, of $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$) generated by the set $\{bX_k \mid b \in B, 1 \leq k \leq n\}$ is finitely generated. Hence $I = (f_1, f_2, \ldots, f_m)$, for some $f_1, f_2, \ldots, f_m \in I$. Let $\{b_{jk} \mid 1 \leq k \leq n\}$ be the set of coefficients of linear monomials of the polynomial (respectively, power series) f_j, $1 \leq j \leq m$. It is easy to verify that $\{b_{jk} \mid 1 \leq j \leq m, 1 \leq k \leq n\}$ generates B as A-module; thus $A \subseteq B$ is a finite ring extension. □

Remark 5.10. Let $A \subseteq B$ be a ring extension, and let X be an indeterminate over B. Note that the ideal $J' = XB[X]$ of $B[X]$ is never finitely generated as an A-module (with the structure induced by the inclusion $\sigma' : A \hookrightarrow B[X]$). As a matter of fact, assume that $\{g_1, g_2, \ldots, g_r\}$ ($\subset B[X]$) is a set of generators of J' as A-module and set $N := \max\{\deg(g_i) \mid i = 1, 2, \ldots, r\}$. Clearly, we have $X^{N+1} \in J' \setminus \sum_{i=1}^{r} Ag_i$, which is a contradiction. Therefore, the previous observation shows that the Noetherianity of the ring $A \bowtie^f J$ does not imply that J is finitely generated as an A-module (with the structure induced by f); for instance $\mathbb{R} + X\mathbb{C}[X]$ ($\cong \mathbb{R} \bowtie^{\sigma'} X\mathbb{C}[X]$, where $\sigma' : \mathbb{R} \hookrightarrow \mathbb{C}[X]$ is the natural embedding) is a Noetherian ring (Proposition 5.9), but $X\mathbb{C}[X]$ is not finitely generated as an $\mathbb{R}$-vector space. This fact shows that condition (a) (or, equivalently, (b) or (c)) of Proposition 5.7 is not necessary for the Noetherianity of $A \bowtie^f J$.

Example 5.11. Let $A \subseteq B$ be a ring extension, J an ideal of B and $\boldsymbol{X} := \{X_1, \ldots, X_r\}$ a finite set of indeterminates over B. We set $B' := B[\boldsymbol{X}]$, $J' := \boldsymbol{X}J[\boldsymbol{X}]$ and we denote by σ' the canonical embedding of A into B'. By a routine argument, it is easy to see that the ring $A \bowtie^{\sigma'} J'$ is naturally isomorphic to the ring $A + \boldsymbol{X}J[\boldsymbol{X}]$. Now, we want to show that, in this case, we can characterize the Noetherianity of the ring $A + \boldsymbol{X}J[\boldsymbol{X}]$, without assuming a finiteness condition on the inclusion $A \subseteq B$ (as in Corollary 5.9 (iii)) or on the inclusion $A + \boldsymbol{X}J[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$. More precisely, *the following conditions are equivalent:*

(i) $A + \boldsymbol{X}J[\boldsymbol{X}]$ *is a Noetherian ring.*

(ii) *A is a Noetherian ring, J is an idempotent ideal of B and it is finitely generated as an A-module.*

(i) $\Rightarrow$ (ii). Assume that $R := A + XJ[X] = A + J'$ is a Noetherian ring. Then, clearly, A is Noetherian, since A is canonically isomorphic to R/J'. Now, consider the ideal L of R generated by the set of linear monomials $\{bX_i \mid 1 \leq i \leq r,\ b \in J\}$. By assumption, we can find $\ell_1, \ell_2, \dots, \ell_t \in L$ such that $L = \sum_{k=1}^{t} \ell_k R$. Note that $\ell_k(0, 0, \dots, 0) = 0$, for all k, $1 \leq k \leq t$. If we denote by b_k the coefficient of the monomial X_1 in the polynomial ℓ_k, then it is easy to see that $\{b_1, b_2, \dots, b_t\}$ is a set of generators of J as an A-module.

The next step is to show that J is an idempotent ideal of B. By assumption, J' is a finitely generated ideal of R. Let

$$g_h := \sum_{j_1+\cdots+j_r=1}^{m_h} c_{h,j_1\dots j_r} X_1^{j_1} \cdots X_r^{j_r}, \quad \text{with } h = 1, 2, \dots, s,$$

be a finite set of generators of J' in R. Set $\overline{j_1} := \max\{j_1 \mid c_{h,j_1 0\dots 0} \neq 0, \text{ for } 1 \leq h \leq s\}$. Take now an arbitrary element $b \in J$ and consider the monomial $bX_1^{\overline{j_1}+1} \in J'$. Clearly, we have

$$bX_1^{\overline{j_1}+1} = \sum_{h=1}^{s} f_h g_h, \quad \text{with } f_h := \sum_{e_1+\cdots+e_r=0}^{n_h} d_{h,e_1\dots e_r} X_1^{e_1} \cdots X_r^{e_r} \in R.$$

Therefore,

$$b = \sum_{h=1}^{s} \sum_{j_1+e_1=\overline{j_1}+1} c_{h,j_1 0\dots 0} d_{h,e_1 0\dots 0}.$$

Since $j_1 < \overline{j_1} + 1$, we have necessarily that $e_1 \geq 1$. Henceforth f_h belongs to J' and so $d_{h,e_1 0\dots 0} \in J$, for all h, $1 \leq h \leq s$. This proves that $b \in J^2$.

(ii) $\Rightarrow$ (i). In this situation, by Nakayama's lemma, we easily deduce that $J = eB$, for some idempotent element $e \in J$. Let $\{b_1, \dots, b_s\}$ be a set of generators of J as an A-module, i.e., $J = eB = \sum_{1 \leq h \leq s} b_h A$. We consider a new set of indeterminates over B (and A) and precisely $Y := \{Y_{ih} \mid 1 \leq i \leq r, 1 \leq h \leq s\}$. We can define a map $\varphi : A[X, Y] \to B[X]$ by setting $\varphi(X_i) := eX_i$, and $\varphi(Y_{ih}) := b_h X_i$, for all $i = 1, \dots, r$, $h = 1, \dots, s$. It is easy to see that φ is a ring homomorphism and $\mathrm{Im}(\varphi) \subseteq R\ (= A + XJ[X])$. Conversely, let

$$f := a + \sum_{i=1}^{r} \left(\sum_{e_{i_1}+\cdots+e_{i_r}=0}^{n_i} c_{i,e_{i_1}\dots e_{i_r}} X_1^{e_{i_1}} \cdots X_r^{e_{i_r}} \right) X_i \in R \text{ (and so } c_{i,e_{i_1}\dots e_{i_r}} \in J\text{)}.$$

Since $J = \sum_{1 \leq h \leq s} b_h A$, then for all $i = 1, \dots, r$ and $e_{i_1}, \dots, e_{i_r}$, with $e_{i_1} + \cdots + e_{i_r} \in \{0, \dots, n_i\}$, we can find elements $a_{i,e_{i_1}\dots e_{i_r},h} \in A$, with $1 \leq h \leq s$, such that $c_{i,e_{i_1}\dots e_{i_r}} = \sum_{h=1}^{s} a_{i,e_{i_1}\dots e_{i_r},h} b_h$. Consider the polynomial

$$g := a + \sum_{i=1}^{r} \sum_{h=1}^{s} \sum_{e_{i_1}+\cdots+e_{i_r}=0}^{n_i} a_{i,e_{i_1}\dots e_{i_r},h} X_1^{e_{i_1}} \cdots X_r^{e_{i_r}} Y_{ih} \in A[X, Y].$$

It is straightforward to see that $\varphi(g) = f$ and so $\mathrm{Im}(\varphi) = R$. By Hilbert's basis theorem, we conclude easily that R is Noetherian.

Remark 5.12. We preserve the notation of Example 5.11.

(1) Note that in the previous example, when $J = B$, we reobtain Corollary 5.9 ((i) $\Leftrightarrow$ (iii)). If $B = A$ and I is an ideal of A, then we simply have that $A + XI[X]$ *is a Noetherian ring if and only if A is a Noetherian ring and I is an idempotent ideal of A*. Note the previous two cases were studied as separate cases by S. Hizem, who announced similar results in her talk at the Fez Conference in June 2008, presenting an ample and systematic study of the transfer of various finiteness conditions in the constructions $A + XI[X]$ and $A + XB[X]$.

(2) The Noetherianity of B it is not a necessary condition for the Noetherianity of the ring $A+XJ[X]$. For instance, take A any field, B the product of infinitely many copies of A, so that we can consider A as a subring of B, via the diagonal ring embedding $a \mapsto (a, a, \dots)$, $a \in A$. Set $J := (1, 0, \dots)B$. Then J is an idempotent ideal of B and, at the same time, a cyclic A-module. Thus, by Example 5.11, $A + XJ[X]$ is a Noetherian ring. Obviously, B is not Noetherian.

(3) Note that, if $A + XJ[X]$ is Noetherian and B is not Noetherian, then $A \subseteq B$ and $A + XJ[X] \subseteq B[X]$ are necessarily not finite. Moreover, it is easy to see that $A + XJ[X] \subseteq B[X]$ is a finite extension if and only if the canonical homomorphism $A \hookrightarrow B[X]/(XJ[X])$ is finite. Finally, it can be shown that last condition holds if and only if $J = B$ and $A \subseteq B$ is finite.

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A sheaf-theoretic bound on the cardinality of a finite ring

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Abstract. By using Pierce's representation of a ring as the ring of global sections of a sheaf of connected rings, an upper bound is given for the cardinality of a finite ring. Examples are given to compare this upper bound to bounds due to Ganesan and Redmond.

Keywords. Finite ring, sheaf, connected ring, associated Boolean ring, idempotent element, prime ideal, zero-divisor, annihilator, integral domain.

AMS classification. 13M99, 13A15, 54B40, 13A02.

1 Introduction

All rings considered in this note are commutative with 1 and typically nonzero; all ring homomorphisms are unital. If R is a ring, then $\mathrm{Spec}(R)$ denotes the set of prime ideals of R; $Z(R)$ the set of zero-divisors in R; and $(0 :_R r)$ the annihilator of r in R. If q is a prime-power, then $\mathbb{F}_q$ denotes the field with exactly q elements; and $|S|$ denotes the cardinality of a set S.

Our starting point is the following remarkable result of Ganesan [4, Theorem I]: if R is a ring such that $2 \le k := |Z(R)| < \infty$, then R is finite and, in fact, $|R| \le k^2$. In reviewing [4] in Mathematical Reviews, Alex Rosenberg wrote that "The methods are elementary." In fact, Ganesan's proof depends only on the First Isomorphism Theorem and Lagrange's theorem. (In detail, if r is a nonzero zero-divisor in R, then $I := (0 :_R r) \subseteq Z(R)$ must be finite, while $R/I \cong Rr \subseteq Z(R)$ is also finite.) Despite its elementary nature, Ganesan's result has had a considerable influence, and a number of improvements to it have appeared. Although we shall not consider noncommutative rings in this paper, note that Ganesan did generalize his result to noncommutative rings [5, Theorem 1]; and Bell [1] extended the result to the context of alternative (not necessarily associative) rings. However, the greatest influence of Ganesan's theorem has been to commutative ring theory. The two most obvious such influences are to the burgeoning development of the theory of zero-divisor graphs, perhaps because finite zero-divisor graphs correspond to finite rings (apart from the obvious counter-examples); and the search for inequalities that, at least for certain classes of rings, improve upon the upper bound in Ganesan's result: cf. [11], [12]. We will say no more here about the former, but instead focus on the latter, by providing a new upper bound for the cardinality of a finite ring: see Corollary 2.10 (a). The results given below reveal that for certain classes of rings, our upper bound is sharper than those in [4], [11] and [12], while for some other classes of rings, the reverse is true.

A key step leading to Corollary 2.10 is taken in Lemma 2.6, where it is shown that each ring R is (to use the terminology of another time [9]) a certain subdirect sum of connected rings. (Recall that a ring A is said to be *connected* if its only idempotent elements are 0 and 1; this is equivalent to requiring that Spec(A), in the Zariski topology, is a connected space [2, Corollary 2, page 104].) This is shown as a consequence of the Pierce representation of any ring as the ring of global sections of a sheaf of connected rings [10]. (For another approach to this representation, see [13, Section 2, especially pages 84–88].) Our use of sheaf theory here will require only the basic definition and facts about sheaves, as in [6, Chapter II, Section 1, especially pages 61–62]. The most important upshot is Theorem 2.8 (a), giving a canonical description up to isomorphism of any direct product of finitely many connected rings; this, in turn, leads to a formula for $|R|$ in Corollary 2.9 and the upper bound in Corollary 2.10 (a).

2 Results

We begin by recording the fundamental result of Ganesan.

Theorem 2.1 (Ganesan [4, Theorem I]). *Let R be a ring such that* $2 \leq k := |Z(R)| < \infty$. *Then R is finite and, in fact,* $|R| \leq k^2$.

We next record examples showing that Ganesan's upper bound is sometimes exact and sometimes only an inequality. Some of the data in Example 2.2 will be referred to later when we discuss other upper bounds for $|R|$. The specific examples given in Example 2.2 have been chosen to facilitate comparison among the various bounds, and the interested reader will have no trouble in generalizing some of the arguments in Example 2.2 (for instance, to various finite special principal ideal rings, idealizations or direct products with more than two factors).

Example 2.2. (a) Let X be transcendental over $\mathbb{F}_q$ for some prime-power q, consider an integer $e \geq 2$, and put $R := \mathbb{F}_q[X]/(X^e)$. Let $k := |Z(R)|$. Then $|R| = k^2$ if and only if $e = 2$ (i.e., if and only if R is isomorphic to the ring of dual numbers over $\mathbb{F}_q$).

(b) Let R be a finite (nonzero) Boolean ring and $k := |Z(R)|$. Then $|R| < k^2$.

(c) Let $q_1 \leq q_2$ be prime-powers, and put $R := \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$. Let $k := |Z(R)|$. Then $k = q_1 + q_2 - 1$ and $|R| < k^2$.

(d) Consider integers $b \geq a \geq 2$, and put $R := \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z}$. Let $k := |Z(R)|$. Then $k = 3 \cdot 2^{a+b-2}$ and $|R| < k^2$.

Proof. (a) Let $x := X + (X^2) \in R$. Then every element $r \in R$ can be expressed uniquely as $\sum_{i=0}^{e-1} a_i x^i$ for some elements $a_0, \ldots, a_{e-1} \in \mathbb{F}_q$. Such an element r is in $Z(R)$ if and only if $a_0 = 0$. It follows that $|R| = q^e$ and $k = |Z(R)| = q^{e-1}$. Then $|R| = k^2$ if and only if $q^e = (q^{e-1})^2$, i.e., if and only if $e = 2(e-1)$. This equation holds if and only if $e = 2$. Moreover, if $e > 2$, Theorem 2.1 yields that $|R| < k^2$.

(b) It is easy to see, via the Chinese Remainder Theorem, that R is isomorphic to the direct product of n copies of $\mathbb{F}_2$, for some positive integer n (cf. [9, Theorem 3.20]), and so $|R| = 2^n$. Since every (idempotent) element of R is either a zero-divisor or 1, we have that $k = |Z(R)| = 2^n - 1$. Thus, $|R| = k^2$ if and only if $2^n = (2^n - 1)^2$, or equivalently, $(2^n - 1)/2^n = 1/(2^n - 1)$. However this equality never happens, since a positive rational number can be written in only one way as a ratio of positive integers that are relatively prime. Thus, by Theorem 2.1 (or directly), $|R| < k^2$ regardless of the value of n.

(c) Since $Z(R) = \{(a, b) \in R \mid \text{either } b = 0 \text{ or } a = 0 \text{ (or both)}\}$, it is clear that $k = q_1 + q_2 - 1$. Thus, the assertion comes down to verifying that $q_1 q_2 < (q_1 + q_2 - 1)^2$, or equivalently, that

$$1 < (q_1 - 1)^2 + (q_2 - 1)^2 + q_1 q_2,$$

and it is easy to verify this inequality regardless of the values of q_1, q_2.

(d) It is easy to check that for any integer $c \geq 2$ and prime number p, we have that $Z(\mathbb{Z}/p^c\mathbb{Z}) = \{m + p^c\mathbb{Z} \in \mathbb{Z}/p^c\mathbb{Z} \mid p \text{ divides } m \text{ in } \mathbb{Z}\}$, and so $|Z(\mathbb{Z}/p^c\mathbb{Z})| = p^{c-1}$. By handling overlaps in counting as in the proof of (c), we find that

$$k = 2^{a-1}2^b + 2^{b-1}2^a - 2^{a-1}2^{b-1} = 2^{a-1}2^{b-1}(2 + 2 - 1) = 3 \cdot 2^{a+b-2},$$

as asserted. Of course, $|R| = 2^a 2^b = 2^{a+b}$. Thus, it remains only to verify that $2^{a+b} < (3 \cdot 2^{a+b-2})^2$, or equivalently, that $2^{4-a-b} < 9$, which is obvious. □

In [11, Section 3], Redmond generalized Theorem 2.1 by showing that if R is a ring which is not an integral domain such that $|(0 :_R r)| < \infty$ for each nonzero element $r \in R$, then R is finite. This led to an upper bound on $|R|$ in case a uniform upper bound was known for $|(0 :_R r)|$, $0 \neq r \in R$. This work was improved in a subsequent paper, as follows.

Theorem 2.3 (Redmond [12, Theorem 6.1]). *Let R be a Noetherian ring which is not an integral domain and K a positive integer such that $|(0 :_R r)| \leq K$ for each nonzero element $r \in R$. Then R is finite and, in fact, $|R| \leq (K^2 - 2K + 2)^2$.*

The next results parallels Example 2.2, this time with a focus on the upper bound of Redmond.

Example 2.4. For each of the types of rings R in Example 2.2, let $k := |Z(R)|$ be as in Example 2.2, and let K be optimal as in Theorem 2.3, namely, the least positive integer such that $|(0 :_R r)| \leq K$ for each nonzero element $r \in R$. Then:

(a) Let R be as in Example 2.2 (a). Then $K = k$, i.e., $K = q^{e-1}$. It follows that $|R| < (K^2 - 2K + 2)^2$. Moreover, the bounds in Theorems 2.1 and 2.3 are related by $k^2 < (K^2 - 2K + 2)^2$.

(b) Let R be as in Example 2.2 (b). Then $K = 2^{n-1} = |R|/2$. Moreover, the bounds in Theorems 2.1 and 2.3 are related by

$$|R| < k^2 = (2^n - 1)^2 < (K^2 - 2K + 2)^2.$$

(c) Let R be as in Example 2.2 (c). Then $K = q_2$. Also, $|R| < (K^2 - 2K + 2)^2$ if and only if $q_1 q_2 < (q_2^2 - 2q_2 + 2)^2$. Moreover, the bounds in Theorems 2.1 and 2.3 are related by

$$k^2 \le (K^2 - 2K + 2)^2 \Leftrightarrow (q_1 + q_2 - 1)^2 \le (q_2^2 - 2q_2 + 2)^2.$$

(d) Let R be as in Example 2.2 (d). Then $K = 2^{a+b-1}$. Moreover, the bounds in Theorems 2.1 and 2.3 are related by

$$|R| < k^2 = (3 \cdot 2^{a+b-2})^2 < (K^2 - 2K + 2)^2.$$

Proof. (a) The first assertion follows from the fact that $|(0 :_R x^{e-1})| = |Rx| = q^{e-1}$ and Example 2.2 (a). The remaining assertions then follow from Example 2.2 (a) and the fact that $k = q^{e-1} \ge 2$ satisfies $k^2 < (k^2 - 2k + 2)^2$. This inequality is clear from calculus, since the real-valued function f of a real variable t given by

$$f(t) := t^4 - 4t^3 + 7t^2 - 4t + 4$$

satisfies $f(t) > 0$ for all $t \ge 2$.

(b) The first assertion follows since $|(0 :_R (1, 0, \ldots, 0))| = 2^{n-1}$ and no proper additive subgroup of R can have larger cardinality than this (the far reaches of Lagrange's theorem!). In view of Example 2.2 (b), the remaining assertions come down to showing that

$$(2^n - 1)^2 < (2^{2n-2} - 2^n + 2)^2$$

for all positive integers n. This reduces to showing that

$$2^{3n-1} + 2^{2n} + 2^{n+2} < 2^{4n-4} + 2^{2n+2} + 3,$$

which is clear.

(c) As we have assumed that $q_1 \le q_2$ and $|(0 :_R (1, 0))| = q_2$, the assertions follow easily from Example 2.2 (c).

(d) The first assertion follows since $|(0 :_R (2 + 2^a\mathbb{Z}, 0))| = 2^{a-1}2^b = 2^{a+b-1}$. In view of Example 2.2 (b), the remaining assertions come down to showing that

$$k^2 < \left(\frac{4}{9}k^2 - \frac{4}{3}k + 2\right)^2 \text{ for all } k \ge 12,$$

or equivalently, that $0 < 16k^4 - 96k^3 + 207k^2 - 432k + 324$ if $k \ge 12$. We leave this easy verification to the reader. □

If R is any ring, let $B(R)$ denote the Boolean ring of idempotents of R, with multiplication being that of R and addition redefined by $a \star b := a + b - 2ab$ for all idempotents $a, b \in R$. The next result contains some elementary but useful facts about the $B(-)$ construction.

Lemma 2.5. *Let R be a ring expressed as a ring direct product $R = \prod_{i \in I} R_i$ where the index set I is nonempty and each ring R_i is nonzero. Then:*

(a) $B(R) = \prod_{i \in I} B(R_i)$ *as rings.*

(b) *If each* R_i *is connected (for instance, quasilocal), then* $B(R) \cong \prod_{i \in I} \mathbb{F}_2$.

Proof. (a) Straightforward.

(b) Any quasilocal ring is connected. It remains to show that if R is a (nonzero) connected ring, then $B(R) \cong \mathbb{F}_2$. This, in turn, follows from the fact that $B(R)$ is a (Boolean) ring with exactly two elements. □

The next result contains the main use of sheaf theory in this note. The only fact that we will use about sheaves is the following easy consequence of the definitions. If F is a sheaf of rings on a topological space X and U is an open set in X, then the canonical ring homomorphism $F(U) \to \prod_{x \in U} F_x$ is an injection, where F_x denotes the stalk of F at x.

Lemma 2.6. *If R is any ring, then the natural ring homomorphism*

$$\alpha : R \to \prod_{x \in \mathrm{Spec}(B(R))} R/xR$$

is an injection.

Proof. According to the Pierce representation, there is a sheaf F of rings on the topological space $X := \mathrm{Spec}(B(R))$ endowed with the Zariski topology such that for all $x \in X$, the stalk of F at x is the connected ring $F_x = R/xR$ (where xR denotes the ideal of R generated by the set x). Thus, by the above comments, for any Zariski-open subset U of X, we have a natural ring homomorphism $F(U) \to \prod_{x \in U} R/xR$. Taking $U := X$, we thus obtain a natural ring homomorphism

$$\beta : R \to \prod_{x \in \mathrm{Spec}(B(R))} R/xR.$$

With the details of the Pierce representation in hand (from [10] or [13]), it is straightforward to check that $\beta = \alpha$. □

Proposition 2.7. *Let R be a ring. Then the following conditions are equivalent:*

(1) *R has only finitely many idempotent elements and $|R/xR| < \infty$ for each $x \in \mathrm{Spec}(B(R))$;*

(2) *$|\mathrm{Spec}(B(R))| < \infty$ and $|R/xR| < \infty$ for each $x \in \mathrm{Spec}(B(R))$;*

(3) *R is finite.*

Proof. Since $B(R)$ is the set of idempotent elements of R and a finite ring can have only finitely many prime ideals, it is easy to see that (3) $\Rightarrow$ (1) $\Rightarrow$ (2). As for (2) $\Rightarrow$ (3), it suffices to note, via Lemma 2.6 and the definition of multiplication of cardinal numbers, that any ring R satisfies $|R| \leq \prod_{x \in \mathrm{Spec}(B(R))} |R/xR|$. □

Proposition 2.7 can be viewed in the spirit of the results of Ganesan and Redmond that were recalled in Theorems 2.1 and 2.3, respectively. To be sure, conditions (1) and (2) in Proposition 2.7 each have two stipulations on R, but the same is true of the characterizations of finite rings implicit in Theorems 2.1 and 2.3 (with the latter result also stipulating that R is Noetherian). Note that one cannot avoid the first of the stipulations in either (1) or (2) of Proposition 2.7, as any (possibly infinite) Boolean ring R has the property that $R/xR \cong \mathbb{F}_2$ for each $x \in \mathrm{Spec}(B(R))$.

Recall that every finite ring is uniquely expressible as an internal direct product of finitely many finite local rings (cf. [14, Theorem 3, page 205; Remark 1, page 208]). For such a ring, the homomorphism α from Lemma 2.6 turns out to be an isomorphism. Actually, the next result establishes somewhat more.

Theorem 2.8. *Let R be a ring expressed as a ring direct product $R = \prod_{i=1}^{n} R_i$ where n is a positive integer and each R_i is a nonzero connected (for instance, quasilocal) ring. As above, consider the natural ring homomorphism $\alpha : R \to \prod_{x \in \mathrm{Spec}(B(R))} R/xR$. Then:*

(a) *α may be identified with the identity map $R \to \prod_{i=1}^{n} R_i$, and so α is an isomorphism.*

(b) $|R| = \prod_{x \in \mathrm{Spec}(B(R))} |R/xR|$.

Proof. (a) By Lemma 2.5 (b), $B(R) \cong \prod_{i=1}^{n} \mathbb{F}_2$, and so $|\mathrm{Spec}(B(R))| = n$. Thus, in view of Lemma 2.6, it suffices to prove that if $x \in \mathrm{Spec}(B(R))$, then $R/xR \cong R_i$. For convenience of notation, identify $B(R)$ with $\prod_{i=1}^{n} \mathbb{F}_2$. Then

$$x = \mathbb{F}_2 \times \cdots \times \mathbb{F}_2 \times 0 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$$

where the 0 is at the ith position for some $i = 1, \ldots, n$. Clearly,

$$xR = R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n,$$

whence $R/xR \cong R_i$, as desired.

(b) As in the proof of Proposition 2.7, one need only combine (a) with the definition of multiplication of cardinal numbers. □

Corollary 2.9. *Let R be a finite ring. Express R (uniquely) as a ring direct product $R = \prod_{i=1}^{n} R_i$ where n is a positive integer and each R_i is a (nonzero finite) local ring. Then $R \cong \prod_{x \in \mathrm{Spec}(B(R))} R/xR$ and $|R| = \prod_{x \in \mathrm{Spec}(B(R))} |R/xR|$.*

Proof. Recall that any local ring is connected. Apply Theorem 2.8. □

The equality describing $|R|$ in Corollary 2.9 should perhaps not be considered as giving an upper bound for $|R|$. For this reason, we proceed to weaken that formula. As a result, in the spirit of the results of Ganesan and Redmond in Theorems 2.1 and 2.3, we will give a new upper bound for a ring with at least 2, but only finitely many, zero-divisors. Since the upper bound of Ganesan (involving $|Z(R)|$) is sometimes exact (for instance, if $R = \mathbb{F}_q[X]/(X^2)$), our formula will be predicated on an additional parameter, which is called b below.

Corollary 2.10. *Let R be a ring. Suppose that $2 \leq k := |Z(R)| < \infty$. Then R and $B(R)$ are finite. Let $b := \max_{x \in \mathrm{Spec}(B(R))} |R/xR|$. Then:*

(a) $|R| \leq b^{|\mathrm{Spec}(B(R))|} \leq b^{\log_2(k+1)}$.

(b) *Express R (uniquely) as an internal direct product $R = \prod_{i=1}^{n} R_i$ where n is a positive integer and each R_i is a (nonzero finite) local ring. Then $|R|$ equals the lesser of the above upper bounds (i.e., the first inequality in (a) becomes an equality) if and only if $|R_1| = \cdots = |R_n|$. In particular, if the n rings $R_1, \ldots, R_n$ are isomorphic to one another, then $|R| = b^{|\mathrm{Spec}(B(R))|}$. Moreover, $|R| = b^{\log_2(k+1)}$ if and only if R is a (finite) Boolean ring.*

Proof. The finitude of R follows from Ganesan's result (Theorem 2.1); the finitude of $B(R)$ is then a triviality, as is that of b. As $|R/xR| \leq b$ for all $x \in \mathrm{Spec}(B(R))$, (a) is an easy consequence of the second assertion of Corollary 2.9 once we show that $|\mathrm{Spec}(B(R))| \leq \log_2(k+1)$. To show the latter fact, note first that $|B(R)| = 2^m$ for some positive integer m, since $B(R)$ is a finite Boolean ring. Hence $|\mathrm{Spec}(B(R))| = m$. On the other hand, $|B(R)| \leq k+1$, since each idempotent element of R is either 1 or an element of $Z(R)$. Thus $2^m \leq k+1$, and so $m \leq \log_2(k+1)$, thus completing the proof of (a).

(b) Let m be as in the proof of (a). As noted above, $|\mathrm{Spec}(B(R))| = m$. Note that $m = n$, as a consequence of Lemma 2.5 (b). Write $\mathrm{Spec}(B(R)) = \{x_1, \ldots, x_n\}$. Put $q_i := |R/x_i R|$ for each $i = 1, \ldots, n$. By the proof of Theorem 2.8 (a), we may relabel the elements of $\mathrm{Spec}(B(R))$ and the direct factors $R_1, \ldots, R_m$ so that $q_1 \leq \cdots \leq q_m$ and $R/x_i R \cong R_i$ for each $i = 1, \ldots, n$. Note that $b = q_n$. By Corollary 2.9,

$$|R| = q_1 \cdot \cdots \cdot q_n \leq b^n = b^{|\mathrm{Spec}(B(R))|};$$

since $q_i \leq b$ for each i, equality holds if and only if $q_1 = \cdots = q_n$, i.e., if and only if $|R_1| = \cdots = |R_n|$.

It remains to prove the "Moreover" assertion. It follows from the above that $|R| = b^{\log_2(k+1)}$ if and only if $q_1 = \cdots = q_n$ and $|\mathrm{Spec}(B(R))| = \log_2(k+1)$. Observe that $|\mathrm{Spec}(B(R))| = \log_2(k+1)$ if and only if $|B(R)| = k+1$, i.e., if and only if each zero-divisor of R is idempotent. This last condition is equivalent to (either $n = 1$, in which case, $R = R_1$ is a finite local, hence connected, ring in which 0 is the only zero-divisor, contradicting the hypothesis that $k \geq 2$, or) $n \geq 2$ such that each nonzero element of each R_i must be (a zero-divisor in R and hence) idempotent. Thus, as each R_i is connected, $|R| = b^{\log_2(k+1)}$ if and only if $R_1 \cong \cdots \cong R_n \cong \mathbb{F}_2$, i.e., if and only if R is Boolean. □

The appearance in Corollary 2.10 (a) of a logarithmic expression in an upper bound for the cardinality of a spectral set should not be surprising: cf. [3, Proposition 2.1 (a)]. For a complete answer to the question of the possible cardinalities of $|\mathrm{Spec}(R)|$, assuming that R is any finite ring with given cardinality n, see [3, Theorem 2.3].

We next compare the bounds of Ganesan and Redmond to the weaker of the upper bounds that were established in Corollary 2.10 (a).

Remark 2.11. Let R be a finite ring, but not a integral domain, with $k := |Z(R)|$; K the least positive integer such that $|(0 :_R r)| \leq K$ for each nonzero element $r \in R$; and $b := \max_{x \in \mathrm{Spec}(B(R))} |R/xR|$. We proceed to compare the upper bounds for $|R|$ from Theorem 2.1, Theorem 2.3 and (the weaker assertion in) Corollary 2.10 (a), namely, the expressions k^2, $(K^2 - 2K + 2)^2$ and $b^{\log_2(k+1)}$, respectively.

(a) Let R be a (finite nonzero) Boolean ring. Then $|R| = b^{\log_2(k+1)}$, by the final assertion in Corollary 2.10 (b). However, by Example 2.4 (b), the bounds from Theorems 2.1 and 2.3 each exceed $|R|$ for any such R.

(b) Let R be as in Example 2.2 (a); i.e., $R = \mathbb{F}_q[X]/(X^e)$. Since R is local, $B(R) = \mathbb{F}_2$ and so $b = |R| = q^e$. Thus, $|R| < b^{\log_2(k+1)}$ (since $b > 1$ and $k + 1 > 2$). As we saw in Example 2.2 (a) and Example 2.4 (a), the bound from Theorem 2.1 exceeds $|R|$ if and only if $e > 2$ and the bound from Theorem 2.3 exceeds $|R|$. In particular, Ganesan's bound exceeds the (weaker) bound from Corollary 2.10 (a) for all rings of the form $R = \mathbb{F}_q[X]/(X^2)$.

How do the bounds compare for rings of the form $R = \mathbb{F}_q[X]/(X^e)$ in general (bearing in mind that q is a prime-power and $e \geq 2$)? For simplicity, we study the case $q = 2$. Recall that $k = K = q^{e-1} = 2^{e-1}$. Hence, the bound from Corollary 2.10 (a) is

$$b^{\log_2(k+1)} = 2^{e\log_2(k+1)} = 2^{e\log_2(q^{e-1}+1)} = 2^{e\log_2(2^{e-1}+1)}.$$

As we have already seen in this remark that, for various rings, there is no general inequality relating the bounds from Theorem 2.1 and Corollary 2.10 (a), we will leave the proof of the next fact to the reader. Let $\beta := 2^{e\log_2(2^{e-1}+1)}$. Then the bound from Theorem 2.1 is less than β for all values of $e \geq 2$ (and $q = 2$). However, the situation relative to the bound from Theorem 2.3 is more complicated. Indeed, using the above information and Example 2.4 (a) (still with $q := 2$), one easily checks that β is less than (resp., exceeds) the bound from Theorem 2.3 if $e = 2$ (resp., $e = 3$).

(c) The discussion in (a) and (b) has already shown that the bound from Corollary 2.10 (a) can, depending on the ring R, exceed both, only one, or neither of the bounds from Theorems 2.1 and 2.3. For this reason and to save space, we will merely state some conclusions relative to the rings from parts (c) and (d) of Example 2.2 without proof.

Let R be as in Example 2.2 (c). Then $b = q_2$ and the upper bound from Theorem 2.10 (a) is

$$\beta := b^{\log_2(k+1)} = q_2^{\log_2(k+1)} = q_2^{\log_2(q_1+q_2)}.$$

It might be expected that this bound typically exceeds the bounds from Theorem 2.1 and 2.3, using the intuition that exponential functions grow quicker than polynomial functions, and a general analysis shows that this is often the case. However, we would point out that β is less than each of the bounds from Theorem 2.1 and 2.3 in case $q_1 = 2 < q_2 = 3$, i.e., for $R = \mathbb{F}_2 \times \mathbb{F}_3$, the key calculation being that $3^{\log_2 5} < 12.82 < 16 < 25$. To be fair, one should also record that Ganesan's bound is less than β whenever q_1 and q_2 are distinct integral powers of 2.

Finally, let R be as in Example 2.2 (d). Unfortunately, we now have two different meanings for the symbol b. So, only in this paragraph, $R = \mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z}$ with $2 \leq a \leq b$ and the symbol b will not necessarily mean $\max_{x \in \mathrm{Spec}(B(R))} |R/xR|$. The upper bound from Theorem 2.10 (a) is

$$\beta := (2^b)^{\log_2(k+1)} = (2^b)^{\log_2(3 \cdot 2^{a+b-2}+1)} = (3 \cdot 2^{a+b-2} + 1)^b.$$

It can be shown that Ganesan's bound is less than β for all values of a and b. As an (easy) exercise, we leave to the reader to determine how β compares to Redmond's bound for rings of the type in Example 2.2 (d).

The author feels that Remark 2.11 gives additional reasons for one to appreciate Ganesan's result. In view of the quotation from Rosenberg in the Introduction, it seems reasonable to ask if there is a "non-elementary" proof of Ganesan's result, perhaps using the Pierce representation. With Lemma 2.6 in hand, we could then ask (given that R is a ring such that $2 \leq |Z(R)| < \infty$) if it is the case that $|\mathrm{Spec}(B(R))| < \infty$; that, whenever $x \in \mathrm{Spec}(B(R))$, it must hold that the connected ring R/xR is a non-domain with only finitely many zero-divisors; and that each such R/xR is finite. Such a program may seem initially to have some promise. For instance, since $Z(R)$ is finite, it is easy to see that $B(R)$ is finite, and so $|\mathrm{Spec}(B(R))| < \infty$. Also, if A is a connected ring, but not a domain, with only finitely many zero divisors, it can be shown that the associated reduced ring A_{red} is also connected and has at most as many zero divisors as A. However, proceeding via R_{red} cannot succeed, since R_{red} may be an integral domain: consider, for instance, $R := \mathbb{Z}/q\mathbb{Z}$ for any prime-power q.

Nevertheless, by considering reduced rings (and avoiding Ganesan's methods), one can produce a fragment of Ganesan's result, by decidedly non-elementary means, and we close by recording that in Proposition 2.12. This is done in the belief that the use of a strong theoretical machine can often lead to some insight without needing to resort to imagination of the kind exhibited in Ganesan's proof. In doing so, I acknowledge the contrary attitude, as exemplified in 1963 by my calculus teacher, William O. J. Moser (who is an expert in combinatorial aspects of group theory, number theory and geometry), when he disparaged such heavy use of theoretical machinery as "dynamiting butterflies". One should, perhaps, add that Proposition 2.12 is also an easy consequence of Theorem 2.1 and the fact that each finite ring is a direct product of finitely many connected rings.

Proposition 2.12. *There is no reduced connected ring R such that* $2 \leq k := |Z(R)| < \infty$.

Proof. Suppose, on the contrary, that such a ring R exists. Since R is a reduced ring, $Z(R)$ is the union of the minimal prime ideals of R [7, Corollary 2.4]. Suppose next that $P_1, \ldots, P_{k+1}$ are pairwise distinct minimal prime ideals of R. By the Prime Avoidance Lemma [8, Theorem 83], we can pick elements $x_i \in P_i \setminus \bigcup_{j=1}^{k+1} P_j$, for $i = 1, \ldots, k+1$. As $x_{i_1} \neq x_{i_2}$ whenever $i_1 \neq i_2$ and $\{x_1, \ldots, x_{k+1}\} \subseteq Z(R)$,

$$|\{x_1, \ldots, x_{k+1}\}| = k + 1 \leq |Z(R)| = k,$$

a contradiction. Thus, R has at most k (pairwise distinct) minimal prime ideals, say, $Q_1, \dots, Q_s$ for some positive integer $s \le k$. Let T denote the total quotient ring of R. Then $T = R_{R \setminus (Q_1 \cup \cdots \cup Q_s)}$ has only the prime ideals $Q_i T$, and so T has (Krull) dimension 0. Being a ring of fractions of a reduced ring, T must also be reduced. Thus, T is a von Neumann regular ring (cf. [8, Exercise 22, page 64]). However, whenever S is a multiplicatively closed subset of a ring R, we have that $\mathrm{Spec}(R_S)$ is homeomorphic to a subspace of $\mathrm{Spec}(R)$ (using the Zariski topology here and later, of course). In particular, $\mathrm{Spec}(T)$ is homeomorphic to a subspace of a connected space, hence is itself connected, and so T is a connected ring. But any nonzero connected von Neumann regular ring is a field (since each of its nonzero principal ideals is generated by an idempotent and thus must be the entire ring). Thus T is a field, and so R, as a subring of T, must be an integral domain, contradicting $|Z(R)| > 1$. □

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Straight rings, II

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Abstract. A (commutative integral) domain A is called a straight domain if $A \hookrightarrow B$ is a prime morphism for each overring B of A; i.e., if B/PB is a torsion-free A/P-module for each $P \in \mathrm{Spec}(A)$. It is known that each straight domain is a going-down domain, but not conversely; and that each locally divided domain is straight. We obtain new characterizations of prime morphisms by using, i.a., weak Bourbaki associated primes and attached primes. Applications include a characterization of straight domains within the universe of quasi-Prüfer domains, as being the going-down domains for which certain related total quotient rings are Artinian. We also characterize the straight domains within the universes of i-domains and of treed domains. Sufficient conditions are given for the "straight domain" property to be inherited by all overrings. Some new classes of going-down domains are introduced, leading to a characterization of divided domains within the class of straight domains.

Keywords. Prime morphism, torsion-free, integral domain, straight domain, weak Bourbaki associated prime, primal ideal, quasi-Prüfer domain, divided domain, i-domain, going-down, incomparability, going-down domain, n-almost valuation domain, conductor overring, Krull dimension.

AMS classification. 13G05, 13B24, 13B99, 13A15, 13F05, 13E10.

1 Introduction and notation

All rings considered below are commutative with 1 and typically nonzero; all ring homomorphisms are unital. We next collect the notation that we use in connection with any ring A: $\mathrm{Z}(A)$ is the set of all zero-divisors of A; $\mathrm{Reg}(A) := A \setminus \mathrm{Z}(A)$ the set of all regular elements of A; $\mathrm{Tot}(A)$ the total quotient ring of A; $\mathrm{Spec}(A)$ (resp., $\mathrm{Min}(A)$) the set of all prime (resp., minimal prime) ideals of A; and $\dim(A)$ the (Krull) dimension of A. The height of a prime ideal P is denoted by $\mathrm{ht}(P)$. The radical of an ideal I of A is denoted by $\sqrt{I}$. An *overring of* A is a subring of $\mathrm{Tot}(A)$ that contains A as a subring, that is, a ring B such that $A \subseteq B \subseteq \mathrm{Tot}(A)$. The integral closure of A (in $\mathrm{Tot}(A)$) is denoted by A'. If I is an ideal of A, then $\mathrm{V}_A(I) := \{P \in \mathrm{Spec}(A) \mid I \subseteq P\}$. If $f : A \to B$ is a ring homomorphism, then ${}^af : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ denotes the canonically induced map; f is called an *i-morphism* if ${}^af$ is an injection. Extending the usage in [14, page 28], we let GD and INC refer to the going-down and incomparability properties, respectively, of ring homomorphisms (not just of ring extensions).

This paragraph summarizes some key material from [8], while the next paragraph indicates the focus of the present sequel. As in [7], a ring homomorphism $f : A \to B$ is called a *prime morphism* if B/PB is torsion-free over A/P for each $P \in \mathrm{Spec}(A)$. A characterization that we use in Section 2 is the following: f is a prime morphism if and only if $PB_P \cap B = PB$ for each $P \in \mathrm{Spec}(A)$ [8, Proposition 2.2 (a)]. A ring

A is called *extensionally straight* if $A \hookrightarrow B$ is a prime morphism for each overring B of A. An extensionally straight domain is called a *straight domain*. A ring A such that A/P is a straight domain for all $P \in \mathrm{Spec}(A)$ is called a *straight ring*. A domain is a straight ring if and only if it is a straight domain [8, Theorem 4.6 (a)]. Since flat ring homomorphism $\Rightarrow$ prime morphism $\Rightarrow$ GD, we have that Prüfer domain $\Rightarrow$ straight domain $\Rightarrow$ going-down domain. Moreover, locally divided ring $\Rightarrow$ straight ring $\Rightarrow$ going-down ring. These implications can be reversed for seminormal weak Baer rings [8, Theorem 3.12], but neither of these implications is reversible in general. Although [8, Example 4.4] constructs a quasilocal two-dimensional going-down domain which is not a straight domain, there is no known example of a straight domain which is not a locally divided domain; equivalently, there is no known example of a quasilocal straight domain which is not a divided domain (in the sense of [6]).

As outlined below, this paper deepens the earlier study in [8], with special emphasis on quasi-Prüfer domains (also known as INC-domains) and i-domains. (Background on INC-domains and i-domains will be recalled as needed.) For partial motivation for this focus, note that the above-mentioned domain in [8, Example 4.4] is an i-domain (and hence an INC-domain). Much of this work examines the relationship between straight domains and divided domains, bearing in mind that [8, Proposition 4.18 (b), (c)] showed that any quasilocal straight domain does have some divided-like behavior. In particular, we develop new characterizations of prime morphisms, characterize the straight domains within several classes of domains, give sufficient conditions for the "straight domain" property to pass to overrings, and introduce new classes of going-down domains that lead to a characterization of divided domains within the class of straight domains.

In view of the torsion-theoretic motivation for the concept of a prime morphism, we devote much of Section 2 to studies involving weak Bourbaki associated primes and attached primes (in the sense of [4], [16]). To complement [8, Proposition 2.2], some useful characterizations of prime morphisms in terms of these concepts are given in Proposition 2.1. One consequence (Corollary 2.3) is a characterization of straight domains within the universe of INC-domains, as being the going-down domains for which certain related total quotient rings are Artinian. Another consequence, Corollary 2.5, shows that if A is a straight domain with quotient field K, I is a nonzero finitely generated ideal of A and P is a prime ideal of A, then the fractional overring $B := (I :_K I)$ satisfies the following two conclusions: B/PB has compact minimal spectrum (in the Zariski topology); and $(B/PB)_P$ $(:= (B/PB)_{A \setminus P})$ has (Krull) dimension 0.

Section 3 begins by complementing the characterizations of prime morphisms in [8] and Proposition 2.1 by adding a few more in Proposition 3.1. One of the relevant concepts there is that of a primal ideal, in the sense of [11]. Upshots include characterizations of the straight domains within the universe of treed domains (Proposition 3.4), within the universe of quasi-Prüfer domains (Proposition 3.5), and within the universe of i-domains (Corollary 3.6). In the second and third of these results, primary ideals play some significant roles.

Section 4 introduces the concept of a surjectively prime morphism. Any surjectively prime morphism is an i-morphism and, for the special case of an integral ring extension, must be a prime morphism. The "surjectively prime" concept is used to find a number of sufficient conditions for an overring of a straight domain to be a straight domain: cf. Proposition 4.3, Corollary 4.4 and Proposition 4.7.

Section 5 introduces some new classes of going-down domains. One of these, the class of quasi-divided domains, figures in a characterization of divided domains within the class of straight domains (Proposition 5.3). Another of these, the class of n-AVDs (or n-almost valuation domains), gives a new universe within which the concepts of "straight domain" and "divided domain" are equivalent (Corollary 5.6).

Section 6 studies the conductor overrings of a quasilocal i-domain A of finite (Krull) dimension. These overrings are the simplest examples of the fractional overrings that were considered in [8, Sections 3 and 5]. Corollary 6.3 shows that for A as above, A is a divided domain if and only if A is a straight domain whose nonzero nonmaximal prime ideals have their heights determined by conductor overrings in a specific way.

In addition to the above notation, we also use standard notation for conductors $(A :_C B)$, occasionally deleting "C" if no confusion is possible; and $\subset$ denotes proper inclusion. Any unexplained material is standard, as in [4], [14].

2 Connections with associated or attached prime ideals

It is well known that the torsion of a module M is closely related to the associated prime ideals of M. Thus, it is not surprising that results on prime morphisms can be obtained by considering associated prime ideals. We will mainly use the weak Bourbaki associated prime ideals and the attached prime ideals (of Northcott). Background on these topics will be recalled or cited as needed. First, we recall some useful notation. Let A be a ring. If X is a subset of $\mathrm{Spec}(A)$ then, as in [7], $\mathfrak{U}(X)$ denotes the union of all the elements of X.

Let M be an A-module. A prime ideal P of A is a *weak Bourbaki associated prime ideal of* M if there exists (a necessarily nonzero element) $x \in M$ such that P is a minimal element of $\mathrm{V}_A(0 :_A x)$ (with respect to inclusion). The set of all weak Bourbaki associated prime ideals of M is denoted by $\mathrm{Ass}_A(M)$; and $\mathrm{Ass}_A(M) \neq \varnothing \Leftrightarrow M \neq 0$. We set $\mathrm{Ass}(A) := \mathrm{Ass}_A(A)$. It is known that $\mathrm{Z}(A) = \mathfrak{U}(\mathrm{Ass}(A))$. Moreover, if I is an ideal of A and M is an A/I-module, then $\mathrm{Ass}_A(M) = {}^a\pi(\mathrm{Ass}_{A/I}(M))$, where $\pi : A \to A/I$ is the canonical ring homomorphism. The reader is referred to [15] and [16] for additional background and references.

We will also need the attached prime ideals, as defined by Northcott and studied subsequently in, i.a., [9], [12], [20]. A prime ideal P of a ring A is an *attached prime ideal of* M if, for every finitely generated ideal $I \subseteq P$, there exists $x \in M$ such that $I \subseteq (0 :_A x) \subseteq P$. The set of all attached prime ideals of M is denoted by $\mathrm{Att}_A(M)$; and $\mathrm{Att}_A(M) \neq \varnothing \Leftrightarrow M \neq 0$. It is known that $\mathrm{Z}(A) = \mathfrak{U}(\mathrm{Att}(A))$. Moreover, if I is an ideal of A and M is an A/I-module, then $\mathrm{Att}_A(M) = {}^a\pi(\mathrm{Att}_{A/I}(M))$, where $\pi : A \to A/I$ is the canonical ring homomorphism.

We next collect four useful facts. Let M be an A-module.

(a) $\mathrm{Ass}_A(M) \subseteq \mathrm{Att}_A(M)$.

(b) Each element of $\mathrm{Att}_A(M)$ is a union of some elements of $\mathrm{Ass}_A(M)$ [20, Section 5].

Let $f : A \to B$ be a ring homomorphism and M a B-module.

(c) $\mathrm{Ass}_A(M) \subseteq {}^af(\mathrm{Ass}_B(M))$, with equality if f is flat [15].

(d) $\mathrm{Att}_A(M) = {}^af(\mathrm{Att}_B(M))$ [12, Proposition 2.1].

The next result establishes some useful characterizations of prime morphisms in terms of associated or attached prime ideals. As usual, if $P \in \mathrm{Spec}(A)$, then $\mathbf{k}(P)$ denotes A_P/PA_P.

Proposition 2.1. *Let $f : A \to B$ be a ring homomorphism. Then the following six conditions are equivalent:*

(1) *f is a prime morphism;*

(2) *${}^af(\mathrm{Ass}_B(B/PB)) = \{P\}$ for each $P \in \mathrm{Spec}(A)$ that survives in B;*

(3) *$\mathrm{Ass}_A(B/PB) = \{P\}$ for each $P \in \mathrm{Spec}(A)$ that survives in B;*

(4) *${}^af(\mathrm{Att}_B(B/PB)) = \{P\}$ for each $P \in \mathrm{Spec}(A)$ that survives in B;*

(5) *$\mathrm{Att}_A(B/PB) = \{P\}$ for each $P \in \mathrm{Spec}(A)$ that survives in B;*

(6) *$A \to B$ induces a ring homomorphism $\mathbf{k}(P) \to \mathrm{Tot}(B/PB)$ for each $P \in \mathrm{Spec}(A)$ that survives in B.*

If (any of the above equivalent conditions) (1)–(6) hold(s), then f satisfies GD and $f^{-1}(W) = P$ for each $W \in \mathrm{Min}(\mathrm{V}(PB))$, whence $f^{-1}(\sqrt{PB}) = P$.

Proof. Since any prime morphism satisfies GD, the final assertion follows from a standard fact about GD (cf. [14, Exercise 37, page 44]).

(6) $\Rightarrow$ (1). Assume (6). We must show that if $P \in \mathrm{Spec}(A)$, then B/PB is a torsion-free A/P-module. This is clear if $PB = B$, and so, without loss of generality, P survives in B. The conclusion follows from a comment in the next-to-last paragraph of the introduction of [8], as (6) ensures that $A/P \to B/PB$ is the restriction of a ring homomorphism $\mathrm{Tot}(A/P) \to \mathrm{Tot}(B/PB)$.

(1) $\Rightarrow$ (6). Assume (1). Let P be a prime ideal of A that survives in B. As (1) ensures that B/PB is a torsion-free A/P-module, the composite of $A/P \to B/PB$ and the inclusion map $B/PB \to \mathrm{Tot}(B/PB)$ sends $\mathrm{Reg}(A/P) = (A/P) \setminus \{0\}$ into $\mathrm{Reg}(B/PB) \subseteq \mathrm{U}(\mathrm{Tot}(B/PB))$. Accordingly, the universal mapping property of rings of fractions yields the desired ring homomorphism $\mathbf{k}(P) \to \mathrm{Tot}(B/PB)$, thus proving (6).

(4) $\Rightarrow$ (2) by (a); (2) $\Rightarrow$ (4) by (b); (4) $\Leftrightarrow$ (5) by (d); (5) $\Rightarrow$ (3) by (a); and (3) $\Rightarrow$ (5) by (b).

(1) $\Rightarrow$ (2). Assume (1), and let P be a prime ideal of A that survives in B. Consider any $Q \in \mathrm{Ass}_B(B/PB)$. We must show that ${}^af(Q) = P$. By the definition of a weak Bourbaki associated prime, Q is a minimal element of $\mathrm{V}_B(PB :_B b)$ for some $b \in B$. Suppose that $a \in A$ is such that $f(a) \in Q$. Then $a^n sb = f(a^n)sb \in PB$ for some

$s \in B \setminus Q$ and some positive integer n, by [4, Proposition 12, page 73]. Note that $sb \notin PB$ (for, otherwise, $s \in (PB :_B b) \subseteq Q$, a contradiction). As $sb \notin PB$ and f is a prime morphism, we have that $a \in P$. Thus, ${}^a f(Q) \subseteq P$. Moreover, $PB \subseteq Q$ gives the reverse inclusion, and so (2) holds.

(2) $\Rightarrow$ (1). Assume (2). Let $P \in \mathrm{Spec}(A)$, $a \in A$ and $b \in B \setminus PB$ such that $ab = f(a)b \in PB$. Our task is to show that $a \in P$. Since $(PB :_B b) \neq B$, we can choose a minimal element, say Q, of $\mathrm{V}_B(PB :_B b)$; of course, $Q \in \mathrm{Ass}_B(B/PB)$. Since $f(a) \in (PB :_B b) \subseteq Q$ and it follows from (2) that ${}^a f(Q) = P$, we can conclude that $a \in P$, as desired. □

We next offer some considerations involving homomorphisms associated to the overrings of an extensionally straight ring. Then we give some topological consequences; observe some divided-like behavior of straight domains; and close the section with a remark summarizing two parallel theories that focus on certain subclasses of prime morphisms defined in terms of associated or attached prime ideals.

Theorem 2.2. *Let A be an extensionally straight ring and B an overring of A such that B is an A-algebra of finite type. Let P be a prime ideal of A that survives in B. Then:*

(a) *The total quotient ring of B/PB is Noetherian and* $\mathrm{Min}(B/PB)$ *is a finite set.*

(b) *If, in addition, the inclusion map $A \to B$ satisfies INC, then the total quotient ring of B/PB is Artinian and* $\mathrm{Min}(B/PB) = \mathrm{Ass}(B/PB)$.

Proof. (a) Denote the fiber of $A \to B$ at P by $F_B(P) := \mathbf{k}(P) \otimes_A B \cong B_P/PB_P$. Since B is a finite-type A-algebra, $F_B(P)$ is a finite-type $\mathbf{k}(P)$-algebra and hence (by the Hilbert Basis Theorem) a Noetherian ring.

Since $A \to B$ is a prime morphism and $PB \neq B$, it follows from Proposition 2.1 that $A/P \to B/PB$ induces a ring homomorphism $\mathbf{k}(P) \to \mathrm{Tot}(B/PB)$. By tensoring this with the canonical map $B \to \mathrm{Tot}(B/PB)$ and multiplying, one obtains a ring homomorphism $\beta : \mathbf{k}(P) \otimes_A B \to \mathrm{Tot}(B/PB)$. On the other hand, the canonical ring homomorphism $\alpha : B/PB \to B_P/PB_P$ is an injection, since [8, Proposition 2.2 (a)] ensures that $PB_P \cap B = PB$ (in light of $A \to B$ being a prime morphism). Note that the inclusion map $B/PB \to \mathrm{Tot}(B/PB)$ factors as the composite of α and β. Note also that α is a flat epimorphism, because it is inferred from the flat epimorphism $A/P \to \mathbf{k}(P)$ by base change. Since any flat epimorphism is essential [15, Proposition 2.1, page 111], it follows that β is an injection. Thus, $F_B(P)$ is a Noetherian overring of B/PB, and so $\mathrm{Tot}(B/PB) = \mathrm{Tot}(F_B(P))$ is a Noetherian ring.

It remains to prove that $\mathrm{Min}(B/PB)$ is a finite set. Since the Noetherian property ensures that $\mathrm{Min}(\mathrm{Tot}(B/PB))$ is a finite set, it suffices to show that the canonical map $\mathrm{Spec}(\mathrm{Tot}(B/PB)) \to \mathrm{Spec}(B/PB)$ restricts to a surjection $\mathrm{Min}(\mathrm{Tot}(B/PB)) \to \mathrm{Min}(B/PB)$. Standard facts imply that each minimal prime ideal of the base ring of any ring extension is lain over by some minimal prime ideal (cf. [14, Exercise 1, page 41; and Theorem 10]). Therefore, the assertion follows because the inclusion map $B/PB \to \mathrm{Tot}(B/PB)$ satisfies GD (as a consequence of the flatness of $\mathrm{Tot}(B/PB)$ over B/PB).

(b) It is well known that $\mathrm{Spec}(F_B(P))$ is order-isomorphic to the set of all the prime ideals of B that lie over P. Since we are assuming that $A \to B$ satisfies INC, it follows that $F_B(P)$ is zero-dimensional. As we saw above that $F_B(P)$ is also Noetherian, it must be Artinian. Hence (passing to a ring of fractions), so is $\mathrm{Tot}(F_B(P)) = \mathrm{Tot}(B/PB)$.

It remains to prove that $\mathrm{Min}(B/PB) = \mathrm{Ass}(B/PB)$. We claim that if C is any ring and $g : C \to \mathrm{Tot}(C)$ the inclusion map, then ${}^a g(\mathrm{Ass}(\mathrm{Tot}(C))) = \mathrm{Ass}(C)$. Consider the multiplicatively closed set $S := \mathrm{Reg}(C) := C \setminus \mathrm{Z}(C)$. By [4, Exercise 17 (d), page 289], the assignment $P \mapsto P_S$ gives a bijection from $\{P \in \mathrm{Ass}(C) \mid P \cap S = \varnothing\}$ to $\mathrm{Ass}(C_S)$, whose inverse map is induced by the canonical contraction map. Of course, $C_S = \mathrm{Tot}(C)$, and *each* $P \in \mathrm{Ass}(C)$ satisfies $P \cap S = \varnothing$ since $C \setminus S = \mathrm{Z}(C) = \mathfrak{U}(\mathrm{Ass}(C))$. The upshot is a bijection between $\mathrm{Ass}(C)$ and $\mathrm{Ass}(\mathrm{Tot}(C))$, whose inverse map is induced by contraction. This proves the claim.

It is known that if C is an Artinian ring, then $\mathrm{Min}(C) = \mathrm{Ass}(C)$. (In detail, when C is Artinian, the corresponding fact follows for the classical notion of associated primes by [4, Corollary 2, page 274], but that classical notion leads to the same $\mathrm{Ass}(C)$ as we built via weak Bourbaki associated primes because C is Noetherian [4, Exercise 17 (g), page 289].) In particular, $\mathrm{Min}(\mathrm{Tot}(B/PB)) = \mathrm{Ass}(\mathrm{Tot}(B/PB))$. Hence, the claim that was established above allows us to conclude that ${}^a g(\mathrm{Min}(\mathrm{Tot}(B/PB))) = \mathrm{Ass}(B/PB)$. Therefore, it suffices to show that ${}^a g(\mathrm{Min}(\mathrm{Tot}(B/PB))) = \mathrm{Min}(B/PB)$. But this *was* shown in the course of proving (a). This completes the proof of (b). □

Following [19], we say that a domain A is an INC-*domain* if the inclusion map $A \to B$ satisfies INC for each overring B of A. An equivalent condition is that A be a quasi-Prüfer domain, in the sense that the integral closure A' of A is a Prüfer domain [19, Proposition 2.26]. The literature offers many classes of domains A for which A' is a Prüfer domain.

Corollary 2.3. *Let A be an* INC*-domain. Then the following conditions are equivalent:*

(1) *A is a going-down domain, and* $\mathrm{Tot}(B/PB)$ *is an Artinian ring whenever B is an overring of finite type over A and P is a prime ideal of A that survives in B;*

(2) *A is a straight domain.*

Proof. (2) ⇒ (1). Recall that any straight domain is a going-down domain. As the inclusion map $A \to B$ satisfies INC for each overring B of A, the assertion now follows from Theorem 2.2 (b).

(1) ⇒ (2). Assume (1). By [8, Proposition 2.4], it is enough to prove that the inclusion map $i : A \to B$ is a prime morphism for each overring B of A which is a finite-type A-algebra. By reworking the proof of Theorem 2.2, we see that if P is a prime ideal of A that survives in B, then $\mathrm{Min}(B/PB) = \mathrm{Ass}(B/PB)$. Next, note from fact (c) earlier in this section that if $\pi : B \to B/PB$ is the canonical surjection, then $\mathrm{Ass}_B(B/PB)$ is a subset of ${}^a\pi(\mathrm{Ass}(B/PB))$ and, hence, is a subset of ${}^a\pi(\mathrm{Min}(B/PB))$. By Proposition 2.1 [(2) ⇒ (1)], it suffices to show that if $q \in \mathrm{Min}(B/PB)$, then ${}^a i({}^a\pi(q)) = P$. An equivalent task is to show that if Q is minimal as a prime ideal of B that contains PB, then $Q \cap A = P$. This, in turn, follows because

the hypothesis that A is a going-down domain ensures that $A \to B$ satisfies GD. The proof is complete. $\square$

Proposition 2.4. *Let A be an extensionally straight ring (with integral closure A'), let B be an overring of A such that the inclusion map $f : A \to B$ satisfies INC, and P be a prime ideal of A that survives in B. Then:*

(a) *The fiber B_P/PB_P is a zero-dimensional ring.*

(b) *The minimal spectrum* $\mathrm{Min}(B/PB)$ *is compact (in the Zariski topology).*

(c) $\mathrm{Min}(A'/QA')$ *is compact and A'_Q/QA'_Q is a zero-dimensional ring for each* $Q \in \mathrm{Spec}(A)$.

(d) *If, in addition, A is a quasilocal (straight) domain and* $Q \in \mathrm{Spec}(A)$, *then $QA' = QA'_Q$, or equivalently, $QA' = aQA'$ for each $a \in A \setminus Q$.*

Proof. (a) Observe that any prime ideal Q of B that lies over P must contain PB. By Zorn's Lemma, Q contains some prime ideal M which is minimal with respect to containing PB. It suffices to show that $Q = M$. As f is a prime morphism, it satisfies GD, and so $M \cap A = P$. Hence, INC ensures that $Q = M$.

(b) Since f is a prime morphism, [8, Proposition 2.2 (a)] yields that the canonical map $g : B/PB \to B_P/PB_P$ is an injection. Moreover, g is a flat epimorphism since it is obtained from the flat epimorphism $A/P \to \mathbf{k}(P)$ by change of base. As is well known [15, Proposition 1.5, page 49], these conditions ensure that the induced map ${}^a g : \mathrm{Spec}(B_P/PB_P) \to \mathrm{Spec}(B/PB)$ is an injection. However, the flatness of g ensures that g satisfies GD (cf. [14, Exercise 37, page 44]). As all minimal prime ideals are lain over in any extension [14, Exercise 1, page 41] and $\dim(B_P/PB_P) = 0$ by (a), the upshot is that the image of ${}^a g$ is $\mathrm{Min}(B/PB)$. Then (b) follows from the fact that any continuous image of a compact space is compact.

(c) It is well known that any integral ring extension satisfies INC. Thus, (c) follows by applying (a) and (b), with $B := A'$.

(d) Since A is a quasilocal going-down domain, it follows from [6, Lemma 2.4 (a)] that $A \subseteq A + QA_Q$ is an integral (and unibranched) ring extension. Hence, $A' + QA'_Q$ is an integral overring of A' (because $QA'_Q = (QA_Q)A'$), and so $QA'_Q \subseteq A'$. Since the "equivalently" assertion results from a routine calculation, it remains only to show that $QA'_Q = QA'$. As one inclusion is trivial, we will show that $QA'_Q \subseteq QA'$. Consider any element $\xi \in QA'_Q$. Then there exists $z \in A \setminus Q$ such that $z\xi \in QA'$. Since the inclusion map $A \to A'$ is a prime morphism and $\xi \in A'$, it follows that $\xi \in QA'$, as desired. $\square$

Corollary 2.5. *Let A be a straight domain with quotient field K, let I be a finitely generated ideal of A, and let P be any prime ideal of A. Then:*

(a) *$(I_P :_K I_P)/P(I_P :_K I_P)$ is a zero-dimensional ring.*

(b) *The ring $(I :_K I)/P(I :_K I)$ has compact minimal spectrum.*

Proof. Since I is a finitely generated ideal of A, we have that $(I :_K I)$ is an integral overring of A and $(I :_K I)_P = (I_P :_K I_P)$. As integral ring extensions satisfy INC and the lying-over property, the assertions follow directly from parts (a) and (b) of Proposition 2.4, with $B := (I :_K I)$. □

Note that Proposition 2.4 (d) implies that any integrally closed quasilocal straight domain must be a divided domain, but we know more generally (cf. [6]) that any integrally closed quasilocal going-down domain is a divided domain. The next two results give some kinds of divided-like behavior for certain quasilocal going-down domains.

Corollary 2.6. *Let A be a quasilocal straight domain and P a prime ideal of A. Then:*

(a) *P is comparable to any ideal of A which is contracted from A'.*

(b) *If $a \in A$, then either $A'a \cap A \subseteq P$ or $P \subseteq A'a \cap A$.*

(c) *P is comparable to each radical ideal of A, to each valuation ideal of A (i.e., each ideal which is the contraction of an ideal of some valuation overring of A), to each common ideal of A and A', to the integral closure I' of any ideal I of A and hence to any integrally closed ideal of A.*

Proof. (a), (b) By Proposition 2.4 (d), $PA' = PA'_P$, or equivalently, $PA' = aPA'$ for each $a \in A \setminus P$. Thus, if I is an ideal of A such that $I \nsubseteq P$, then $PA' \subseteq IA'$, and so $P \subseteq IA' \cap A$. Then (a) follows easily; and (b) is a special case of (a).

(c) In view of (a), it is enough to prove that P is comparable to I'. Recall that if I is an ideal of A, then the integral closure of I is defined to be

$$I' := \{a \in A \mid \text{there exist an integer } n > 0 \text{ and elements } a_i \in I^i,\ i = 1, \dots, n, \text{ such that } a^n + a_1 a^{n-1} + \cdots + a_n = 0\}.$$

It is well known that $I' = \cap IV \cap A$, where the index V runs through the set of all valuation overrings of A. In particular, I' is a contracted ideal from A', and so (c) follows from (a). □

In view of [5], the set of all the contracted ideals of a given domain A is the set of all the valuation ideals of A (resp., all the integrally closed ideals of A) if and only if A' is a valuation domain, i.e., if and only if A is a quasilocal i-domain. Now consider any quasilocal i-domain A with integral closure V. If I is an ideal of A and $P \in \operatorname{Spec}(A)$, then IV is comparable to any prime ideal of V which lies over P, and so $I' = IV \cap A$ is comparable to P. Thus, the converse of Corollary 2.6 (a) is not valid, since there exists a quasilocal i-domain which is not a straight domain.

Let A be a domain and R an overring of A. If $P \in \operatorname{Spec}(A)$, another type of integral closure $\overline{P}$ of P in R was defined in [3, page 63], as follows:

$$\overline{P} := \{r \in R \mid p(r) = 0 \text{ for some } p(X) = X^n + p_1 X^{n-1} + \cdots + p_n \in P[X]\}.$$

Let $\overline{A} := A' \cap R$, the integral closure of A in R. Then $\overline{P} = \sqrt{P\overline{A}}$ by [3, Lemma 5.14]. It follows that if A is a going-down domain (more generally, if $A \subseteq A'$ satisfies GD), then $\overline{P} = \cap\{Q \in \operatorname{Spec}(\overline{A}) \mid Q \cap A = P\}$.

Proposition 2.7. *Let A be a quasilocal going-down domain and R an overring of A such that A is integrally closed in R. Then $P = PA_P \cap R$ for each $P \in$ Spec(A). Hence, a prime ideal P of A is divided if $A_P \subseteq R$.*

Proof. Without loss of generality, P is nonzero and distinct from the maximal ideal of A. As A is integrally closed in R, it follows from the above remarks that $\overline{P} = P$. By reworking the proof of [6, Lemma 2.4 (a)], we can show that $PA_P \cap R \subseteq \overline{P}$. Hence, $P = PA_P \cap R$. The final assertion is then clear. □

The final remark of the section considers some similar theories that result from tweaking the definition of extensionally straight rings.

Remark 2.8. We describe the basics of two theories that achieve somewhat greater stability by focussing on particular types of prime morphisms. Let $f : A \to B$ be a ring homomorphism. We say that f is an Ass-*homomorphism* if ${}^af(\mathrm{Ass}_B(B/IB)) \subseteq \mathrm{Ass}_A(A/I)$ for each ideal I of A. Similarly, we say that f is an Att-*homomorphism* if ${}^af(\mathrm{Att}_B(B/IB)) \subseteq \mathrm{Att}_A(A/I)$ for each ideal I of A. It follows from Proposition 2.1 that all Ass-homomorphisms and all Att-homomorphisms are prime morphisms. It can be shown that each flat ring homomorphism is an Att-homomorphism and that all Ass-homomorphisms and all Att-homomorphisms are torsion-free. Moreover, the classes of Ass-homomorphisms and Att-homomorphisms are each stable under composition (an improvement over the situation for arbitrary prime morphisms as reported in [8, Proposition 2.6]).

Let us say that a ring A is an *extensionally strong straight ring* if the inclusion map $A \to B$ is an Att-homomorphism for each overring B of A. The definitions of *strong straight domain* and *strong straight ring* parallel the corresponding definitions of straight domains and straight rings. Since each Att-homomorphism is a prime morphism, each strong straight domain is a straight domain. Since each flat homomorphism is an Att-homomorphism, each Prüfer domain is a strong straight domain. Any domain of dimension at most 1 is a strong straight domain. If A is a Noetherian domain, then: A is a strong straight domain $\Leftrightarrow$ $\dim(A) \leq 1$ $\Leftrightarrow$ A is a straight domain $\Leftrightarrow$ A is a going-down domain $\Leftrightarrow$ A is a weak straight domain $\Leftrightarrow$ A is a locally divided domain.

3 Prime, primal and primary

We begin the section by giving some new characterizations of prime morphisms, complementing some results of [8]. Recall that an ideal I of a ring A is said to be *primal* if the set of all $x \in A$ such that $(I : x) \supset I$ (equivalently, such that the canonical image of x in A/I is a zero-divisor of A/I) is an ideal P, necessarily prime. This prime ideal P is then called the *adjoint ideal* of the primal ideal I [11, Preliminaries], and we will say that I is P*-primal*. Note that $I \subseteq P$ for any P-primal ideal $I \neq A$.

Let $A \subseteq B$ be a ring extension (or $A \to B$ an injective ring homomorphism) and J an ideal of the ring B. If $Q \in$ Spec(A), let $\mathrm{Sat}_Q(J)$ denote the canonical inverse image of J_Q in B; i.e., $\mathrm{Sat}_Q(J) = \{x \in B \mid$ there exists $s \in A \setminus Q$ such that $sx \in J\}$. We next show that this concept is related to the background on weak

Bourbaki associated primes that was recalled in Section 2, together with the fact that $\sqrt{J} = \cap \{Q \mid Q \in \mathrm{Ass}_A(J)\}$.

Proposition 3.1. *Let $f : A \to B$ be an injective ring homomorphism and $P \in \mathrm{Spec}(A)$ such that $PB \neq B$. Then the following six conditions are equivalent:*

(1) *$f : A \to B$ is prime at P;*

(2) *If $b \in B \setminus PB$, then $P = (PB :_A b)$;*

(3) *${}^a\!f(\mathrm{Ass}_A(PB)) = \{P\}$;*

(4) *$\mathrm{Sat}_P(PB) = PB$;*

(5) *PB is the intersection of some family $\{J_i\}_{i \in I}$ of Q_i-primal ideals J_i of B such that $Q_i \in \mathrm{Spec}(B)$ and $Q_i \cap A = P$ for each $i \in I$.*

(6) *$(PB :_A b) \subseteq P$ for each $b \in B \setminus PB$.*

Moreover, when (5) holds, $J_i \cap A = P$ for each $i \in I$.

Proof. If (1) holds, with $b \in B \setminus PB$ and $a \in (PB :_A b)$, then $ab \in PB$ implies $a \in P$; thus, (1) $\Rightarrow$ (2). If (2) holds, with $a \in A \setminus P$ and $b \in B \setminus PB$ such that $ab \in PB$, then $a \in P$, which is absurd; thus, (2) $\Rightarrow$ (1). Then (1) is clearly equivalent to (4) and to (6). Moreover, (1) $\Leftrightarrow$ (3) by the proof of Proposition 2.1.

(1) $\Rightarrow$ (5). Assume (1). Using [11, Theorem 3.5 and Lemma 2.2], we see that $PB = \cap\{\mathrm{Sat_Q(PB)} \mid \mathrm{Q} \in \mathcal{X}_{\mathrm{PB}}\}$, where $\mathcal{X}_{PB}$ denotes the set of all maximal elements in the set of all prime ideals that are the unions of elements in $\mathrm{Ass_A(PB)}$. Moreover, each $\mathrm{Sat_Q(PB)}$ is a Q-primal ideal. As Q is a union of some elements in $\mathrm{Ass_A(PB)}$, we have $Q \cap A = P$, by (3). To obtain (5), note that $J_i \subseteq Q_i$ for all i.

(5) $\Rightarrow$ (1). Assume (5). Consider $a \in P$ and $b \in B$ such that $ab \in PB$ and $b \notin PB$. Then there exists an index i such that $b \notin J_i$ and $ab \in J_i$. Since J_i is a Q_i-primal ideal, $a \in Q_i$, whence $a \in Q_i \cap A = P$, thus yielding (1). □

Recall that a ring A is called *extensionally straight* if the inclusion map $A \to B$ is a prime morphism for each overring B of A. Bearing in mind that an empty intersection of ideals of an overring B is conventionally taken to be B, we see that the preceding proposition has the following immediate consequence.

Corollary 3.2. *Let A be a ring. Then A is extensionally straight if and only if, for each $P \in \mathrm{Spec}(A)$ and each overring B of A, PB is the intersection of some Q_i-primal ideals such that $Q_i \in \mathrm{Spec}(B)$ and $Q_i \cap A = P$ for each i.*

Proposition 3.1 specializes as follows in case A is an i-domain.

Corollary 3.3. *Let A be an i-domain, B an overring of A, and $P \in \mathrm{Spec}(A)$ such that $PB \neq B$. Then the following conditions are equivalent:*

(1) *The inclusion map $A \to B$ is prime at P;*

(2) *PB is a Q-primal ideal of B for some $Q \in \mathrm{Spec}(B)$ such that $Q \cap A = P$;*

(3) $\mathrm{Ass}_A(B/PB)$ *has only one element;*

(4) PB *is a primary ideal of* B *such that* $\sqrt{PB} \cap A = P$.

Moreover, when (2) *holds,* $\mathrm{Z}(PB) = Q$.

Proof. The assertions follow easily from Proposition 3.1 and the proof of Proposition 2.1, since A being a going-down domain ensures that each minimal prime ideal of PB lies over P. □

We next recall some material from [22, Section 1]. Let A be a domain. Let $\mathrm{Specass}(A)$ (resp., $\mathrm{Specp}(A)$) denote the set of all prime ideals P of A such that P is a minimal prime ideal of an ideal of A of the form $(Aa :_A Ab)$ for some $a, b \in A$ (resp., of the form Aa for some $a \in A$). Also, we let $\mathrm{Specatt}(A)$ denote the set of all prime ideals P of A for which there is some $a \in A$ such that for each finitely generated ideal $I \subseteq P$ of A, there is some $b \in A$ such that $I \subseteq (Aa :_A Ab) \subseteq P$. Then $\mathrm{Specp}(A) \subseteq \mathrm{Specass}(A) \subseteq \mathrm{Specatt}(A)$; and each nonzero prime ideal of A is the union of some elements of $\mathrm{Specp}(A)$ (resp., of $\mathrm{Specatt}(A)$; resp., of $\mathrm{Specp}(A)$). Moreover $A = \cap\{A_P \mid P \in \mathrm{Specass}(A)\}$.

Recall that the t-operation on the set of all nonzero ideals J of a domain A is defined by $J_t := \cup I_v$, where I runs through the set of all finitely generated ideals $I \subseteq J$ of A (and $I_v := (I^{-1})^{-1}$). An ideal J is called a t-ideal if $J = J_t$. According to [22, Proposition 1.23], any element of $\mathrm{Specatt}(A)$ is a prime t-ideal of A. We let $\mathrm{Spect}(A)$ denote the set of all prime t-ideals of A; then $\mathrm{Specass}(A) \subseteq \mathrm{Spect}(A)$.

The following observation will be useful in the proof of the next result. If $A \to B$ is an injective ring homomorphism and $\{P_i\}_{i \in I}$ is a directed family of prime ideals of A, then the prime ideal $P := \cup\{P_i \mid i \in I\}$ of A satisfies $\mathrm{Sat}_P(PB) \subseteq \cup\{\mathrm{Sat}_{P_i}(P_i B) \mid i \in I\}$.

We next characterize straight domains within the universe of treed domains.

Proposition 3.4. *Let* A *be a domain. Then the following conditions are equivalent:*

(1) A *is a treed domain and, for each overring* B *of* A*, the inclusion map* $A \to B$ *is prime at each* $P \in \mathrm{Specp}(A)$ *(resp., at each* $P \in \mathrm{Spect}(A)$*; resp., at each* $P \in \mathrm{Specass}(A)$*; resp., at each* $P \in \mathrm{Specatt}(A)$*);*

(2) A *is a straight domain.*

Proof. Any straight domain is a going-down domain, and hence a treed domain. Thus, if A is treed, it follows from the above remarks that each prime ideal of A is the union of a linearly ordered set contained in $\mathrm{Spect}(A)$ (resp., $\mathrm{Specass}(A)$, resp., $\mathrm{Specatt}(A)$). In view of Proposition 3.1 [(1) $\Leftrightarrow$ (4)], the conclusion easily follows from the above observation (and the fact that $PB = \cup P_i B$). □

We next show that in order to check the "straight domain" property for an INC-domain A, it is enough to consider the overrings B of A such that the inclusion map $A \to B$ makes B a finite(ly generated) A-module. We say that such an overring is *finite*. We also say that an overring B of A is *of finite type* if the inclusion map $A \to B$

makes B an A-algebra of finite type. Notice that a domain A is an i-domain if and only if A is a quasi-Prüfer domain such that the inclusion map $A \to A'$ is unibranched.

Proposition 3.5. *Let A be a quasi-Prüfer domain. Then the following conditions are equivalent:*

(1) *For each finite overring B of A, the inclusion map $A \to B$ is a prime morphism;*

(2) *For each finite overring B of A and each $P \in \mathrm{Spec(A)}$, we have $PB = Q_1 \cap \cdots \cap Q_n$, where, for all i, Q_i is a P_i-primary ideal and P_i lies over P;*

(3) *A is a treed domain and, for each finite overring B of A and each $P \in \mathrm{Specatt}(A)$ (resp., $P \in \mathrm{Specass}(A)$; resp., $P \in \mathrm{Specp}(A)$), we have $PB = Q_1 \cap \cdots \cap Q_n$, where, for all i, Q_i is a P_i-primary ideal and P_i lies over P;*

(4) *A is a straight domain.*

Moreover, if either (2) *or* (3) *holds, then $Q_i \cap A = P$ for all i.*

Proof. By [8, Proposition 2.4], A is a straight domain if and only if $A \hookrightarrow B$ is a prime morphism for each overring B of finite type over A. In this case, the fact that $A \subseteq B$ satisfies INC means that $A \hookrightarrow B$ is quasi-finite [26, Corollary 1.8]. By Zariski's Main Theorem, $A \hookrightarrow B$ can be factored as $A \to A' \to B$, where $A \to A'$ is finite and $A' \to B$ is an open immersion, that is, a flat epimorphism of finite presentation [24, Corollaire 2, page 42]. Hence A is a straight domain if and only if $A \hookrightarrow B$ is a prime morphism for each finite overring B, by [8, Propositions 2.1 (b) and 2.6 (a)]. In other words, (1) $\Leftrightarrow$ (4).

Next, an appeal to [4, Corollary 3, page 327] shows the following. Let B be a finite overring of A, and let $P \in \mathrm{Spec(A)}$. Then there are only finitely many prime ideals $P_1B_P, \dots, P_nB_P$ of B_P that lie over PA_P. Moreover, $PB_P = Q'_1 \cap \cdots \cap Q'_n$ where each $Q'_i = \mathrm{Sat}_{P_iB_P}(PB_P)$ is primary. As each P_iB_P contains PB_P, we have that $Q'_i \subseteq P_iB_P$. Each P_iB_P is maximal in B_P/PB_P, and the ideals P_iB_P are the maximal ideals of B_P that contain PB_P. Hence, each Q'_i is P_iB_P-primary. Assume, in addition, that the inclusion map $A \to B$ is prime at P. It follows that $PB = \mathrm{Sat}_P(PB) = (Q'_1 \cap B) \cap \cdots \cap (Q'_n \cap B)$ is an intersection of finitely many P_i-primary ideals, where each P_i lies over P. Thus, (1) $\Rightarrow$ (2). The converse is a consequence of Proposition 3.1. Finally, the proof of Proposition 3.4 shows that (2) $\Leftrightarrow$ (3). □

We can now infer a characterization of the straight domains which are i-domains.

Corollary 3.6. *Let A be an i-domain with integral closure A'. Then A is a straight domain if and only if PB is a primary ideal of B for each finite overring B of A and each $P \in \mathrm{Spec}(A)$ (resp., $P \in \mathrm{Specass}(A)$). If these equivalent conditions hold, then PA' is a primary ideal of A' for each $P \in \mathrm{Spec}(A)$.*

Proof. Note that any i-domain is a treed domain. If PB is a primary ideal of B, its radical lies over P when $A \hookrightarrow B$ is finite. Therefore, the first assertion follows by combining Proposition 3.5 and Corollary 3.3. To infer the final assertion from the first assertion, express A' as the direct limit of the finite overrings B of A, and then use the fact that A'/PA' is the direct limit of the rings B/PB. □

Corollary 3.7. *Let A be a quasi-Prüfer domain. Then the following conditions are equivalent:*

(1) *PB is a primary ideal of B for each $P \in \mathrm{Spec}(A)$ and each finite overring B of A;*

(2) *A is a straight i-domain.*

Proof. (2) ⇒ (1) by Corollary 3.6. Conversely, assume (1). It follows that PA' is a primary ideal of A' for each $P \in \mathrm{Spec}(A)$. Reasoning as above, we see that $A \hookrightarrow A'$ is unibranched, for $\sqrt{PA'} \cap A = P$. (To see this, note that if $Q \in \mathrm{Spec}(A')$ is such that $Q \cap A = P$, then $PA' \subseteq Q$ implies $\sqrt{PA'} \subseteq Q$, and so by incomparability, $\sqrt{PA'} = Q$.) Thus, A is an i-domain, and so (2) follows by Corollary 3.6. □

4 More properties of straight domains

The following observation of Roby [25, Théorème 4, page 11] will lead us to some new concepts and new facts concerning straight domains. If $f : A \to B$ is an epimorphism in the category of commutative rings and $Q \in \mathrm{Spec}(B)$ lies over $P \in \mathrm{Spec}(A)$, then $Q := \mathrm{Sat}_P(PB)$ is a prime ideal of B, whence ${}^af : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an injection. Thus, if A is an extensionally straight ring and B is an epimorphic overring of A, we see, using Proposition 3.1 [(1) ⇔ (4)], that $Q = PB$.

Next, recall from [8] (or [7]) that a ring homomorphism $f : A \to B$ is called *prime-producing* if, for each $P \in \mathrm{Spec}(A)$, either $PB \in \mathrm{Spec}(B)$ or $PB = B$. Each prime-producing homomorphism is a prime morphism. The above result of Roby and further considerations prompt us to introduce the following definition. We say that a ring homomorphism $f : A \to B$ is a *surjectively prime morphism* if $f^{-1}(Q)B = Q$ for each $Q \in \mathrm{Spec}(B)$. Clearly, any surjectively prime morphism f is prime at each $P \in \mathrm{Im}({}^af)$ and is an i-morphism.

Lemma 4.1. *Let A be a straight domain and B an overring of A such that the inclusion map $f : A \to B$ is a surjectively prime morphism. Then $f : A \to B$ is an i-morphism and B is a straight domain.*

Proof. By the above remarks, f is an i-morphism. To show that B is a straight domain, consider any overring R of B and any prime ideal Q of B. Put $P := Q \cap A$. By our earlier work, $QR = PR$ is an intersection of P_i-primal ideals Q_i such that $P = P_i \cap A$. Then $P_i \cap B = Q$, because $A \to B$ is an i-morphism. It follows that B is a straight domain. □

Let $f : A \to B$ be a ring homomorphism and $P \in \mathrm{Spec}(A)$. Note that $\mathrm{Sat}_P(PB) \subseteq \sqrt{PB}$ characterizes the property that f satisfies going-down to P. In this way, we see that the next result gives some properties of a very special type of homomorphism that satisfies GD.

Lemma 4.2. *Let B be an overring of a domain A such that the inclusion map $f : A \to B$ satisfies INC and GD. Suppose that* $\mathrm{Sat}_P(PB) \in \mathrm{Spec}(B)$ *whenever* $Q \in \mathrm{Spec}(B)$ *and* $P := Q \cap A$. *Then:*

(a) $\sqrt{PB} = Q = \mathrm{Sat}_P(PB)$ *whenever* $Q \in \mathrm{Spec}(B)$ *and* $P := Q \cap A$.

(b) $f : A \to B$ *is an i-morphism.*

(c) *If, in addition, $f : A \to B$ is a prime morphism, then f is surjectively prime.*

Proof. Since f satisfies GD, we get $PB \subseteq \mathrm{Sat}_P(PB) \subseteq \sqrt{PB} \subseteq \mathrm{Sat}_P(PB)$, so that $\sqrt{PB} = \mathrm{Sat}_P(PB) \subseteq Q$. Then $\mathrm{Sat}_P(PB) \subseteq Q$, $\mathrm{Sat}_P(PB) \cap A = P$, and the incomparability of f combine to yield $Q = \mathrm{Sat}_P(PB)$, thus proving (a). Then (b) is an easy consequence of (a); and (c) follows from the fact that $\mathrm{Sat}_P(PB) = PB$ when $A \to B$ is a prime morphism. □

Proposition 4.3. *Let A be a straight domain which is also a quasi-Prüfer domain. Let B be an overring of A such that* $\mathrm{Sat}_P(PB)$ *is a prime ideal of B whenever* $Q \in \mathrm{Spec}(B)$ *and* $P := Q \cap A$. *Then the inclusion map $A \hookrightarrow B$ is a surjectively prime morphism and B is a straight domain.*

Proof. In view of the above two lemmas, it is enough to observe that A being quasi-Prüfer implies that $A \hookrightarrow B$ satisfies INC; and that $\mathrm{Sat}_P(PB) = PB$ since A is a straight domain. □

Corollary 4.4. *Let A be an i-domain. Then the following conditions are equivalent:*

(1) *The inclusion map $A \to B$ is surjectively prime for each finite overring B of A;*

(2) *PB is a radical ideal of B for each finite overring B of A and each* $P \in \mathrm{Spec}(A)$ *(resp.,* $P \in \mathrm{Specp}(A)$*);*

(3) *A is a straight domain and* $\mathrm{Sat}_P(PB)$ *is a prime ideal of B for each finite overring B of A and each* $P \in \mathrm{Spec}(A)$.

Moreover, if the above equivalent conditions hold, then each integral overring of A is a straight domain.

Proof. (1) ⇒ (2). Straightforward.

(2) ⇒ (1). Assume (2). Let B be a finite overring of A and let $P \in \mathrm{Spec}(A)$. By (2), $\sqrt{PB} = PB$; let Q denote the unique prime ideal of B which is minimal over PB. Then $PB = Q$, since each minimal prime ideal of PB must lie over P because $A \subseteq B$ satisfies GD. It follows that $A \to B$ is surjectively prime, and so $A \to B$ is a prime morphism (because $A \subseteq B$ satisfies the lying-over property). Therefore, A is a straight domain, by Proposition 3.5 [(1) ⇔ (4)].

(1) ⇒ (3). Assume (1). Let $P \in \mathrm{Spec}(A)$ and let B be a finite overring of A. Then (1) implies that $\mathrm{Sat}_P(PB)$ is a prime ideal of B, since PB is a prime ideal that lies over P. Moreover, A is a straight domain, as we saw in the above proof that (2) ⇒ (1).

(3) ⇒ (1). Apply Proposition 4.3.

Finally, assume that the above equivalent conditions hold. By Lemma 4.1, each finite overring of A is a straight domain. But each integral overring B of A is a direct limit of finite overrings of A, and so an application of [8, Proposition 3.14] shows that B is a straight domain, to complete the proof. □

We pause to note that the preceding corollary admits a slight generalization, as follows. Let A be a going-down domain such that PB is a radical ideal for each $P \in \mathrm{Spec}(A)$ and each overring B of A. Then A is a straight domain. The crux of the proof is that $PB \subseteq \mathrm{Sat}_P(PB) \subseteq \sqrt{PB}$.

The next remark collects some facts about epimorphisms, some of which will figure in the proof of Proposition 4.6.

Remark 4.5. Let A be a straight domain with quotient field K. Then:

(a) If an inclusion map $A \to B$ is a flat epimorphism, then B is an extensionally straight ring. For a proof, first recall that $A_P \to B_Q$ is an isomorphism for each $Q \in \mathrm{Spec}(B)$, where $P := Q \cap A$ [15, Lemme 1.2, page 109]; as $\mathrm{Tot}(B)$ is a von Neumann regular ring [23, Lemme 2.5], the assertion follows from [8, Proposition III.5]. Also, under the above conditions, if A is a (locally) divided domain, then so is B; the crux of the proof is that each ideal J of B is of the form $(J \cap A)B$.

(b) Let B be an overring of A such that the inclusion map $A \to B$ is locally an epimorphism, i.e., $A_P \to B_Q$ is an epimorphism for each $Q \in \mathrm{Spec}(B)$, where $P := Q \cap A$. Then B is a straight domain by Lemma 4.1. Also, [21, Proposition 2.9] yields that $PB = \mathrm{Sat}_P(PB) = \sqrt{PB}$, since $A \subseteq B$ satisfies GD.

If A is a domain with quotient field K, then A is called *anodal* (or *u-closed*) if the relations $u^2 - u,\ u^3 - u^2 \in A$ for $u \in K$ imply $u \in A$. The reader is referred to, for instance, [21] for the definitions and properties of u-integral morphisms and the u-closure of A.

Proposition 4.6. *Let A be a domain with u-closure U. Then:*

(a) *If A is a straight domain, U is a straight domain.*

(b) *If A is a locally divided domain, U is a locally divided domain.*

(c) *If A is an i-domain, then A is a locally u-closed domain, hence u-closed.*

Proof. By [21, Corollary 2.23], any u-integral morphism is locally an epimorphism. By Remark 4.5 (b), the u-closure U of a straight domain A is a straight domain, because the inclusion map $A \to U$ is u-integral. This proves (a). As for the analogue for the locally divided case in (b), the proof is similar. Finally, for (c), we see via [21, Theorem 2.26] that $U = A$ if A is an i-domain and A is u-closed; as the i-domain property localizes, A is then a locally u-closed domain. □

To close the section, we give one more result where a property such as INC is enough to force the "straight domain" condition to be inherited by a certain type of overring.

Proposition 4.7. *Let B be an overring of a straight domain A such that the inclusion map $A \to B$ is prime-producing and satisfies INC. Then B is a straight domain.*

Proof. Let Q be a prime ideal of B, and put $P := Q \cap A$. Since PB is a prime ideal of B and $PB \subseteq Q$, we have $PB = Q$. An application of Lemma 4.1 completes the proof. □

5 Some going-down domains defined by ideal-theoretic properties

In this section, we introduce some new classes of going-down domains. They are generalizations of the class of divided domains and figure in a characterization of divided domains within the class of straight domains.

Definition 5.1. An almost-divided domain (an ADD) (respectively, a quasi-divided domain) is a domain A such that, for each prime ideal P of A, there is an integer $n > 0$ such that $P^n A_P = P^n$ (respectively, $P^n A_P \subseteq P$). Moreover, a domain A is called *n-divided* if n is a positive integer such that $P^n A_P = P^n$ for each $P \in \mathrm{Spec}(A)$.

The following facts are clear. The 1-divided domains are the same as the divided domains. If $m \leq n$ are positive integers, then each m-divided domain is an n-divided domain. In particular, each divided domain is n-divided, for each positive integer n. Also, each ADD is a quasi-divided domain. Each n-divided domain is an ADD and, hence, a quasi-divided domain.

Proposition 5.2. *Each quasi-divided domain is a quasilocal going-down domain.*

Proof. Let A be a quasi-divided domain. Let $M \in \mathrm{Spec}(A)$ and s a positive integer such that $M^s A_M \subseteq M$. If N is an ideal of A, then either $N \subseteq M$ or $M^s \subseteq NM$, and so either $N \subseteq M$ or $M^s \subseteq N$. In particular, it cannot be the case that M and N are distinct maximal ideals of A. Thus, A is quasilocal.

To show that A is a going-down domain, we adapt part of the proof of [6, Proposition 2.1]. If the assertion fails, there exist $P \in \mathrm{Spec}(A)$, an overring B of A, and Q minimal among prime ideals of B that contain P such that $PB \cap (A \setminus P)(B \setminus Q) \neq \varnothing$. Thus, $\sum_{i=1}^{k} p_i b_i = ab$ for some elements $p_i \in P$, $b_i \in B$, $a \in A \setminus P$ and $b \in B \setminus Q$. By hypothesis, there exists a positive integer n such that $P^n A_P \subseteq P$. Raising the previous equation to the nth power and dividing by a^n, we find that $b^n \in P^n A_P B \subseteq PB \subseteq Q$, whence $b \in Q$, the desired contradiction. □

We next give a result that was promised at the beginning of this section.

Proposition 5.3. *Let A be a domain. Then A is a divided domain if and only if A is a straight domain and a quasi-divided domain.*

Proof. The "only if" assertion follows from the above comments and the fact that each divided domain is a straight domain. For the converse, it suffices to note that if A is a straight domain and $P \in \mathrm{Spec}(A)$, then $PA_P = P + P^n A_P$ for each positive integer n [8, Theorem 4.20 (c)]. □

We next recall a definition from [2] and introduce an n-variant of it.

Definition 5.4 ([2, Definition 5.5]). Let A be a domain with quotient field K. Then A is called an *almost valuation domain* (in short, an AVD) if, for each $x \in K \setminus \{0\}$, there exists a positive integer n such that either x^n or x^{-n} belongs to A (equivalently, there exist positive integers m, n such that either $x^n \in A$ or $x^{-m} \in A$). For any positive integer n, we call A an n-AVD if, for each $x \in K \setminus \{0\}$, either $x^n \in A$ or $x^{-n} \in A$.

By [2, Theorem 5.6], a domain A is an AVD if and only if its integral closure A' is a valuation domain and $A \subseteq A'$ is a root extension. In particular, any AVD is a quasilocal i-domain. One interesting way to construct AVDs is the following. Let B be a domain, M a maximal ideal of B, and D a subring of $K := B/M$; let k denote the quotient field of D. Then, by [17, Theorem 2.2], the pullback $A := D \times_K B$ is an AVD if and only if B and D are AVDs and $k \subseteq K$ is a root extension.

Note that each n-AVD is an AVD. Also, it is clear that a 1-AVD is the same as a valuation domain. An example of a 2-AVD was provided in [1, page 2454], namely, the domain $A := \mathbb{Z}_2[[Y^2, Y^3]]$. As this domain A is quasilocal and of Krull dimension 1, it is a divided domain and, hence, a straight domain. In fact, we will show in Corollary 5.6 that within the universe of n-AVDs, there is no difference between the concepts of "divided domain" and "straight domain".

We will unify and slightly generalize two results of D. D. Anderson and Zafrullah [2, Theorem 6.12 and Theorem 6.14]. Let I be an ideal of a ring A and n a positive integer. Recall that I_n denotes the ideal of A generated by the elements of the form x^n where x runs through I. It is clear that $I_n \subseteq I^n$.

Proposition 5.5. *Let A be an n-AVD. Then $P_n A_P \subseteq P$ for each $P \in \operatorname{Spec}(A)$.*

Proof. $P_n A_P$ is generated as an A-module by the set of elements of the form x^n/s, where $x \in P$ and $s \in A \setminus P$. Consider any elements $x \in P$ and $s \in A \setminus P$. If $a := (s/x)^n \in A$, then $s^n = ax^n \in P$, whence $s \in P$ (since P is prime), a contradiction. As A is an n-AVD, it must be the case that $b := (x/s)^n \in A$. Therefore, $bs^n = x^n \in P^n \subseteq P$, which implies that $b \in P$; that is, $x^n/s^n \in P$. Hence $x^n/s = (x^n/s^n)s^{n-1} \in P$ and the assertion follows. □

We show next that for n-AVDs, the "quasi-divided domain" condition can be deleted from the statement of Proposition 5.3.

Corollary 5.6. *Let A be an n-AVD. Then A is a divided domain if and only if A is a straight domain.*

Proof. Any divided domain is a straight domain. It remains to prove that any straight domain A that is an n-AVD must be a quasi-divided domain. We will show that $PA_P \subseteq P$ for each $P \in \operatorname{Spec}(A)$. Let k be a positive integer. Note that any nonzero element of $P_k A_P$ is a sum of nonzero elements of the form $x = a^k b/s$, where $a \in P$, $b \in A$, and $s \in A \setminus P$. Given such data x, a, b and s, we next use [8, Proposition 4.18 (a)] (with $B := A$). This result applies because A is assumed to be a straight domain. Also, being an AVD, A is quasilocal. The upshot is an equation $x = \tau + a^{2k}(b^2/s^2)g(x)$, for some $\tau \in P$ and some polynomial $g \in PA[X]$. Thus, $x = \tau + a^{k+1}v$, for some $v \in A_P$. It follows that $x \in P + P_{k+1}A_P$, and so $P_k A_P \subseteq P + P_{k+1}A_P$. As $P_1 = P$, iterating the argument enough times leads to $PA_P = P_1 A_P \subseteq P_n A_P$. An application of Proposition 5.5 completes the proof. □

Let I be an ideal of a ring A and n a positive integer. Let $I_{(n)}$ denote the set of elements of the form $\epsilon_1 x_1^n + \cdots + \epsilon_p x_p^n$ where each $x_k \in I$ and each ϵ_k is a unit of A.

Proposition 5.7. *Let A be a ring and n a positive integer such that $n!$ is invertible in A. (For instance, suppose that the ring A contains a field whose characteristic is either 0 or a prime number p such that $0 < n < p$.) If I is an ideal of A, then $I^n = I_{(n)} = I_n$.*

Proof. Since $I_n \subseteq I^n$, it is enough to prove that $I^n \subseteq I_{(n)} \subseteq I_n$. Consider any subset $\{x_1, \ldots, x_n\}$ of n elements of A. For each subset H of $\{1, 2, \ldots, n\}$, let $x_H := \sum_{i \in H} x_i$. Note that $x_H \in I$ if each $x_i \in I$. It is known that $\sum_H (-1)^{n-|H|}(x_H)^n = n! x_1 \cdots x_n$. The assertion now follows easily. □

Recall that the Grothendieck characteristic $c(A)$ of a quasilocal domain (A, M) is defined to be the nonnegative integer p such that $M \cap \mathbb{Z} = p\mathbb{Z}$; that is, the characteristic of the field A/M. If $p = 0$, then $\mathbb{Q} = \mathbb{Z}_{M \cap \mathbb{Z}} \hookrightarrow A_M = A$; if $p > 0$, then $\mathbb{Z}/p\mathbb{Z} \hookrightarrow A/M$. We say that a quasilocal domain A *avoids* a positive integer n if either $c(A) = 0$ or the prime number $c(A) > n$. It follows from the above comments that if A avoids n, then $n!$ is a unit in A. With this background in hand, we can now close the section with a characterization of valuation domains in terms of the "n-AVD" concept. To motivate Proposition 5.8, note that the 2-AVD $A := \mathbb{Z}_2[[Y^2, Y^3]]$ has $c(A) = 2$, does not avoid 2, and is not a valuation domain.

Proposition 5.8. *Let A be a domain. Then the following conditions are equivalent:*

(1) *There exists a positive integer n such that A is an n-AVD that avoids n;*

(2) *A is a valuation domain.*

Proof. It is easy to see that if (2) holds, then (1) holds with $n := 1$. Conversely, assume (1). According to [13, Théorème 8.2], if B is any ring and $b \in B$, the given positive integer n leads to an equation of the form

$$n!b = \sum_{h=0}^{n-1} (-1)^{n-1-h} \binom{n-1}{h} [(b+h)^n - h^n].$$

Take $B := A'$, the integral closure of A. Then, for any $b \in A'$, each $(b+h)^n \in A$ and each $h^n \in A$ since $A \subseteq A'$ is an n-root extension. The displayed equation now yields that $n!b \in A$. But we noted above that $n!$ is a unit in A (since A avoids n), and so $b \in A$. It follows that $A' = A$. To complete the proof, use the fact that A being an AVD implies that A' is a valuation domain. □

6 Conductor overrings of i-domains

Let A be a domain with quotient field K. A *conductor overring* of A is a ring of the form $(I :_K I)$ for some ideal I of A. In this section, we study certain conductor overrings of a quasilocal i-domain. The upshot in Corollary 6.3, which can be viewed as a companion for Proposition 5.3, is a characterization of the divided domains within the universe of quasilocal straight i-domains of finite (Krull) dimension.

Recall that a quasilocal domain A is an i-domain if and only if A' is a valuation domain V, or equivalently, if and only if each overring of A is quasilocal [19, Corollary 2.15, Proposition 2.34]. We will show in Proposition 6.1 (a) that if (A, M) is a quasilocal i-domain, then $(M^n :_K M^n) \subseteq A'$ for each positive integer n. The proof will depend, in part, on facts about weak Bourbaki associated primes that were recalled in Section 2. Also, recall that if A is a domain with quotient field K, then $A' = \cup\{(I :_K I) \mid I \in \mathcal{I}_f\}$, where $\mathcal{I}_f$ denotes the set of all nonzero finitely generated ideals of A.

Proposition 6.1. *Let (A, M) be a quasilocal i-domain with quotient field K; let (V, M') denote the integral closure of A. (Note that V is a valuation domain.) Then:*

(a) *$(P^n :_K P^n) \subseteq V_P$ for each prime ideal P of A and each positive integer n. In particular, $(M^n :_K M^n) \subseteq V$ and so $(M^n :_K M^n)$ is integral over A for each positive integer n.*

(b) *Assume, in addition, that A has finite (Krull) dimension. If B is a (necessarily quasilocal) overring of A and its maximal ideal N lies over P in A, then $A_P \subseteq B \subseteq V_P := V_{A\setminus P}$ and V_P is the integral closure of B. Moreover,* $\dim(A_P) = \dim(B) = \dim(V_P)$.

(c) *Let P be a nonzero nonmaximal prime ideal of A. Then $A_P = (M^n :_K M^n)_P$ for each positive integer n. Moreover, P is a common ideal of A and $(M :_K M)$. Suppose that $P \subset M^n$ for some positive integer n (this holds for $n = 1$). Then the inclusion map $A \to (M^n :_K M^n)$ is prime at P if and only if P is a prime ideal of $(M^n :_K M^n)$, or equivalently, if and only if $P = (P :_K M^n)$.*

(d) *Suppose, in addition, that M is not a principal ideal of A. Then $(M :_K M) = \{u \in K \mid M \subseteq I_u\}$, where $I_u := (A :_A u)$ for any $u \in K$.*

Proof. (a) We consider the case $P = M$. By [10, Lemma 3.1.9], $I := M^n V$ satisfies $(I :_K I) = V_Q$, where Q is the prime ideal of V such that Q/I is the set of zero-divisors in V/I. Now, Q is the union of the prime ideals P' that are minimal over $(M^n V : r')$ for some $r' \in V$ (i.e., of the weak Bourbaki prime ideals P' associated to I). Any such ideal P' is contained in M' and contains MV, whence $P' \cap A = M$, and so $P' = M'$ as $A \subseteq V$ satisfies INC. Thus, $Q = M'$. Therefore, $(M^n : M^n) \subseteq (M^n V : M^n V) = V_{M'} = V$.

Now, let P be any nonzero prime ideal of A. As A_P is an i-domain with integral closure V_P, we see from $(P^n : P^n) \subseteq (P^n A_P : P^n A_P) \subseteq V_P$ (where the last inclusion follows from the above case) that $(P^n : P^n) \subseteq V_P$, as asserted.

(b) Since the inclusion map $A \to B$ is an i-morphism, $\dim(B) \le \dim(A) < \infty$. As $P = N \cap A$, we have $A_P \subseteq B_N = B$. Note next that the inclusion map $A_P \to B$ is an i-morphism that satisfies GD (since A_P is an i-domain and, hence, a going-down domain). It follows easily that $\dim(A_P) = \dim(B)$. Then $V_P = (A')_P = (A_P)' \subseteq B'$, and B' has the same finite (Krull) dimension as the valuation domain V_P. Hence $B' = V_P$, and so $A_P \subseteq B \subseteq B' = V_P$. The final assertion is clear because integral ring extensions preserve (Krull) dimension.

(c) If $x \in (M^n :_K M^n)$, then $s^n x \in M^n \subset A$ for any $s \in M \setminus P$, and so $(M^n :_K M^n) \subseteq A_P$. Thus, $(M^n :_K M^n)_P \subseteq A_P$. The reverse inclusion is clear, and so $(M^n :_K M^n)_P = A_P$. Suppose next that $P \subset M^n$. We claim that P is a common ideal of A and $(M^n :_K M^n)$.

To simplify the notation in this paragraph, we put $B := (M^n :_K M^n)$. As $M^n \not\subseteq P$ and $M^n \subseteq (A :_K B)$, we have that $(A :_K B) \not\subseteq P$. By a well-known result (cf. [14, Exercise 41 (b), page 46]), it follows that there is a unique prime ideal of B that lies over P. Therefore, $PB \cap A = P$. As $P \subseteq M^n$, we have $PB = P(M^n :_K M^n) \subseteq M^n \subseteq A$, and so $PB = PB \cap A = P$. In other words, P is an ideal of B, thus proving the above claim.

It now follows easily that the inclusion map $A \to (M^n :_K M^n)$ is prime at P if and only if P is a prime ideal of $(M^n :_K M^n)$. (Use [8, Proposition 2.2 (c)] for the "only if" assertion and the basic definitions for the "if" assertion.) The final equivalence in the assertion follows immediately from [18, Corollary 1.5].

(d) Note that if $\xi \in (M :_K M)$, then $M \subseteq I_\xi$. If $D := \{u \in K | M \subseteq I_u\}$, it follows that $(M :_K M) \subseteq D$. To prove the reverse inclusion, let $u \in D$. Then Mu is an ideal of A. If $Mu = A$, then M is invertible, hence a principal ideal of A [14, Theorem 59], which is absurd. Hence $Mu \subseteq M$ for each $u \in D$, and so $D \subseteq (M :_K M)$. □

The following is an interesting consequence of Proposition 6.1(c). If (A, M) is a quasilocal i-domain, then the inclusion map $A \to (M :_K M)$ is a prime morphism if and only if $\operatorname{Spec}(A) \setminus \{M\} = \operatorname{Spec}(M :_K M) \setminus \{N\}$, where N denotes the unique maximal ideal of $(M :_K M)$.

Proposition 6.2. *Let (A, M) be a finite-dimensional quasilocal i-domain with integral closure V. (Note that V is a valuation domain.) Let P be a nonzero prime ideal of A and let n be a positive integer. Then:*

(a) $\dim(A) \ge \dim(P^n :_K P^n) \ge \operatorname{ht}(P)$.

(b) $P^n A_P = P^n \Leftrightarrow \operatorname{ht}(P) = \dim(P^n :_K P^n) \Leftrightarrow V_P$ *is the integral closure of* $(P^n :_K P^n)$.

(c) $\dim(A) = \dim(P^n :_K P^n) \Leftrightarrow (P^n :_K P^n) \subseteq V \Leftrightarrow (P^n :_K P^n)$ *is integral over A.*

Proof. For simplicity of notation, we let $B := (P^n :_K P^n)$ in the proof. If $P = M$, all the assertions follow from the final conclusion in Proposition 6.1 (a). Suppose henceforth that $P \subset M$.

(a) If $0 \subset P_1 \subset \cdots \subset P_{n-1} \subset M$ is the prime spectrum of A, we have $P = P_k$ with $1 \leq k \leq n-1$. By [18, Proposition 1.3 (a), (b)],

$$0 \subset ((P_1 \cap P^n) :_K P^n) \subset \cdots \subset ((P_{k-1} \cap P^n) :_K P^n)$$

is a strictly increasing chain of prime ideals of B, dominating $0 \subset \cdots \subset P_k$. As $P^n B = P^n \neq B$, we have $PB \neq B$, and so the fact that $A \subseteq B$ satisfies GD (because A is a going-down domain) provides $N \in \operatorname{Spec}(B)$ such that $N \cap A = P$. It follows via going-down that $\dim(B) \geq \operatorname{ht}(P)$. Moreover, by Proposition 6.1 (b), we have $A_Q \subseteq B \subseteq V_Q$, where $Q := N' \cap A$, N' denotes the maximal ideal of B, and V_Q is the integral closure of B. To conclude the proof of (a), note that

$$\dim(B) = \dim(B') = \dim(V_Q) \leq \dim(V) = \dim(A') = \dim(A).$$

(b) We continue using the notation that was introduced in the proof of (a). As $N \subseteq N'$, we have that $P = N \cap A \subseteq N' \cap A = Q$, whence $V_Q \subseteq V_P$. Suppose first that $P^n A_P = P^n$. Then $A_P \subseteq (P^n :_K P^n) = B$, and so $V_Q = B' \supseteq (A_P)' = (A')_P = V_P \supseteq V_Q$, whence $V_P = V_Q = B'$. However, it follows easily that $\dim(V_P) = \operatorname{ht}(P)$ since the inclusion map $A \to B$ is an i-morphism that satisfies GD. Thus, $\operatorname{ht}(P) = \dim(B') = \dim(B)$.

Suppose next that $\dim(B) = \operatorname{ht}(P)$. By the above reasoning, $\operatorname{ht}(P) = \dim(V_P)$, and so $\dim(V_Q) = \dim(B') = \dim(B) = \operatorname{ht}(P) = \dim(V_P)$. As V_P is a localization of (the finite-dimensional valuation domain) V_Q, we must have $V_P = V_Q = B'$.

Finally, it remains to show that if $B' = V_P$, then $P^n A_P = P^n$. Assume that $B' = V_P$. We will show that $A_P \subseteq (P^n :_K P^n)$. Let $r \in A \setminus P$. Then r is a unit in A_P, hence a unit in $V_P = B'$, and hence by integrality (more specifically, the Lying-over Theorem) a unit in $B = (P^n :_K P^n)$. It follows that $(1/r)P^n = P^n$ for all $r \in A \setminus P$, and so $A_P \subseteq (P^n :_K P^n)$.

(c) We continue using the notation that was introduced in the proof of (a). It is clear that: $B \subseteq V \Rightarrow B$ integral over $A \Rightarrow \dim(A) = \dim(B)$. It remains only to prove that if $\dim(A) = \dim(B)$, then $B \subseteq V$. Suppose that $\dim(A) = \dim(B)$. It follows easily from Proposition 6.1(b) that the maximal ideal of B must lie over M and that $B \subseteq V_M = V$, as desired. □

Corollary 6.3. *Let (A, M) be a quasilocal i-domain of finite (Krull) dimension, and let K be the quotient field of A. Then:*

(a) *The following conditions are equivalent:*

  (1) *A is an almost-divided domain;*

  (2) *For each $P \in \operatorname{Spec}(A) \setminus \{0, M\}$, there is a positive integer n such that $\operatorname{ht}(P) = \dim(P^n :_K P^n)$.*

(b) *A is a divided domain if and only if A is a straight domain and for each nonzero nonmaximal prime ideal P of A, there exists a positive integer n such that* $\operatorname{ht}(P) = \dim(P^n :_K P^n)$.

Proof. For (a), use Proposition 6.2 (b). As for (b), it follows from (a) that the assertion comes down to the following: A is a divided domain if and only if A is a straight domain and an almost-divided domain. The "if" assertion of this formulation follows from the proof of Proposition 5.3, while the "only if" assertion follows by combining Proposition 5.3 with the fact that each divided domain is a quasi-divided domain. □

Remark 6.4. Consider a two-dimensional quasilocal i-domain A, with integral closure a valuation domain V and quotient field K. Let P denote the only height 1 prime ideal of A. There are two cases: (1) A is not an almost-divided domain; (2) A is an almost-divided domain.

By the earlier results in this section, (1) $\Leftrightarrow$ $\dim(P^n :_K P^n) = 2$ for each positive integer n $\Leftrightarrow$ $(P^n :_K P^n)$ is integral over A for each positive integer n. If (1) holds, then A is not a divided domain.

(2) $\Leftrightarrow$ $\dim(P^n :_K P^n) = 1$ for some positive integer n. We do not know if (2) implies that A is a divided domain. We close with a more general open question: does there exist an almost-divided domain which is not a divided domain?

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On TV-domains

Said El Baghdadi

Abstract. We give some new characterizations of TV-domains and answer some open questions on these domains.

Keywords. Divisorial ideal, t-ideal, Divisorial domain, w-divisorial domain, TV-domain, Prüfer v-multiplication domain.

AMS classification. 13A15, 13G05, 13F05.

1 Introduction

In 1988 E. Houston and M. Zafrullah [12] introduced the notion of TV-domains, i.e., domains in which each t-ideal is divisorial. This concept is extremely useful in studying some classical integral domains such as Mori and Krull domains. This work was also a starting point of the investigations of several analogous concepts, see for instance [9], [15], [16] and [5]. In this note we give some new characterizations of TV-domains and answer some questions left open in [12].

Let D be an integral domain with quotient field K. Let $F(D)$ denote the set of all nonzero fractional ideals of D and $f(D)$ denote the subset of finitely generated members of $F(D)$. We shall use the language of star operations. The reader is referred to [7, Sections 32, 34] and [13] for the properties of star operations, which we shall use freely. Let $I \in F(D)$. Recall that the v-closure of I is given by $I_v := (I^{-1})^{-1}$ and the t-closure of I by $I_t = \cup\{J_v \mid J \text{ is a finitely generated nonzero subideal of } I\}$. A fractional ideal I is said to be a v-ideal or divisorial (respectively, a t-ideal) if $I = I_v$ (respectively, $I = I_t$). The ideal I is v-finite or of finite type if $I_v = J_v$ for some $J \in f(D)$. The v- and t-closure are the best known non trivial star operations. Recall that a star operation $*$ on D is of finite type if $I^* = \cup\{J^* \mid J \text{ is a finitely generated nonzero subideal of } I\}$ for all $I \in F(D)$. Noting that if $J \in F(D)$ is finitely generated $J_t = J_v$ we conclude that the t-operation is of finite type. A prime ideal that is also a t-ideal is called a t-prime ideal. The set of (integral) t-ideals of D has maximal elements under inclusion, called t-maximal ideals, and these ideals are prime. We denote by $t\text{-Max}(D)$ the set of t-maximal ideals of D.

Recently a new star operation called the w-operation has received much more interest. The w-operation on D is defined by $I_w = \cap\{ID_M \mid M \in t\text{-Max}(D)\}$ (equivalently, $I_w = \cup\{(I : H) \mid H \in f(D) \text{ and } H_v = D\}$ for all $I \in F(D)$). The w-operation is of finite type. For more details on the w-operation, see [11], [17] and [18]. For each $I \in F(D)$, we have $I \subseteq I_w \subseteq I_t \subseteq I_v$, the inclusions may be strict (cf. [19, page 105] and [15, Proposition 1.2]).

An integral domain D is a Prüfer v-multiplication domain (PVMD) if every finitely generated nonzero fractional ideal I of D is t-invertible (i.e., $(II^{-1})_t = D$). It is well known that D is a PVMD if and only if for each t-maximal ideal M of D, D_M is a valuation domain [8, Theorem 5]. Finally, when a property is satisfied by the quotient rings of a domain D at each of its t-maximal ideals, we say that such a property is t-local.

2 Main results

Recall that a domain D is divisorial (respectively, w-divisorial) if $v = d$ (respectively, $v = w$) on D. Note that for a domain, divisorial $\Rightarrow$ w-divisorial $\Rightarrow$ TV, these implications cannot be reversed in general, see [15, Example 2.7, (2)–(3)]. A domain D such that $w = t$ is called a TW-domain [15]. Thus a TV-domain is a w-divisorial domain if and only if it is a TW-domain. In [10] (respectively, [5]), it was shown that the property of divisoriality (respectively, w-divisoriality) is local (respectively, t-local). Moreover, a complete characterization of these properties in terms of their localization were established, see [3, Proposition 5.4] and [5, Theorem 1.5]. In [12], the authors proved TV analogues of several results of [10]. They remarked that some properties of divisorial domains do not carry over to those of TV-domains and left some questions open. Almost all of these questions have been solved and developed, but the question of whether the TV-property is t-local remained open. The main result of this paper is to give an affirmative answer to this problem. More precisely, we give a new characterization of the TV-property in terms of its t-localization.

Lemma 2.1. *Let D be an integral domain and M a prime ideal of D. Then the following are equivalent:*

(1) *For every family $\{I_\alpha\}_\alpha$ of divisorial integral ideals of D such that $\cap I_\alpha \neq 0$, $\cap I_\alpha \subseteq M \Rightarrow I_\alpha \subseteq M$ for some α.*

(2) *For every family $\{I_\alpha\}_\alpha$ of divisorial (fractional) ideals of D such that $\cap I_\alpha \neq 0$, $(\cap I_\alpha)D_M = \cap(I_\alpha D_M)$.*

(3) *For every nonzero ideal I of D, $(ID_M)^{-1} = I^{-1}D_M$. In particular, $(ID_M)_v = I_v D_M$.*

Proof. (1) $\Rightarrow$ (2). Clearly $(\cap I_\alpha)D_M \subseteq \cap(I_\alpha D_M)$. For the opposite inclusion, let $x \in \cap(I_\alpha D_M)$. Then, for each α, there exists $s_\alpha \in D \setminus M$ such that $s_\alpha x \in I_\alpha$. Set $J_\alpha = (I_\alpha :_D x)$, for each α. The J_α's form a collection of divisorial ideals of D such that $J_\alpha \nsubseteq M$ for all α (since $s_\alpha \in J_\alpha$). We claim that $\cap J_\alpha \neq 0$. Let $y \in D$ such that $0 \neq y \in \cap I_\alpha$ and set $x = z/t$, where $0 \neq t, z \in D$. Then $ytx = yz \in \cap I_\alpha$ and hence $yt \in J_\alpha$ for all α. Hence $\cap J_\alpha \neq 0$. Thus $\cap J_\alpha \nsubseteq M$. Let $s \in \cap J_\alpha$ such that $s \notin M$. Write $x = (xs)/s$. Since $s \in \cap J_\alpha$, $sx \in \cap I_\alpha$. So $x \in (\cap I_\alpha)D_M$. Hence $\cap(I_\alpha D_M) \subseteq (\cap I_\alpha)D_M$.

(2) $\Rightarrow$ (1). Suppose that there exists a family $\{I_\alpha\}_\alpha$ of integral divisorial ideals of D such that $\cap I_\alpha \neq 0$ and $\cap I_\alpha \subseteq M$, but $I_\alpha \nsubseteq M$ for all α. Then $(\cap I_\alpha)D_M = \cap(I_\alpha D_M) = D_M$, which is impossible.

(2) $\Leftrightarrow$ (3). See [1, Lemma 5.5]. □

We call an ideal I of D transportable through a family X of prime ideals of D if $(ID_M)^{-1} = I^{-1}D_M$ for all $M \in X$ ([20, page 437]). It is well known that nonzero finitely generated ideals are transportable through $\mathrm{Spec}(D)$. Note that an ideal transportable through a prime ideal M satisfies the equivalent conditions of Lemma 2.1.

The following theorem gives an affirmative answer to the open question in [12, page 298] mentioned above.

Theorem 2.2. *Let D be an integral domain. Then the following are equivalent:*

(1) *D is a TV-domain;*

(2) *D is a t-locally TV-domain and every nonzero ideal is transportable through* t-$\mathrm{Max}(D)$.

Proof. (1) $\Rightarrow$ (2). Let M be a t-maximal ideal of D. We first show that M satisfies (1) of Lemma 2.1. Let $\{I_\alpha\}_\alpha$ be a family of divisorial integral ideals of D such that $\cap I_\alpha \neq 0$ and $\cap I_\alpha \subseteq M$, but $I_\alpha \nsubseteq M$ for all α. Let $A = \cap I_\alpha$. Then A is a t-ideal of D and $A \subseteq M$, which is impossible by [12, Lemma 1.2]. By Lemma 2.1, it follows that every ideal is transportable through t-$\mathrm{Max}(D)$. To complete the proof, let I be a t-ideal of D_M. Then $I \cap D$ is a t-ideal of D and so $J := I \cap D = J_v$ because D is a TV-domain. But then $I = J_v D_M = (JD_M)_v = I_v$. Thus every t-ideal of D_M is a v-ideal.

(2) $\Rightarrow$ (1). Consider the map $* : F(D) \to F(D)$, $I^* = \cap\{(ID_M)_t \mid M \in t\text{-}\mathrm{Max}(D)\}$. It is not hard to check that $*$ is a star operation on D. Whence $* \leq v$. Let $I \in F(D)$. Then $(ID_M)_t = (ID_M)_v = I_v D_M$, for each $M \in t$-$\mathrm{Max}(D)$. Hence $I_v \subseteq I^*$. Consequently, $* = v$. Now, let $x \in I_v = I^*$. Then for every $M \in t$-$\mathrm{Max}(D)$, there exists $J_M \in f(D)$, $J_M \subseteq I$, such that $x \in (J_M D_M)_v$. Moreover, we have $(J_M D_M)_v = (J_M)_v D_M = (J_M)_t D_M$. Hence $x \in (J_M)_t D_M$ for all $M \in t$-$\mathrm{Max}(D)$. Set $J = \sum J_M$. Then $x \in J_t D_M$ for all $M \in t$-$\mathrm{Max}(D)$. Thus $x \in \cap\{J_t D_M \mid M \in t\text{-}\mathrm{Max}(D)\} = J_t \subseteq I_t$. Hence $I_v \subseteq I_t$. Therefore D is a TV-domain. □

Remark 2.3. (1) The TV-property of a domain D need not be inherited by a quotient ring of D at a (t-) prime ideal. To see this, recall that a valuation domain is divisorial if and only if its maximal ideal is a principal ideal [10, Lemma 5.2]. The same characterization for valuation TV-domains holds [12, Remark 1.5]. The example as in [10, Remark 5.4] will do the job. That is, we take V a rank two valuation domain with maximal principal ideal M and a minimal prime N such that V_N is not discrete.

(2) A domain D which satisfies the transportable property through t-$\mathrm{Max}(D)$ is not necessarily a TV-domain. Indeed, let V be a valuation domain with idempotent maximal ideal M. The transportable property through t-$\mathrm{Max}(V) = \{M\}$ in V is trivial. But V is not a TV-domain since M is not principal.

(3) A TV-domain which is t-locally divisorial is a w-divisorial domain, see [5, Theorem 1.5].

Recall that a nonempty family X of nonzero prime ideals of D is of finite character if each nonzero element of D belongs to at most finitely many members of X and that X is said to be independent if no two members of X contain a common nonzero prime ideal (cf. [1]). The domain D has finite character (respectively, t-finite character) if $\mathrm{Max}(D)$ (respectively, t-$\mathrm{Max}(D)$) is of finite character. If the set $\mathrm{Max}(D)$ is independent of finite character, the domain D is called an h-local domain. A domain D such that t-$\mathrm{Max}(D)$ is independent of finite character is called in [1] a weakly Matlis domain; hence D is a weakly Matlis domain if it has t-finite character and each t-prime ideal is contained in a unique t-maximal ideal. In a TV-domain the t-finite character property is satisfied [12, Theorem 1.3], but a t-prime ideal may be contained in more than one t-maximal ideal [12, Example 4.1], and hence a TV-domain is not in general a weakly Matlis domain. We recall the following lemma from [5]:

Lemma 2.4 ([5, Lemma 1.2]). *Let D be an integral domain. The following conditions are equivalent:*

(1) *D is a weakly Matlis domain;*

(2) *For each t-maximal ideal M of D and a collection $\{I_\alpha\}_\alpha$ of w-ideals of D such that $\cap_\alpha I_\alpha \neq 0$, if $\cap_\alpha I_\alpha \subseteq M$, then $I_\alpha \subseteq M$ for some α.*

Corollary 2.5. *Let D be an integral domain such that each t-prime ideal is contained in a unique t-maximal ideal. Then the following are equivalent:*

(1) *D is a TV-domain;*

(2) *D is a t-locally TV-domain and has t-finite character.*

Proof. (1) ⇒ (2). This follows from Theorem 2.2 and [12, Theorem 1.3].
(2) ⇒ (1). This is a consequence of Lemmas 2.1 and 2.4 and Theorem 2.2. □

Remark 2.6. (1) It was mentioned in [12, Remark 4.2] that by a similar proof like that one of [10, Theorem 3.6], we can show that a TV-domain in which each t-prime ideal is contained in a unique t-maximal ideal is a t-locally TV-domain. This is also justified by the fact that weakly Matlis domains satisfy the transportable property through t-maximal ideals (Lemmas 2.1 and 2.4). However, in [12, Example 4.1], the authors give an example of a Noetherian domain (and hence a TV-domain) which has a t-prime ideal contained in two t-maximal ideals. Note that a Noetherian domain D satisfies the transportable property through $\mathrm{Spec}(D)$. So there is a Noetherian domain that is not a weakly Matlis domain. It would be instructive to have some more examples of TV-domains that are not weakly Matlis.

(2) An important class of t-locally TV-domains are the generalized Krull domains. In fact they are t-locally divisorial domains [4, Remark 3.10]. In particular, generalized Dedekind domains are (t-)locally TV-domains. Since in a PVMD $w = t$ [14, Theorem 3.1], then a PVMD is a TV-domain if and only if it is w-divisorial. Thus by [5, Theorem 3.5], a generalized Krull domain D is a TV-domain if and only if each t-prime ideal of D is contained in a unique t-maximal ideal of D. However, a generalized Krull domain need not be a TV-domain, see the example below.

Example 2.7. A t-locally TV-domain which has t-finite character need not be a TV-domain. Consider a finite family of rank one DVRs $\{V_i\}_{i=1}^n$, $n \geq 2$, with the same quotient field. Let $\{M_i\}_i$ denote their corresponding maximal ideals. Then $D = \cap V_i$ is a Dedekind domain with a finite set of maximal ideals, say $\{m_i\}_i$. Let K denote the quotient field of D. Set $T = K[[t]] = K + M$ and $R = D + M$. By [6, Theorem 4.1], R is a generalized Dedekind domain, and hence (t-)locally TV-domain. The (t-)maximal ideals of R are $\{m_i + M\}_{i=1}^n$. In particular, R has (t-)finite character. We claim that R is not a TV-domain. Otherwise, R will be a divisorial domain since it is a Prüfer domain. But the ideal M is contained in the maximal ideals $m_i + M$ with $n \geq 2$, which is impossible by [10, Theorem 2.4].

Another question in [12] which needs more development is the case of integrally closed TV-domains. As an analogue of the fact that an integrally closed divisorial domain is Prüfer [10, Theorem 5.1], one may suspect that an integrally closed TV-domain is a PVMD. However, in [12, Remark 3.2] the authors give an example showing that this is not the case. Nevertheless, by a suitable choice of integrality we can get a PVMD. Recall that the pseudo-integral closure of a domain D is the overring $\tilde{D} = \{(J_v : J_v) \mid J \in f(D)\}$. A domain D is said to be pseudo-integrally closed if $\tilde{D} = D$. A pseudo-integrally closed domain is integrally closed. For more details, see [2].

Theorem 2.8. *Let D be an integral domain. The following conditions are equivalent:*

(1) *D is a pseudo-integrally closed TV-domain;*

(2) *D is an integrally closed w-divisorial domain;*

(3) *D is a weakly Matlis PVMD and each t-maximal ideal of D is t-invertible.*

Proof. (1) $\Rightarrow$ (2). Let $I \in f(D)$. Let $x \in (II^{-1})^{-1}$, then $xII^{-1} \subseteq D$ which implies that $xI_v \subseteq I_v$. Thus $x \in \tilde{D} = D$. Hence $(II^{-1})^{-1} = D$, that is $(II^{-1})_v = D$. Since D is a TV-domain, $(II^{-1})_t = D$. Hence D is a PVMD. Whence $w = t$ on D. Therefore, D is an integrally closed w-divisorial domain.

(2) $\Leftrightarrow$ (3). See [5, Theorem 3.2].

(3) $\Rightarrow$ (1). By (2) $\Leftrightarrow$ (3), D is a TV-domain and one can easily check that a PVMD is pseudo-integrally closed (see [2]). □

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Integral basis of cubic number fields

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Abstract. In this paper, for a cubic number field K, a p-integral basis of K is given for each prime integer p. The discriminant d_K and an integral basis of K are then obtained from its p-integral bases.

Keywords. p-integral bases, index, Newton polygon.

AMS classification. 11Yxx.

Introduction

Let K be a cubic number field defined by an irreducible polynomial $P(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}[X]$, α a complex root of P, $\mathbb{Z}_K$ the ring of integers of K, d_K its discriminant and $\operatorname{ind}(P) = [\mathbb{Z}_K : \mathbb{Z}[\alpha]]$ the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$. It is well known that $\triangle = N_{K/\mathbb{Q}}(P'(\alpha)) = (\operatorname{ind}(P))^2 d_K$, where $\triangle$ is the discriminant of P.

Let p be a prime integer. A p-integral basis of K is a set of integral elements $\{w_1, w_2, w_3\}$ such that p does not divide the index $[\mathbb{Z}_K : \Lambda]$, where $\Lambda = \sum_{i=1}^{3} \mathbb{Z} w_i$. In that case, we say that Λ is a p-maximal order of K. A triangular p-integral basis of K is a p-integral basis of K $(1, w_2, w_3)$ such that $w_2 = \frac{\alpha + x_1}{p^{r_1}}$ and $w_3 = \frac{\alpha^2 + y_2\alpha + x_2}{p^{r_2}}$. In Theorem 1.1, for every prime p, a triangular p-integral basis of K is given. For every prime p and $(x, m) \in \mathbb{Z}^2$, let $x_p := \frac{x}{p^{v_p(x)}}$ and $x[m]$ denote the remainder of the Euclidean division of x by m.

In this paper, an improvement of the index theorem, announced in [3], is made. This will allow us to give a new detailed proof of [1, Theorem 2.1]. Remark 3.4 provides a method to generalize these results to any cubic number field defined by an irreducible polynomial $P(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}[X]$.

1 Newton polygon

Let p be a prime integer such that p^2 divides $\triangle$ and $\phi(X)$ is an irreducible divisor of $P(X)$ modulo p. Set $m = \deg(\phi(X))$ and let

$$P(X) = a_0(X)\phi(X)^t + a_1(X)\phi(X)^{t-1} + \cdots + a_t(X)$$

be the $\phi(X)$-adic development of $P(X)$ (every $a_i(X) \in \mathbb{Z}[X]$, $\deg a_i(X) < m$). To any coefficient $a_i(X)$ we attach the integer $u_i = v_p(a_i(X))$ and the point of the plane $P_i = (i, u_i)$, if $u_i < \infty$.

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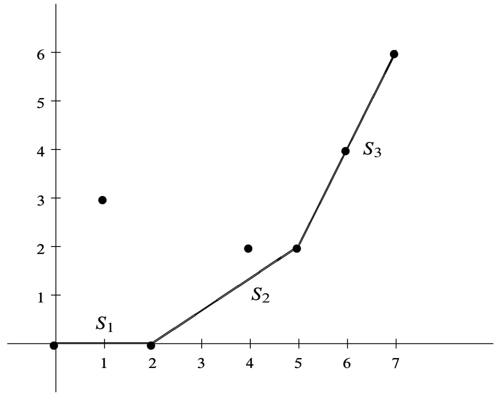
The ϕ-Newton polygon of $P(X)$ is the lower convex envelope of the set of points $P_i = (i, u_i)$, $u_i < \infty$, in the cartesian plane. This (open) polygon is denoted by $N_\phi(P)$. For instance, for a ϕ-development of degree 7 with $u_i = 0, 3, 0, \infty, 2, 2, 4, 6$ for $i = 0, 1, \dots, 7$, the polygon is N: the $\phi(X)$-Newton polygon of $P(X)$.



The *length*, $\ell(N_\phi(P))$, and *height*, $h(N_\phi(P))$, of the polygon are the respective lengths of the projection to the horizontal and vertical axes. Clearly,

$$\deg P(X) = m\ell(N_\phi(P)) + \deg a_0(X).$$

The ϕ-Newton polygon is the union of different adjacent *sides* $S_1, \dots, S_t$ with increasing slopes $\lambda_1 < \lambda_2 < \cdots < \lambda_t$. We shall write $N_\phi(P) = S_1 + \cdots + S_t$. The points joining two different sides are called the *vertexes* of the polygon. The polygon determined by the sides of positive slope of $N_\phi(P)$ is called the *principal part ϕ-polygon* of $P(X)$ and denoted by $N_\phi^+(P)$. The length and height of $N_\phi^+(P)$ are the respective lengths of the projection to the horizontal and vertical axes.

For instance, the polygon of the figure has three sides S_1, S_2, S_3 with slopes $0 < 2/3 < 2$ and $N_\phi^+(P) = S_2 + S_3$. For every side S of the principal part $N_\phi^+(P)$. The *length*, $\ell(S)$, and *height*, $h(S)$, of S are the respective lengths of the projection to the horizontal and vertical axis. The *slope* of S is the quotient $h(S)/\ell(S)$. The positive integer $d(S) := \gcd(h(S), \ell(S))$ is called the *degree* of S. Denote $d := d(S)$ the degree of S, $h := h(S)/d$ and $e := \ell(S)/d$ positive coprime integers such that h/e is the slope of S. Let $s = \lfloor \frac{n}{m} \rfloor$, where $n = \deg(P)$, $m = \deg(\phi)$ and $\lfloor \frac{n}{m} \rfloor$ is the integral part of $\frac{n}{m}$. For every $1 \le j \le s$, let H_j be the length of the projection of P_j to the horizontal axis, h_j its integral part and $t_j = \text{red}\left(\frac{a_j(X)}{p^{h_j}}\right)$ ($t_j = 0$ if $P_j \notin S$). If i is the abscissa of the initial point of S, let $P_S(Y)$ be the *residual polynomial* attached to S to be

$$P_S(Y) := t_i Y^d + t_{i+e} Y^{d-1} + \cdots + t_{i+(d-1)e} Y + t_{i+de} \in \mathbb{F}_\phi[Y].$$

The following theorem is an improvement and a special case for cubic number fields of the theorem of index (in [3, p. 328], it was supposed that $\bar{f} = X^r$ modulo p and $N^+(f)$ is one side).

Theorem 1.1. *Let p be a prime integer, $f = X^3 + aX^2 + bX + c$ a polynomial of $\mathbb{Z}[X]$, and $\bar{f} = X^r \times \phi(X)$ its factorization modulo p, where $r \geq 2$, $v_p(a) = 0$ or $v_p(b) \leq 1$ or $v_p(c) \leq 2$. Let N be the X-Newton polygon of f and N^+ its principal part. Then $\mathrm{ind}_N(f) = h_1 + h_2 :=$ the number of points with integer coordinates that lie below the polygon N and whose abscissas satisfy $1 \leq j \leq 2$, excluding those on both axes.*

If $N^+ = S_1 + S_2$ such that for every i, $P_{S_i}(Y)$ is square free, then $v_p(\mathrm{ind}(f)) = \mathrm{ind}_N(f)$ and $(1, \frac{\alpha+a}{p^{h_1}}, \frac{\alpha^2+a\alpha}{p^{h_2}})$ is a p-integral basis of $\mathbb{Z}_K$.

Proof. Since $v_p(a) = 0$ or $v_p(b) \leq 1$ or $v_p(c) \leq 2$, we have $h_1 = 0$ and let $h_2 = h$, $w_h = \frac{\alpha^2+a\alpha}{p^h}$. Then $ch_{w_h}(X) = X^3 + \frac{b}{p^h}X^2 + \frac{ac}{p^{2h}}$ is the characteristic polynomial of the endomorphism l_{w_h} of K, defined by the multiplication of w_h. To show that $w_h \in \mathbb{Z}_K$, it suffices to show that $v_p(b) \geq h$ and $v_p(ac) \geq 2h$.

(i) $v_p(a) \geq 1$, $v_p(b) \geq 2$ and $v_p(c) = 2$. Since $h_2 = 1$, $w_1 \in \mathbb{Z}_K$, $v_p(\mathrm{ind}(f)) = 1$ and $(1, \alpha, \frac{\alpha^2+a\alpha}{p})$ is a p-integral basis of $\mathbb{Z}_K$.

(ii) $v_p(a) \geq 1$, $v_p(b) \geq 2$ and $v_p(c) = 1$. Since $h_2 = 0$, $v_p(\mathrm{ind}(f)) = 0$ and $(1, \alpha, \alpha^2)$ is a p-integral basis of $\mathbb{Z}_K$.

(iii) $v_p(a) = 0$, $v_p(b) \geq 1$ and ($2v_p(b) = v_p(c)$ and $v_p(b^2 - 4ac) = 0$) or $2v_p(b) > v_p(c)$. Then $h_2 = \lfloor \frac{v_p(c)}{2} \rfloor$ and $v_p(\mathrm{ind}(f)) = h_2$. Since $2v_p(b) \geq v_p(c)$, p^{h_2} divides b and p^{2h_2} divides ac. Thus, $w_{h_2} \in \mathbb{Z}_K$ and $(1, \alpha, \frac{\alpha^2+a\alpha}{p^{h_2}})$ is a p-integral basis of $\mathbb{Z}_K$.

(iv) $v_p(a) = 0$, $v_p(b) \geq 1$ and $2v_p(b) < v_p(c) \leq 2$. Then $h_2 = v_p(b)$, $v_p(\mathrm{ind}(f)) = v_p(b)$, $w_{h_2} \in \mathbb{Z}_K$ and $(1, \alpha, \frac{\alpha^2+a\alpha}{p^{h_2}})$ is p-integral basis of $\mathbb{Z}_K$. □

2 *p*-integral basis of a cubic number field

In this section, a new detailed proof, based on Newton polygon, of Theorem 2.1 announced in [1] is given.

Lemma 2.1. *Let $K = \mathbb{Q}[\alpha]$, where α is a complex root of an irreducible polynomial $P(X) = X^3 + bX + c \in \mathbb{Z}[X]$, p be a prime integer and $w = \frac{z\alpha^2+y\alpha+x}{p^i} \in K$. Then $ch = X^3 + \frac{A_2(w)}{p^i}X^2 + \frac{A_1(w)}{p^{2i}}X + \frac{A_0(w)}{p^{3i}}$ is the characteristic polynomial of l_w the endomorphism of K defined by $l_w(x) = wx$, where*

$$A_0(w) = -(x^3 - 2bx^2z + 3cxyz + b^2xz^2 + c^2z^3 + bxy^2 - cy^3),$$

$$A_1(w) = 3x^2 - 4bxz + b^2z^2 + 3cyz + by^2),$$

$$A_2(w) = 2bz - 3x.$$

In particular, w is integral if and only if, for every $1 \leq j \leq 2$, $\frac{A_j(w)}{p^{ji}} \in \mathbb{Z}$.

The following theorem gives us a triangular p-integral basis of K, $v_p(\triangle)$ and $v_p(d_K)$ for every prime integer p.

Theorem 2.2. *Let $K = \mathbb{Q}[\alpha]$, where α is a complex root of an irreducible polynomial $P(X) = X^3 + bX + c \in \mathbb{Z}[X]$ and $p \geq 5$ be a prime integer. Under the above hypotheses, a p-integral (resp., a 2-integral, resp., a 3-integral) basis of $\mathbb{Z}_K$ is given in Table A (resp., Table B, resp., Table C) below.*

case	conditions	$v_p(\triangle)$	p-integral basis	$v_p(d_K)$
1	$v_p(c) = 2, v_p(b) \geq 2$	4	$(1, \alpha, \frac{\alpha^2}{p})$	2
2	$v_p(c) = 2, v_p(b) = 1$	3	$(1, \alpha, \frac{\alpha^2}{p})$	1
3	$v_p(c) = 1, v_p(b) \geq 1$	2	$(1, \alpha, \alpha^2)$	2
4	$v_p(c) = 0, v_p(b) \geq 1$	0	$(1, \alpha, \alpha^2)$	0
5	$v_p(b) = 0, v_p(c) \geq 1$	0	$(1, \alpha, \alpha^2)$	0
6	$v_p(bc) = 0$	??	$(1, \alpha, \frac{\alpha^2+t\alpha-2t^2}{p^r})$ $r = \lfloor \frac{v_p(\triangle)}{2} \rfloor$, $2bt = -3c\ [p^{2r+1}]$	$v_p(\triangle)\ [2]$

Table A

case	conditions	$v_3(\triangle)$	3-integral basis	$v_3(d_K)$
1	$v_3(c) = 2, v_3(b) = 2$	6	$(1, \alpha, \frac{\alpha^2}{3})$	4
2	$v_3(c) = 2, v_3(b) \geq 3$	7	$(1, \alpha, \frac{\alpha^2}{3})$	5
3	$v_3(c) \geq 2, v_3(b) = 1$	3	$(1, \alpha, \frac{\alpha^2}{3})$	1
4	$v_3(c) = 1, v_3(b) = 1$	3	$(1, \alpha, \alpha^2)$	3
5	$v_3(c) = 1, v_3(b) \geq 2$	5	$(1, \alpha, \alpha^2)$	5
6	$v_3(b) = 0$	0	$(1, \alpha, \alpha^2)$	0
7	$v_3(c^2 - 1) \geq 2, v_3(b) \geq 2,$	3	$(1, \alpha, \frac{\alpha^2-c\alpha+1}{3})$	1
8	$v_3(c^2+b-1) \geq 2, b = 3\ [9]$	3	$(1, \alpha, \frac{\alpha^2-c\alpha+1}{3})$	1
9	$v_3(c^2+b-1) = 1, b = 3\ [9]$	3	$(1, \alpha, \alpha^2)$	3
10	$v_3(c^2+b-1) = 1, b = 6\ [9]$	4	$(1, \alpha, \alpha^2)$	4
11	$v_3(c^2+b-1) = 2, b = 6\ [9]$	5	$(1, \alpha, \frac{\alpha^2-c\alpha+1}{3})$	3
12	$v_3(c^2+b-1) \geq 3, b = 6\ [9]$	≥ 6	$(1, \frac{\alpha+c}{3}, \frac{\alpha^3+y\alpha+x}{3^r})$ $r = \lfloor \frac{\triangle}{2} \rfloor - 1$, $x = 2b_3\ [3^r]$, $2b_3 y = -c\ [3^r]$	$v_3(\triangle) - 2r$

Table B

case	conditions	$v_2(\triangle)$	2-integral basis	$v_2(d_K)$
1	$v_2(c) = 2,\, v_2(b) \geq 2$	4	$(1, \alpha, \frac{\alpha^2}{2})$	2
2	$v_2(c) = 2,\, v_2(b) = 1$	4	$(1, \alpha, \frac{\alpha^2}{2})$	2
3	$v_2(c) = 2,\, b = 1\,[4]$	2	$(1, \alpha, \alpha^2)$	2
4	$v_2(c) = 2,\, b = 3\,[4]$	2	$(1, \alpha, \frac{\alpha^2+\alpha}{2})$	0
5	$v_2(c) = 1,\, v_2(b) \geq 1$	2	$(1, \alpha, \alpha^2)$	2
6	$v_2(c) = 1,\, b = 3\,[4]$	3	$(1, \alpha, \alpha^2)$	3
7	$v_2(c) = 1,\, b = 1\,[4]$, $\triangle = 2r + 1$	$2r+1$	$(1, \alpha, \frac{\alpha^2 - t\alpha - 2t^2}{2^{r-1}})$, $(bt = -3c_2\,[2^r])$	3
8	$v_2(c) = 1, b = 1\,[4]$, $\triangle = 2r, \triangle_2 = 3\,[4]$	$2r$	$(1, \alpha, \frac{\alpha^2 - t\alpha - 2t^2}{2^{r-1}})$, $(bt = -3c_2\,[2^{r-1}])$	2
9	$v_2(c) = 1, b = 1\,[4]$, $\triangle = 2r, \triangle_2 = 1\,[4]$	$2r$	$(1, \alpha, \frac{\alpha^2 - t\alpha - 2t^2}{2^{r}})$, $(bt = -3c_2\,[2^r])$	0
10	$v_2(b) \geq 1, v_2(c) = 0$	0	$(1, \alpha, \alpha^2)$	0
11	$v_2(bc) = 0$	0	$(1, \alpha, \alpha^2)$	0

Table C

Proof. First, $\triangle = -(27c^2 + 4b^3)$ and the proof is based on the Newton polygon. For every prime p and for every $P = X^3 + a_1X^2 + a_2X + a_3$, let $u_i = v_p(a_i)$, $\bar{P}(X)$ the reduction of P modulo p, N the X-Newton polygon of P and N^+ its principal part.

(i) Case 1. If $v_p(c) = 2$ and $v_p(b) \geq 2$, then $N = S$ is one side such that $F_S(Y) = Y + c_p$. Thus, $v_p(\mathrm{ind}(P)) = \mathrm{ind}_N(P) = 1$, $(1, \alpha, \frac{\alpha^2}{p})$ is a p-integral basis of $\mathbb{Z}_K$. Let $b = p^2B$ and $c = p^2C$, where $v_p(C) = 0$ and $v_p(B) \geq 0$. Then $\triangle = -p^4(4B^3p^2 + 27C^2)$. It follows that: if $p \neq 3$, then $v_p(\triangle) = 4$ and $v_p(d_K) = 2$. For $p = 3$, if $v_3(b) \geq 3$, then $v_3(\triangle) = 7$ and $v_p(d_K) = 5$. If $v_3(b) = 2$, then $v_3(\triangle) = 6$ and $v_3(d_K) = 4$.

(ii) Case 2. If $v_p(c) = 2$ and $v_p(b) = 1$, then $N = S_1 + S_2$ such that every $F_{S_i}(Y)$ is of degree 1. Thus, $v_p(\mathrm{ind}(P)) = \mathrm{ind}_N(P) = 1$, $(1, \alpha, \frac{\alpha^2}{p})$ is a p-integral basis of $\mathbb{Z}_K$. Let $b = pB$ and $c = p^2C$ ($v_p(BC) = 0$). Then $\triangle = -p^3(4B^3 + 27C^2p)$. It follows that: if $p \neq 2$, then $v_p(\triangle) = 3$ and $v_p(d_K) = 1$. For $p = 2$, then $v_2(\triangle) = 4$ and $v_p(d_K) = 2$.

(iii) Case 3. If $v_p(c) \geq 1$ and $v_p(b) = 0$, then $\bar{F}(X) = X(X^2 + b)$. It follows that: if $p \neq 2$, then $\bar{F}$ is square free, $(1, \alpha, \alpha^2)$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = v_p(\triangle) = 0$.

For $p = 2$, let $P(X) = F(X + 1) = X^3 + AX^2 + BX + C$ such that $A = 3$, $B = 3 + b$ and $C = 1 + b + c$. Then $v_2(A) = 0$, $v_2(B) \geq 1$ and $v_2(C) \geq 1$.

(a) $v_p(c) \geq 2$.
If $b = 1\,[4]$, then $v_2(C) = 1$, $(1, \alpha, \alpha^2)$ is a 2-integral basis of $\mathbb{Z}_K$ and $v_2(d_K) = v_2(\triangle) = 2$.

If $b = 3\,[4]$, then $v_2(B) = 1$ and $v_2(C) \geq 2$. So, $(1, \alpha, \frac{\alpha^2+\alpha}{2})$ is a 2-integral basis of $\mathbb{Z}_K$, $v_2(\triangle) = 2$ and $v_2(d_K) = 0$.

(b) $v_p(c) = 1$.
If $b = 3\,[4]$, then $v_2(C) = 1$, $(1, \alpha, \alpha^2)$ is a 2-integral basis of $\mathbb{Z}_K$ and $v_2(d_K) = v_2(\triangle) = 2$.
If $b = 1\,[4]$, then $v_2(B) \geq 2$ and $v_2(C) \geq 2$. Let $t \in \mathbb{Z}$ such that $v_2(bt+3c_2) = s$ $(bt = -3c_2 + 2^s K)$ and $P(X) = F(X+t) = X^3 + AX^2 + BX + C$ such that $A = 3t$, $B = 3t^2 + b$ and $C = t^3 + bt + c$. Then $b^2B = \frac{-\triangle}{4} - 9.2^{s+1}Kc_2$ modulo 2^{2s} and $b^3C = c_2\frac{\triangle}{4} - \frac{\triangle}{4}.2^sK - 9.2^{2s}K^2c_2$ modulo 2^{3s}. It follows that:

(1) If $v_p(\triangle) = 2r+1$, then for $s = r$, we have $v_2(B) = r+1$, $v_2(C) = 2r-1$. Thus, $v_2(\text{ind}(f)) = r-1$, $v_2(d_K) = 3$ and $(1, \theta, \frac{\theta^2+3t\theta}{2^{r-1}})$ is a 2-integral basis of $\mathbb{Z}_K$, where $\theta = \alpha - t$.

(2) If $v_p(\triangle) = 2r$, then for $s = r-1$, we have $v_2(B) = r$ and $b^3C = 2^{2r-2}c_2(\triangle_2 - K^2)$ modulo 2^{3r-3}. Since $K^2 = 1$ modulo 4, it follows that: if $\triangle_2 = 1$ modulo 4, then $v_2(C) \geq 2r$, $v_2(\text{ind}(f)) = r$, $v_2(d_K) = 0$ and $(1, \theta, \frac{\theta^2+3t\theta}{2^r})$ is a 2-integral basis of $\mathbb{Z}_K$, where $\theta = \alpha - t$.
If $\triangle_2 = 1$ modulo 4, then $v_2(C) = 2r-1$, $v_2(\text{ind}(f)) = r-1$, $v_2(d_K) = 2$ and $(1, \theta, \frac{\theta^2+3t\theta}{2^{r-1}})$ is a 2-integral basis of $\mathbb{Z}_K$.

(iv) Case 4. $v_p(c) = 1$ and $v_p(b) \geq 1$. Since $v_p(c) = 1$, $N = S$ is one side, $F_S(Y)$ is of degree 1. Therefore, $v_p(\text{ind}(P)) = \text{ind}_N(P) = 0$, $(1, \alpha, \alpha^2)$ is a p-integral basis of $\mathbb{Z}_K$.

Let $b = pB$ and $c = pC$ ($v_p(C) = 0$ and $v_p(B) \geq 0$). Then $\triangle = -p^2(4B^3p^2 + 27C^2)$. It follows that: if $p \neq 3$, then $v_p(\triangle) = v_p(d_K) = 2$. For $p = 3$, if $v_3(b) \geq 2$, then $v_3(\triangle) = v_3(d_K) = 5$. If $v_3(b) = 1$, then $v_3(\triangle) = v_3(d_K) = 3$ and $v_p(d_K) = v_p(\triangle) = 2$.

(v) Case 5. If $v_p(c) = 0$ and $v_p(b) \geq 1$, then $\bar{F}(X) = X^3 + c$. So, if $p \neq 3$, then $\bar{F}$ is square free, $(1, \alpha, \alpha^2)$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = v_p(\triangle) = 0$.

For $p = 3$, let $P(X) = F(X-c) = X^3 + AX^2 + BX + C$, where $A = -3c$, $B = 3c^2 + b$ and $C = -c(c^2 + b - 1)$. Then $v_3(A) = 1$, $v_3(B) \geq 1$ and $v_3(C) \geq 1$. It follows that:

(a) If $v_3(b) \geq 2$ and $v_3(c^2 - 1) \geq 2$, then $v_3(B) = 1$ and $v_3(C) \geq 2$. Thus, $v_3(\text{ind}(F)) = 1$, $(1, \alpha, \frac{\alpha^2 - c\alpha + 1}{3})$ is a 3-integral basis of $\mathbb{Z}_K$ and $v_2(\triangle) = 3$ and $v_2(d_K) = 1$.

(b) If $v_3(b) = 1$ and $v_3(c^2 + b - 1) = 1$, then $v_3(C) = 1$, $(1, \alpha, \alpha^2)$ is a 3-integral basis of $\mathbb{Z}_K$ and if $b = 3\,[9]$, then $v_2(d_K) = v_2(\triangle) = 3$. If $b = 3\,[9]$, then $v_2(d_K) = v_2(\triangle) = 4$

(c) If $b = 3\,[9]$ and $v_3(c^2 + b - 1) \geq 2$, then $v_3(B) = 1$ and $v_3(C) \geq 2$. Thus,

$v_3(\text{ind}(F)) = 1$, $(1, \alpha, \frac{\alpha^2 - c\alpha + 1}{3})$ is a 3-integral basis of $\mathbb{Z}_K$, $v_2(\triangle) = 3$ and $v_2(d_K) = 1$.

(d) If $b = 6\,[9]$ and $v_3(c^2 + b - 1) = 2$, then $v_3(\text{ind}(F)) = 1$, $(1, \alpha, \frac{\alpha^2 - c\alpha + 1}{3})$ is a 3-integral basis of $\mathbb{Z}_K$, $v_2(\triangle) = 5$ and $v_2(d_K) = 3$.

(e) If $b = 6\,[9]$ and $v_3(c^2+b-1) \geq 3$, then $\frac{\alpha+c}{3} \in \mathbb{Z}_K$. Let $r = \lfloor \frac{\triangle}{2} \rfloor - 1$ and $(x, y) \in \mathbb{Z}^2$ such that $x = 2b_3$, $2b_3 y = -c$ modulo 3^r and $w = \frac{\alpha^3 + y\alpha + x}{3^r}$. Replacing the $A_i(w)$ as defined in Lemma 2.1, we have $A_2(w) = 0$ modulo 3^r, $A_1(w) = 0$ modulo 3^{2r} and $A_0(w) = 0$ modulo 3^{3r}. Finally $w \in \mathbb{Z}_K$, $v_3(\text{ind}(f)) = r$, $(1, \frac{\alpha+c}{3}, \frac{\alpha^3+y\alpha+x}{3^r})$ is a 3-integral basis of $\mathbb{Z}_K$ and $v_3(d_k) = v_3(\triangle)$ [2].

(vi) Case 6, $v_p(bc) = 0$, $\bar{F}(X) = X^3 + bX + c$. If $p \in \{2, 3\}$, then $v_p(\triangle) = 0$. Thus, $(1, \alpha, \alpha^2)$ is a 2-integral basis of $\mathbb{Z}_K$ and $v_2(d_K) = v_2(\triangle) = 0$.

For $p \geq 5$, if $\triangle \neq 0\,[p]$, then $\gcd(F, F') = 1$, $(1, \alpha, \alpha^2)$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = v_p(\triangle) = 0$.

If $\triangle = 0\,[p]$, then $\gcd(F, F') = 2bX + 3c$. In that case, let $t \in \mathbb{Z}$ such that $v_p(2bt + 3c) = v_p(\triangle) = s$ and $P(X) = F(X + t) = X^3 + AX^2 + BX + C$, where $v_p(A) = 0$, $B = 3t^2 + b$ and $C = t^3 + bt + c$. Let $2bt = -3c + p^s K$ ($s \geq v_p(\triangle)$. Then $4a^2 B = 0$ modulo $\triangle$ and $8b^3 C = -b\triangle$ modulo $\triangle^2$. Thus, $v_p(\text{ind}(F)) = \lfloor \frac{v_p(\triangle)}{2} \rfloor$ and $(1, \alpha, \frac{\alpha^2 - 2t\alpha + t^2}{p^r})$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = v_p(\triangle)$ [2], where $r = \lfloor \frac{\triangle}{2} \rfloor$. □

3 An integral basis of a cubic number field

Remark 3.1. (1) Let p be a prime integer such that p^2 divides $\triangle$. For every $2 \leq i \leq 3$, let $w_{i,p} = \frac{L_i^p(\alpha)}{p^{r_{i,p}}}$, where $L_i^p(X) \in \mathbb{Z}[X]$ is a monic polynomial of degree $i - 1$ such that $F = (1, w_{2,p}, w_{3,p})$ is a triangular p-integral basis of K. Then $r_{2,p} \leq r_{3,p}$, $v_p(\triangle) = r_2 + r_3$ and $v_p(d_K) = v_p(\triangle) - 2(r_2 + r_3)$.

(2) Let $p_1, \ldots, p_r$ be the primes such that every p_i^2 divides $\triangle$. For every $2 \leq i \leq 3$, denote $d_i = \prod_{j=1}^r p_j^{r_{ij}}$, where for every j, $w_{i,j} = \frac{L_i^{p_j}(\alpha)}{p^{r_{ij}}}$ and $(1, w_{2,j}, w_{3,j})$ is a p_j-integral basis of K. Then $1 \mid d_2 \mid d_3$ are the elementary divisors of $\mathbb{Z}_K/\mathbb{Z}[\alpha]$. In particular, d_3 is the conductor of the order $\mathbb{Z}[\alpha]$ and $d_2 d_3 = \mp\,\text{ind}(f)$.

(3) We can always assume that a triangular p-integral basis has the property: if $r_2 = r_3$, then we can take $w_3 = \alpha w_2$.

One can recover a triangular integral basis from different triangular p-integral bases for all p as follows.

Proposition 3.2. *Let $p_1, \ldots, p_s$ be the prime integers such that p^2 divides $\triangle$ and 1, d_2 and d_3 are the elementary divisors of the Abelian group $\mathbb{Z}_K/\mathbb{Z}[\alpha]$. For every j, let $F_j = (1, w_{2,j}, w_{3,j})$ be a triangular p_j-integral basis of K, i.e., $w_{i,j} = \frac{L_i^j(\alpha)}{p_j^{r_{ij}}}$ such* *that every $L_i^j(X)$ is a monic polynomial of $\mathbb{Z}[X]$ of degree i. Then $B = (1, w_2, w_3)$ is a triangular integral basis of K, where every $w_i = \frac{L_i(\alpha)}{d_i}$, $L_i(X) = L_i^j(X)$ modulo $p_j^{r_{ij}}$.*

Proof. Since $\mathrm{ind}(f) = d_2 d_3$, we need only to check that every $w_i \in \mathbb{Z}_K$. Let $2 \le i \le 3$. Since for every i the integers $(\frac{d_i}{p_j^{r_{ij}}})_{1 \le j \le s}$ are pairwise coprime, there exist integers $t_1, \dots, t_s$ such that $\sum_{j=1}^{s} t_j \frac{d_i}{p_j^{r_{ij}}} = 1$. Hence, $\frac{L_i(\alpha)}{d_i} = \sum_{j=1}^{s} t_j \frac{L_i(\alpha)}{p_j^{r_{ij}}} \in \mathbb{Z}_K$, because all $\frac{L_i(\alpha)}{p_j^{r_{ij}}} \in \mathbb{Z}_K$. □

Corollary 3.3. (i) *Let $(b, c) \in \mathbb{Z}^2$ such that $f = X^3 + bX + c$ is an irreducible polynomial and for every prime p, $v_p(b) \le 1$ or $v_p(c) \le 2$.*

If $b \ne 6$ modulo 9 or $c^2 + b - 1 \ne 0$ modulo 27, then $(1, \alpha, \frac{\alpha^2 + y\alpha + x}{d_2})$ is an integral basis of $\mathbb{Z}_K$, where d_2 is defined in Remarks 3.4 (use Lemma 2.1 to compute x and y).

If $b = 6$ modulo 9 and $c^2 + b - 1 = 0$ modulo 27, then $(1, \frac{\alpha + u}{3}, \frac{\alpha^2 + y\alpha + x}{d_2})$ is an integral basis of $\mathbb{Z}_K$, where d_2 is defined in Remarks 3.4 and $u = 0\,[p]$ for every prime $p \ne 3$, $u = c\,[3]$ (use Lemma 2.1 to compute x and y).

(ii) *Let $b = \prod_{i=1}^{r} p_i^{e_i}$, $c = (\prod_{i=1}^{r} p_i)^2$ and $\pi = \prod_{i=1}^{r} p_i$, where every $e_i \ge 1$ and $b \ne 6$ modulo 9. Then $(1, \alpha, \frac{\alpha^2}{\pi})$ is an integral basis of $\mathbb{Z}_K$ and $d_K = \pi . \prod_{e_i \ge 2} p_i$.*

Remark 3.4. (1) Let $f = X^3 + bX + c$ be an irreducible polynomial. If $v_p(b) \ge 2$ and $v_p(c) \ge 3$, then $K = \mathbb{Q}[\frac{\alpha}{p^k}]$, where $k = \gcd(k_1, k_2)$, k_1 and k_2 are the quotient of the Euclidean division of $v_p(b)$ and $v_p(c)$ by 2 and 3, respectively, and $H(X) = X^3 + b'X + c' \in \mathbb{Z}[X]$ is the minimal polynomial of $\frac{\alpha}{p^k}$, where $\frac{b}{p^{2k}}$ and $\frac{c}{p^{3k}}$. So, we can assume that for every prime p, $v_p(b) \le 1$ or $v_p(c) \le 2$.

(2) Let $K = \mathbb{Q}[\alpha]$, where α is a complex root of an irreducible polynomial $P = X^3 + aX^2 + bX + c$. Then $f = X^3 + BX + C$ is the minimal polynomial of $\theta = 3\alpha + a$, where $B = (9b - 3a^2)$ and $C = -9ab + 27c + 2a^3$. So, to compute an integral basis of $\mathbb{Z}_K$, it suffices to replace α by θ and the polynomial P by $f = X^3 + BX + C$.

Example 3.5. (1) Let $f = X^3 + 300X + 150000$ and $K = \mathbb{Q}[\alpha]$, where α a complex root of f. Since $\triangle = -2^9.3^3.5^6.29.97$ is the discriminant of f, for every prime $p \ge 7$, $(1, \alpha, \alpha^2)$ is a p-integral basis of $\mathbb{Z}_K$.

Since $v_2(300) = 2$, $v_2(150000) = 4$, $v_5(300) = 2$ and $v_5(150000) = 5$, we have $\theta = \frac{\alpha}{10} \in \mathbb{Z}_K$. Let $H = X^3 + 3X + 150$ be the minimal polynomial of θ. Then $\triangle = -2^3.3^3.29.97$ is the discriminant of H. Thus, for every prime $p \ne 2$, $(1, \theta, \theta^2)$ is a p-integral basis of $\mathbb{Z}_K$.

On the other hand, $v_2(30) = 1$ and $v_2(150) = 1$, then $(1, \theta, \theta^2)$ is an integral basis of $\mathbb{Z}_K$.

(2) Let $b \in \mathbb{Z}$ such that $v_3(b) = 0$, $P = X^3 + 3bX^2 + bX + 9b^2$ an irreducible polynomial and $K = \mathbb{Q}[\alpha]$, where α a complex root of P. Then $(1, \theta, \frac{\theta^2}{b^*})$ is an integral basis of $\mathbb{Z}_K$, where b^* is the product of all primes dividing b.

(3) let $P = X^3 + 8X^2 + 10X + 6$ and $K = \mathbb{Q}[\alpha]$, where α a complex root of P. Then $f = X^3 - 102X + 466$ is the minimal polynomial of $\theta = 3\alpha + 8$. Since $\triangle = -2^2.3^7.5.37$ is the discriminant of f, for every prime $p \geq 5$, $(1, \theta, \theta^2)$ is a p-integral basis of $\mathbb{Z}_K$. Since $v_2(102) = 1$ and $v_2(466) = 1$, then $v_2(\text{ind}(f)) = 0$. For $p = 3$, $B = -102 = 6$ modulo 3 and $C^2 + B - 1 = 0$ modulo 27. Then $(1, \frac{\theta+1}{3}, \frac{\theta^2-7\theta-98}{9})$ is an integral basis of $\mathbb{Z}_K$.

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Some new approaches to integer-valued polynomial rings

Jesse Elliott

Abstract. We present some new results on and approaches to integer-valued polynomial rings. One of our results is that, for any PvMD D, the domain $\mathrm{Int}(D)$ of integer-valued polynomials on D is locally free as a D-module if $\mathrm{Int}(D_{\mathfrak{p}}) = \mathrm{Int}(D)_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of D. This fact allows us in particular to strengthen the main results of *J. Algebra* 318 (2007), 68–92, to prove, for example, that the multivariable integer-valued polynomial ring $\mathrm{Int}(D^n)$ decomposes as the n-th tensor power of $\mathrm{Int}(D)$ over D for any such PvMD D. We also present a survey of some new techniques for studying integer-valued polynomial rings – such as universal properties, tensor product decompositions, pullback constructions, and Bhargava rings – that may prove useful.

Keywords. Integer-valued polynomials, PvMD, Krull domain, domain extensions.

AMS classification. 13F20, 13F05, 13G05, 13B02.

1 Introduction

Because they possess a rich theory and provide an excellent source of examples and counterexamples, integer-valued polynomial rings have attained some prominence in the theory of non-Noetherian commutative rings. As with ordinary polynomial rings, much of commutative algebra has some bearing on the subject. Today there remain many open problems concerning, for example, their module structure, ideal structure, prime spectrum, and Krull dimension.

Their definition is simple: for any integral domain D with quotient field K, any set $\underline{X}$, and any subset $\underline{E}$ of $K^{\underline{X}}$, the ring of *integer-valued polynomials on* $\underline{E}$ is the subring

$$\mathrm{Int}(\underline{E}, D) = \{f(\underline{X}) \in K[\underline{X}] : f(\underline{E}) \subset D\}$$

of the polynomial ring $K[\underline{X}]$. As is standard in the literature, we write $\mathrm{Int}(D^{\underline{X}}) = \mathrm{Int}(D^{\underline{X}}, D)$ and $\mathrm{Int}(D) = \mathrm{Int}(D, D)$. We also write $\mathrm{Int}(D^n) = \mathrm{Int}(D^{\underline{X}})$ if $\underline{X}$ is a set of cardinality n.

Historically, many of the results currently known about integer-valued polynomial rings were first proved for number rings and later generalized to larger classes of domains. Research on integer-valued polynomial rings began with some challenging questions surrounding their module structure, and in particular with the search for module bases for them. In more recent years it was discovered that integer-valued polynomial rings also have intricate ideal structures and prime spectra, and questions surrounding their ring-theoretic properties were formulated. From this research several natural settings in which to study integer-valued polynomial rings have emerged. In particular, they have been studied in connection with Dedekind domains, almost Dedekind and Prüfer domains, Noetherian domains, Krull domains, PvMD's, and Mori domains, among many other classes of domains. It is a general thesis of this article that insight into integer-valued polynomial rings may be gained not only from a module-theoretic or ring-theoretic point of view, but also from a category-theoretic viewpoint. For example, at the most basic level they can be characterized via universal properties, and these universal properties motivate questions concerning integer-valued polynomial rings that have not been considered before.

We begin in Section 2 by examining a condition first studied in [11], and later studied in [8] and given the name *polynomial regularity*. This condition generalizes another important condition, namely, the condition $\mathrm{Int}(S^{-1}D) = S^{-1}\mathrm{Int}(D)$, appearing throughout the literature.

In Section 3 we examine *int primes*, as defined in [5], and their relation to *t-maximal ideals*. We prove that the int primes of an arbitrary *H-domain* (as defined in [12]) are precisely its prime conductor ideals having finite residue field. We also show that, for any Krull domain D, or more generally for any PvMD D such that $\mathrm{Int}(D_\mathfrak{p}) = \mathrm{Int}(D)_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of D, the domain $\mathrm{Int}(D)$ of integer-valued polynomials on D is locally free as a D-module. This fact allows us in particular to strengthen several of the main results in [8], including [8, Theorem 1.3]. It also represents some progress toward a classification of those domains D such that $\mathrm{Int}(D)$ is flat as a D-module. Moreover, it allows us to prove in Section 4 that the multivariable integer-valued polynomial ring $\mathrm{Int}(D^n)$ decomposes as the n-th tensor power of $\mathrm{Int}(D)$ over D for any such PvMD D. The tensor product decomposition offers a new approach to studying integer-valued polynomial rings and is discussed further in Section 4.

In Section 5 we motivate some universal characterizations of integer-valued polynomial rings. One of the main ideas is this. In contrast to characterizing the polynomially dense subsets of a given domain, we consider the problem of characterizing those domains that contain a given domain as a polynomially dense subset. We say that an extension A of a domain D is *polynomially complete* if D is a polynomially dense subset of A, that is, if every polynomial with coefficients in the quotient field of A mapping D into A actually maps all of A into A. It turns out that, for any set $\underline{X}$, the domain $\mathrm{Int}(D^{\underline{X}})$ is the free polynomially complete extension of D generated by the set $\underline{X}$, provided only that D is not a finite field [8, Proposition 2.4].

In Section 6 we present some connections to $A + XB[X]$ and other pullback constructions and to Bhargava rings. Finally, in the last section, we define some important D-module lattices contained in $\mathrm{Int}(D)$ and exhibit their relation to the *factorial ideals* $n!_E^D$, as defined in [6, Definition 1.2]. The main results in these last two sections are at this stage somewhat philosophical in nature, although experts in ring-theoretic pullback constructions or module lattices who are also interested in the study of integer-valued polynomial rings may find these approaches useful.

2 The condition $\mathrm{Int}(D, A) = A\,\mathrm{Int}(D)$

The condition $\mathrm{Int}(S^{-1}D) = S^{-1}\mathrm{Int}(D)$ for a multiplicative subset S of D is well known to be extremely important to the study of integer-valued polynomial rings. If D is Noetherian or even Mori then this condition is known to hold for all multiplicative subsets S of D [4, Proposition 2.1], but even for almost Dedekind domains the condition is rather subtle, as seen for example by [17, Theorem 2.4].

The condition $\mathrm{Int}(S^{-1}D) = S^{-1}\mathrm{Int}(D)$ can be subsumed under another condition that appears in the literature. As in [8, Section 3], we say that an extension A of a domain D, by which we mean a domain containing D together with its D-algebra structure, is *polynomially regular* if $\mathrm{Int}(D, A) = A\mathrm{Int}(D)$, where $A\mathrm{Int}(D)$ denotes the A-module (or, equivalently, the A-algebra) generated by $\mathrm{Int}(D)$. Since $\mathrm{Int}(D, S^{-1}D) = \mathrm{Int}(S^{-1}D)$ by [3, Corollary I.2.6], it follows that $\mathrm{Int}(S^{-1}D) = S^{-1}\mathrm{Int}(D)$ if and only if $S^{-1}D$ is a polynomially regular extension of D. Thus the condition $\mathrm{Int}(S^{-1}D) = S^{-1}\mathrm{Int}(D)$ can be subsumed under the polynomial regularity condition.

The condition of polynomial regularity appears in the literature as early as 1993 in an important paper by Gerboud [11], although the condition is not given a name there. There, in another guise, appears the following result, whose proof is trivial.

Proposition 2.1. *For any polynomially regular extension A of a domain D, the following conditions are equivalent.*

(1) *D is a* polynomially dense *subset of A, that is,* $\mathrm{Int}(D, A) = \mathrm{Int}(A)$.

(2) $\mathrm{Int}(A)$ *is equal to the A-module generated by* $\mathrm{Int}(D)$.

(3) $\mathrm{Int}(A) \supset \mathrm{Int}(D)$.

Note that, for an extension A of a domain D that is not necessarily polynomially regular, conditions (1) and (2) each imply condition (3), but (1) need not imply (2) and (2) need not imply (1). For example, if D is a domain containing a multiplicative subset S for which $\mathrm{Int}(S^{-1}D) \neq S^{-1}\mathrm{Int}(D)$, then $A = S^{-1}D$ is an extension of D satisfying (1) but not (2). Conversely, the extension $A = \mathbb{Z}[T/2]$ of $D = \mathbb{Z}[T]$ satisfies condition (2) but not condition (1), by [8, Example 7.3].

The following is another result from [11].

Theorem 2.2. *Let D be a Dedekind domain. Then any extension of D is polynomially regular, and for any extension A of D, the domain D is a polynomially dense subset of A if and only if the extension A of D has trivial residue field extensions, and is unramified, at every nonzero prime ideal of D with finite residue field.*

The latter of the two equivalent conditions of the theorem is to be understood as follows: for every nonzero prime ideal $\mathfrak{p}$ of D with finite residue field and every prime ideal $\mathfrak{P}$ of A lying over $\mathfrak{p}$ one has $A/\mathfrak{P} = D/\mathfrak{p}$ and $\mathfrak{p}A_{\mathfrak{P}} = \mathfrak{P}A_{\mathfrak{P}}$.

The result above was recently extended to flat extensions of Krull domains [8, Theorem 1.3]. In particular, it was shown that every flat extension of a Krull domain is polynomially regular. In the next section we will prove a stronger version of that theorem, Theorem 3.8.

3 Int primes and the condition $\mathrm{Int}(D) \subset S^{-1}D[X]$

Another condition important for studying integer-valued polynomial rings is the condition $\mathrm{Int}(D) \subset S^{-1}D[X]$, or equivalently $S^{-1}\mathrm{Int}(D) = S^{-1}D[X]$, for multiplicative subsets S of D. Recall that a *conductor ideal* of D is an ideal of the form $(aD :_D bD)$

for some $a, b \in D$ with $a \neq 0$. Define $d_n(\underline{X}) \in \mathbb{Z}[\underline{X}] = \mathbb{Z}[X_0, X_1, \ldots, X_n]$ by

$$d_n(\underline{X}) = \prod_{0 \le j < i \le n} (X_i - X_j).$$

The following result follows immediately from [17, Theorem 1.5].

Proposition 3.1. *For any multiplicative subset S of an integral domain D, one has* $\mathrm{Int}(D) \subset S^{-1}D[X]$ *if and only if every conductor ideal of D that contains $d_n(D^{n+1})$ for some positive integer n meets S.*

Following [5], we say that a prime ideal $\mathfrak{p}$ of a domain D is a *polynomial prime* of D if $\mathrm{Int}(D) \subset D_{\mathfrak{p}}[X]$, or equivalently if $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$; otherwise $\mathfrak{p}$ is said to be an *int prime* of D. For a nonzero ideal I of D, we let I_v denote the *divisorial closure* $(I^{-1})^{-1}$ of I, where $I^{-1} = (D :_K I)$, where K is the quotient field of D. A nonzero ideal I of D is said to be *divisorial* if $I = I_v$, and I is said to be a *t-ideal* if $J_v \subset I$ for every nonzero finitely generated ideal $J \subset I$. Every divisorial ideal of D is a t-ideal, and if the converse holds then D is said to be a *TV-domain*. An ideal I of D is *t-maximal* if it is maximal among the proper t-ideals of D. Every t-maximal ideal is prime, and by an application of Zorn's lemma, every proper t-ideal is contained in some t-maximal ideal. A domain D is said to be an *H-domain* if every t-maximal ideal of D is divisorial, or equivalently if every t-maximal ideal of D is a conductor ideal of D [12]. The H-domains are a large class of domains properly containing the Mori domains and the TV-domains, and in fact one has Noetherian $\Rightarrow$ Mori $\Rightarrow$ TV $\Rightarrow$ H [20].

A prime ideal $\mathfrak{p}$ of D is said to be a *Bourbaki associated prime* of a D-module M if $\mathfrak{p}$ equals an annihilator of some element of M. A prime ideal $\mathfrak{p}$ is said to be a *weak Bourbaki associated prime* of M if $\mathfrak{p}$ is minimal over an annihilator of some element of M. The sets of all such primes $\mathfrak{p}$ of D are denoted $\mathrm{Ass}(M)$ and $\mathrm{wAss}(M)$, respectively. A prime $\mathfrak{p}$ lies in $\mathrm{wAss}(K/D)$ (resp., $\mathrm{Ass}(K/D)$) if and only if $\mathfrak{p}$ is minimal over (resp., equal to) some conductor ideal of D.

If $\mathfrak{p}$ is a prime t-ideal of D such that $\mathfrak{p}D_{\mathfrak{p}}$ is a t-ideal (hence a t-maximal ideal) of $D_{\mathfrak{p}}$, then $\mathfrak{p}$ is said to be a *t-localizing*, or *well behaved*, prime of D [2, 19]. A prime ideal $\mathfrak{p}$ of D is said to be a *strong Krull prime*, or *Northcott attached prime*, of a D-module M if for every finitely generated ideal $I \subset \mathfrak{p}$ there exists an $m \in M$ such that $I \subset \mathrm{ann}(m) \subset \mathfrak{p}$ [7]. We let $\mathrm{sKr}(M)$ denote the set of all strong Krull primes of D. One has $\mathrm{wAss}(M) \subset \mathrm{sKr}(M)$ for any D-module M by [7, Proposition 2]. By [19, Proposition 1.1] (or by [15, Proposition 2.2]), one has the following.

Proposition 3.2. *Let D be an integral domain with quotient field K. A prime ideal $\mathfrak{p}$ of D is t-localizing if and only if $\mathfrak{p} \in \mathrm{sKr}(K/D)$, that is, if and only if for every finitely generated ideal $I \subset \mathfrak{p}$ there exists a conductor ideal J of D such that $I \subset J \subset \mathfrak{p}$.*

In particular, one has

$$\mathrm{Ass}(K/D) \subset \mathrm{wAss}(K/D) \subset \mathrm{sKr}(K/D) \subset t\text{-}\mathrm{Spec}(D)$$

for any domain D, where t-$\mathrm{Spec}(D)$ denotes the set of prime t-ideals of D.

By [5, Proposition 1.2], every int prime of a domain is a t-ideal. The proof in fact leads to the following generalization of that result.

Proposition 3.3. *Let D be an integral domain with quotient field K, and let $\mathfrak{p}$ be any prime ideal of D with finite residue field. Each of the following conditions implies the next:*

(a) $\mathfrak{p} \in \mathrm{Ass}(K/D)$.

(b) $\mathfrak{p}$ *is an int prime of* D.

(c) *There exists a conductor ideal I of D contained in $\mathfrak{p}$ such that the residue fields of the prime ideals of D containing I have bounded cardinality.*

(d) $\mathfrak{p} \in \mathrm{wAss}(K/D)$.

(e) $\mathfrak{p} \in \mathrm{sKr}(K/D)$.

(f) $\mathfrak{p}$ *is t-maximal.*

Moreover, if D is an H-domain, or more generally a domain such that every t-maximal ideal of D with finite residue field is a conductor ideal, then these six conditions are equivalent.

Proof. Suppose that (a) holds. Then $\mathfrak{p} = (aD :_D bD)$ for some $a, b \in D$ with $a \neq 0$. Letting $\{u_1, u_2, \ldots, u_q\}$ be a system of representatives modulo $\mathfrak{p}$, we see that the polynomial $f(X) = \frac{b}{a}\prod_{i=1}^{q}(X - u_i)$ lies in $\mathrm{Int}(D)$ but not in $D_{\mathfrak{p}}[X]$, and therefore (b) holds. Thus (a) implies (b). Suppose that (b) holds. By Proposition 3.1 there exists a positive integer n such that $d_n(D^{n+1})$ is contained in some conductor ideal I contained in $\mathfrak{p}$. Let $\mathfrak{q}$ be any prime ideal of D containing I. Since $d_n(D^{n+1}) \subset \mathfrak{q}$, it follows that, among any $n+1$ elements of D, at least two of them must be congruent modulo $\mathfrak{q}$, whence the residue field of $\mathfrak{q}$ has cardinality at most n. Thus (b) implies (c). If (c) holds, then every prime ideal containing I, including $\mathfrak{p}$, is maximal, and therefore minimal over I, whence (d) holds. Next, we have (d) $\Rightarrow$ (e) $\Rightarrow$ (f) since $\mathrm{wAss}(K/D) \subset \mathrm{sKr}(K/D) \subset t\text{-}\mathrm{Spec}(D)$. Finally, if every t-maximal ideal of D with finite residue field is a conductor ideal, then (f) implies (a) and the six conditions are equivalent. □

Corollary 3.4. *For any H-domain D, the int primes of D are precisely the prime conductor ideals of D with finite residue field.*

Note that Corollary 3.4 is already known to hold for the Noetherian domains; see [3, Theorem I.3.14].

Recall that an integral domain D is said to be a *PvMD* if $D_{\mathfrak{p}}$ is a valuation domain for every t-maximal ideal $\mathfrak{p}$ of D. The domain $\mathrm{Int}(D)$ is said to have a *regular basis* if it has a free D-module basis consisting of exactly one polynomial of each degree. Proposition 3.3 can also be used to prove the following.

Proposition 3.5. *Let D be a PvMD such that* $\mathrm{Int}(D_{\mathfrak{p}}) = \mathrm{Int}(D)_{\mathfrak{p}}$ *for every prime ideal $\mathfrak{p}$ of D. Then* $\mathrm{Int}(D_{\mathfrak{p}})$ *has a regular basis for every prime ideal $\mathfrak{p}$ of D, and therefore the D-module* $\mathrm{Int}(D)$ *is locally free, hence faithfully flat.*

Proof. If $\mathrm{Int}(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$, then $\mathrm{Int}(D_{\mathfrak{p}})$ clearly has a regular basis. Suppose on the other hand that $\mathrm{Int}(D_{\mathfrak{p}}) \neq D_{\mathfrak{p}}[X]$. Then $\mathrm{Int}(D)_{\mathfrak{p}} \neq D_{\mathfrak{p}}[X]$, whence $\mathfrak{p}$ is an int prime of D. Therefore $\mathfrak{p}$ is a t-maximal ideal of $D_{\mathfrak{p}}$ with finite residue field, by Proposition 3.3 (or alternatively by [5, Proposition 2.1]). Since D is a PvMD, it follows that $D_{\mathfrak{p}}$ is a valuation domain, and since $\mathrm{Int}(D_{\mathfrak{p}}) \neq D_{\mathfrak{p}}[X]$, it follows from [3, Proposition I.3.16] that the maximal ideal $\mathfrak{p}D_{\mathfrak{p}}$ is principal. But in that case it is known that $\mathrm{Int}(D_{\mathfrak{p}})$ has a regular basis; see [3, Exercise II.16], for example. □

For any Krull domain (or in fact for any Mori domain) D, one has $\mathrm{Int}(D_{\mathfrak{p}}) = \mathrm{Int}(D)_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of D, by [4, Proposition 2.1]. Thus we have the following.

Corollary 3.6. *For any Krull domain D, the D-module* $\mathrm{Int}(D)$ *is locally free, hence faithfully flat.*

Remark 3.7. The referee has pointed out an alternative proof of Corollary 3.6. Let D be a Krull domain. If $\mathfrak{p}$ is a height one prime ideal of D, then $D_{\mathfrak{p}}$ is a DVR and $\mathrm{Int}(D_{\mathfrak{p}})$ is well known to be free over $D_{\mathfrak{p}}$ in that case, and by [3, Proposition I.2.8] one has $\mathrm{Int}(D)_{\mathfrak{p}} = \mathrm{Int}(D_{\mathfrak{p}})$. If $\mathfrak{p}$ has height at least two, then as a special case of [14, Theorem 3] we have $\mathrm{Int}(D)_{\mathfrak{p}} = \mathrm{Int}(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$. Thus, for any Krull domain D and any prime ideal $\mathfrak{p}$ of D, the $D_{\mathfrak{p}}$-module $\mathrm{Int}(D)_{\mathfrak{p}} = \mathrm{Int}(D_{\mathfrak{p}})$ is free.

As a consequence of Corollary 3.6 above, we can prove a strengthening of [8, Theorem 1.3]. An extension A of a Krull domain D is said to be *divisorial* if $A = \bigcap_{\mathfrak{p} \in X^1(D)} A_{\mathfrak{p}}$, where $X^1(D)$ denotes the set of height one prime ideals of D. For example, any flat extension of a Krull domain is divisorial, and $\mathrm{Int}(D^n)$ is a divisorial extension of D for any Krull domain D and any positive integer n. (A divisorial extension of a Krull domain D is equivalently a *t-linked extension* of D, or an extension of D satisfying the condition *PDE*, both as defined in [1], for example.) Given Corollary 3.6 above, we obtain from [8, Corollary 6.13 and Theorem 1.3] the following.

Theorem 3.8. *Any divisorial extension of a Krull domain D is polynomially regular, and for any such extension A of D, the domain D is a polynomially dense subset of A if and only if the extension A of D has trivial residue field extensions, and is unramified, at every height one prime ideal of D with finite residue field.*

Note that if D is a UFD, then $\mathrm{Int}(D)$ is free and in fact has a regular basis, by [10, Theorem 3.6 Corollary 1]. If D is a Dedekind domain then $\mathrm{Int}(D)$ is a non-finitely generated projective module and is therefore free, but it may or may not have a regular basis. Remarkably, we do not know of an example of a domain D such that $\mathrm{Int}(D)$ is not free. However, we conjecture that such an example does exist.

Flatness is another question altogether. Does there exist a domain D such that $\mathrm{Int}(D)$ is not flat as a D-module? For which domains D, if not for all domains, is $\mathrm{Int}(D)$ flat as a D-module? These are open questions.

4 Tensor product decompositions

The question of the flatness of $\mathrm{Int}(D)$ over D is motivated by considering that for any domain D and any set $\underline{X}$ there exists a canonical D-algebra homomorphism

$$\theta_{\underline{X}} : \bigotimes_{X \in \underline{X}} \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D^{\underline{X}}),$$

where the (possibly infinite) tensor product is a tensor product of D-algebras. One might hope, if not expect, that this homomorphism be an isomorphism, at least if integer-valued polynomials are expected to behave anything like ordinary polynomials in this regard. After all, one does have $\mathrm{Int}(\mathrm{Int}(D^{\underline{X}})^{\underline{Y}}) = \mathrm{Int}(D^{\underline{X} \amalg \underline{Y}})$ for any infinite domain D and any sets $\underline{X}$ and $\underline{Y}$, in perfect analogy with ordinary polynomial rings over anything but a finite field; and indeed for several large classes of domains it turns out that $\theta_{\underline{X}}$ is an isomorphism for all $\underline{X}$, as is discussed below. However, it has not been proved that $\theta_{\underline{X}}$ is always an isomorphism, nor has there been found a counterexample. Proving injectivity is equivalent to showing that the given tensor product is D-torsion-free, and the easiest way to do that seems to be to prove that $\mathrm{Int}(D)$ is flat as a D-module.

If D is a domain such that the homomorphism $\theta_{\underline{X}}$ above is an isomorphism for any set $\underline{X}$, we say that D is *polynomially composite*. Polynomial compositeness is useful for studying integer-valued polynomial rings of several variables in terms of those of a single variable, but that is not its only use. In Proposition 5.2, we will see that the polynomial compositeness of a domain D implies that $\mathrm{Int}(D^{\underline{X}})$ for any set $\underline{X}$ has a universal property that may not otherwise hold. In fact, it also implies the existence of the structure on $\mathrm{Int}(D)$ of a *monoid object in the category D-D-birings*, also known as a *D-D-biring triple*, or a *D-plethory*. In particular, for any polynomially composite domain D the set $\mathrm{Hom}_D(\mathrm{Int}(D), A)$ is endowed with a natural D-algebra structure for any D-algebra A, and the endofunctor $\mathrm{Hom}_D(\mathrm{Int}(D), -)$ of the category of D-algebras has a left adjoint. (The same is always true for $D[X]$, but the endofunctor $\mathrm{Hom}_D(D[X], -)$ is just the identity functor.) We will take up these topics elsewhere [9].

A domain D is said to be *Newtonian* if there exists an infinite sequence $\{a_n\}_{n=0}^{\infty}$ of distinct elements of D such that the polynomials $f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k}$ lie in $\mathrm{Int}(D)$ for all nonnegative integers n [6, Section 3]; in that case the polynomials $f_n(X)$ form a regular basis for $\mathrm{Int}(D)$. For example, every local ring with infinite residue field or principal maximal ideal is Newtonian. The following proposition provides the simplest examples of polynomially composite domains.

Proposition 4.1. *Let D be an integral domain. Each of the following conditions implies the next:*

(a) *D is a UFD or a Newtonian domain.*

(b) $\mathrm{Int}(D)$ *has a regular basis.*

(c) $\mathrm{Int}(D)$ *is free as a D-module.*

(d) *D is polynomially composite.*

Proof. If D is a UFD, then $\mathrm{Int}(D)$ has a regular basis by [10, Theorem 3.6 Corollary 1], or by [3, Exercise II.23]. Thus (a) implies (b). Clearly (b) implies (c), and (c) implies (d) by [8, Proposition 6.8]. □

To state our next result, we recall some further definitions from [8]. A domain D is *absolutely polynomially regular* if every extension of D is polynomially regular. For example, every Dedekind domain is absolutely polynomially regular, by Theorem 2.2, while $\mathbb{Z}[T]$ is not, by [8, Example 7.3]. A prime ideal $\mathfrak{p}$ of a domain D is said to be *Newtonian* if $D_\mathfrak{p}$ is Newtonian and $\mathrm{Int}(D_\mathfrak{p}) = \mathrm{Int}(D)_\mathfrak{p}$. Finally, a domain D is *almost Newtonian* if every maximal (or prime) ideal of D is Newtonian. For example, every Dedekind domain is almost Newtonian. Note that every Newtonian domain is almost Newtonian, and every almost Newtonian domain is absolutely polynomially regular.

Proposition 4.2. *Let D be an integral domain. Each of the following conditions implies the next:*

(a) *D is a Krull domain or an almost Newtonian domain.*

(b) $\mathrm{Int}(D)$ *is flat over D, and D is a locally finite intersection of flat, polynomially regular, and absolutely polynomially regular overrings.*

(c) $\mathrm{Int}(D)$ *is flat over D, and every flat extension of D is polynomially regular.*

(d) *D is polynomially composite.*

Proof. This follows readily from Corollary 3.6 and [8, Corollary 6.9]. □

In fact, we also have the following.

Proposition 4.3. *Let D be an integral domain. Each of the following conditions implies the next:*

(a) *D is a PvMD such that* $\mathrm{Int}(D_\mathfrak{m}) = \mathrm{Int}(D)_\mathfrak{m}$ *for every maximal ideal* $\mathfrak{m}$ *of D, or D is almost Newtonian.*

(b) *Every maximal ideal* $\mathfrak{m}$ *of D such that* $\mathrm{Int}(D_\mathfrak{m}) \neq D_\mathfrak{m}[X]$ *is Newtonian.*

(c) $\mathrm{Int}(D_\mathfrak{m})$ *has a regular basis, and* $\mathrm{Int}(D_\mathfrak{m}) = \mathrm{Int}(D)_\mathfrak{m}$, *for every maximal ideal* $\mathfrak{m}$ *of D.*

(d) $\mathrm{Int}(D_\mathfrak{m})$ *is free as a $D_\mathfrak{m}$-module, and* $\mathrm{Int}(D_\mathfrak{m}) = \mathrm{Int}(D)_\mathfrak{m}$, *for every maximal ideal* $\mathfrak{m}$ *of D.*

(e) *D is polynomially composite.*

Proof. Condition (a) implies condition (b) by [8, Lemma 5.2] and the proof of Proposition 3.5. The remainder of the proposition follows from [8, Proposition 6.10]. □

5 Universal properties

Although the domain $\mathrm{Int}(D)$ is not functorial in D, it can nevertheless be characterized by universal properties. We say that an extension A of a domain D is *polynomially* *complete* if D is a polynomially dense subset of A, that is, if $\mathrm{Int}(D, A) = \mathrm{Int}(A)$, or, what is the same, if every polynomial with coefficients in the quotient field of A mapping D into A actually maps all of A into A. It turns out that, for any set $\underline{X}$, the domain $\mathrm{Int}(D^{\underline{X}})$ is the free polynomially complete extension of D generated by the set $\underline{X}$, provided only that D is not a finite field [8, Proposition 2.4]. In other words, the functor $\underline{X} \longmapsto \mathrm{Int}(D^{\underline{X}})$ is a left adjoint for the forgetful functor from the category of polynomially complete extensions of D (with morphisms as the D-algebra homomorphisms) to the category of sets, in the sense that the natural map

$$\mathrm{Hom}_D(\mathrm{Int}(D^{\underline{X}}), A) \longrightarrow \mathrm{Hom}(\underline{X}, A)$$

is a bijection for any set $\underline{X}$ and any polynomially complete extension A of D. Thus, rather than only consider the subsets of a given domain that are polynomially dense, it is natural also to consider those domains containing a given domain as a polynomially dense subset. There are several existing theorems in the literature that do this. Examples from this article are Theorem 2.2 (appearing first in [11]) and Theorem 3.8.

As shown by Proposition 2.1, if A is a polynomially regular extension of a domain D, then A is a polynomially complete extension of D if and only if $\mathrm{Int}(A) \supset \mathrm{Int}(D)$. Generally, if the containment $\mathrm{Int}(A) \supset \mathrm{Int}(D)$ holds, then, as in [8], we will say that the extension A of D is *weakly polynomially complete*. Polynomial completeness implies weak polynomial completeness, but the converse is not true in general, as shown by the extension $\mathbb{Z}[T/2]$ of $\mathbb{Z}[T]$. Nevertheless, there are conditions besides polynomial regularity under which these two conditions are equivalent. The most general such condition we know of is given in [8, Proposition 3.3], restated below.

Proposition 5.1. *Let A be an extension of a domain D. Suppose that $A = \bigcap_{i \in I} A_i$, where the A_i are overrings of A that are polynomially regular extensions of D. Then the following conditions are equivalent:*

(1) *A is a polynomially complete extension of D.*

(2) $\mathrm{Int}(A) = \bigcap_{i \in I} A_i \mathrm{Int}(D)$.

(3) *A is a weakly polynomially complete extension of D.*

Note that an extension A of D is weakly polynomially complete if and only if for every $a \in A$ there exists a (unique) D-algebra homomorphism $\mathrm{Int}(D) \longrightarrow A$ sending the variable X to a. This equivalence, together with results like Proposition 2.1, indicates that weak polynomial completeness is a natural condition to consider.

Interestingly, there is a third condition weaker than polynomial completeness but *a priori* stronger than weak polynomial completeness that in some sense bridges a gap between the two conditions. As in [8] we say that an extension A of a domain D is *almost polynomially complete* if $\mathrm{Int}(A^n) \supset \mathrm{Int}(D^n)$ for every positive integer n, or, equivalently, if for every $a_1, a_2, \ldots, a_n \in A$ there exists a (unique) D-algebra homomorphism $\mathrm{Int}(D^n) \longrightarrow A$ sending X_i to a_i for all i. We note that there exist almost polynomially complete extensions of some domains that are not polynomially complete. The extension $\mathbb{Z}[T/2]$ of $\mathbb{Z}[T]$ again is an example.

As we remarked earlier, provided that D is not a finite field, the domain $\mathrm{Int}(D^{\underline{X}})$ is the free polynomially complete extension of D generated by $\underline{X}$, for any set $\underline{X}$. Likewise, by [8, Proposition 7.7], the domain $\mathrm{Int}(D^{\underline{X}})$ is the free almost polynomially complete extension of D generated by $\underline{X}$. There also exists a free weakly polynomially complete extension of D generated by $\underline{X}$. Let $\mathrm{Int}_{\mathrm{w}}(D^{\underline{X}})$ be the intersection of all D-subalgebras of $\mathrm{Int}(D^{\underline{X}})$ containing $D[\underline{X}]$ that are closed under pre-composition by elements of $\mathrm{Int}(D)$. The domain $\mathrm{Int}_{\mathrm{w}}(D^{\underline{X}})$ is the smallest weakly polynomially complete extension of D containing $D[\underline{X}]$. By [8, Proposition 7.8], it is also the free weakly polynomially complete extension of D generated by $\underline{X}$. Although we can only prove that $\mathrm{Int}(D^{\underline{X}}) = \mathrm{Int}_{\mathrm{w}}(D^{\underline{X}})$ for all $\underline{X}$ for certain classes of domains D, such as the Krull domains, we do not know a counterexample. Nevertheless, by [8, Propositions 7.8 and 7.9] we have the following.

Proposition 5.2. *Let D be a domain. The following conditions are equivalent:*

(1) $\mathrm{Int}(D^{\underline{X}})$ *is the free weakly polynomially complete extension generated by $\underline{X}$, for any set $\underline{X}$.*

(2) $\mathrm{Int}(D^{\underline{X}}) = \mathrm{Int}_{\mathrm{w}}(D^{\underline{X}})$ *for every set $\underline{X}$.*

(3) *Every weakly polynomially complete extension of D is almost polynomially complete.*

Moreover, if D is polynomially composite, then all of the above conditions hold.

As indicated in Section 4, we do not know an example of a domain D that is not polynomially composite. To prove that a given domain D is not polynomially composite, it would suffice by Proposition 5.2 to find a weakly polynomially complete extension of D that is not almost polynomially complete.

6 Pullback constructions and Bhargava rings

Another way in which to characterize integer-valued polynomial rings is via the following pullback construction.

Proposition 6.1. *Let D be a domain with quotient field K, let $\underline{X}$ be a set, and let $\underline{E}$ be a subset of $K^{\underline{X}}$. Let $\mathrm{eval}_{\underline{E}} : K[\underline{X}] \longmapsto K^{\underline{E}}$ be the evaluation homomorphism $f(\underline{X}) \longmapsto (f(\underline{a}))_{\underline{a} \in \underline{E}}$. Then $\mathrm{Int}(\underline{E}, D)$ is the pullback of $D^{\underline{E}}$ along $\mathrm{eval}_{\underline{E}}$. In fact, the commutative square*

$$\begin{array}{ccc} \mathrm{Int}(\underline{E}, D) & \longrightarrow & D^{\underline{E}} \\ \downarrow & & \downarrow \\ K[\underline{X}] & \xrightarrow{\mathrm{eval}_{\underline{E}}} & K^{\underline{E}} \end{array}$$

is both cartesian and co-cartesian.

The proof of the above proposition is routine, so we omit it.

Generally speaking, more information can be obtained about pullbacks along homomorphisms that are surjective onto a field. See [13, Theorem 2.4] for a typical example. Although the homomorphism $\mathrm{eval}_{\underline{E}}$ along which $\mathrm{Int}(\underline{E}, D)$ is a pullback is not surjective onto a field unless $\underline{E}$ is a singleton, some of the known techniques may still extend to this situation.

It has been suggested that there are connections between integer-valued polynomial rings and the so-called $A + XB[X]$ domains, both of which have been studied extensively in the literature. Here we make one such connection explicit. First, we note that the domain $D + XK[X]$ is the pullback of D along the K-algebra homomorphism $\mathrm{eval}_0 : K[X] \longrightarrow K$ sending $f(X)$ to $f(0)$; in fact the square

$$\begin{array}{ccc} D + XK[X] & \longrightarrow & D \\ \downarrow & & \downarrow \\ K[X] & \xrightarrow{\ \mathrm{eval}_0\ } & K \end{array}$$

is cartesian and co-cartesian. We first observe that the domain $\mathrm{Int}(E, D)$, for any subset E of K, may be represented as an intersection over E of subrings of $K[X]$ isomorphic to $D + XK[X]$.

Proposition 6.2. *Let D be a domain with quotient field K, and let E be a subset of K. One has $\mathrm{Int}(E, D) = \bigcap_{a \in E} D + (X - a)K[X]$. Moreover, for all $a \in E$ the map $D + (X - a)K[X] \longrightarrow D + XK[X]$ acting by $f(X) \longmapsto f(X + a)$ is a D-algebra isomorphism.*

Proof. If $E = \{0\}$, then $\mathrm{Int}(E, D)$ is just the pullback $D + XK[X]$. For any $a \in E$ the map $f(X) \longmapsto f(X + a)$ is a D-algebra isomorphism from $\mathrm{Int}(\{a\}, D)$ to $\mathrm{Int}(\{0\}, D)$. It follows that $\mathrm{Int}(\{a\}, D) = D + (X - a)K[X - a] = D + (X - a)K[X]$. Therefore

$$\mathrm{Int}(E, D) = \bigcap_{a \in E} \mathrm{Int}(\{a\}, D) = \bigcap_{a \in E} D + (X - a)K[X].$$

This completes the proof. □

We may reinterpret Proposition 6.2 diagrammatically as follows.

Proposition 6.3. *Let D be a domain with quotient field K, and let E be a subset of K. Let $t_E : K[X] \longrightarrow K[X]^E$ be the K-algebra homomorphism acting by $f(X) \longmapsto (f(X + a))_{a \in E}$. Then the commutative squares*

$$\begin{array}{ccccc} \mathrm{Int}(E, D) & \longrightarrow & (D + XK[X])^E & \longrightarrow & D^E \\ \downarrow & & \downarrow & & \downarrow \\ K[X] & \xrightarrow{\ t_E\ } & K[X]^E & \xrightarrow{\ \mathrm{eval}_0^E\ } & K^E \end{array}$$

are both cartesian and co-cartesian.

As with $A + XB[X]$ domains, we may make explicit connections between integer-valued polynomial rings and *Bhargava rings* [18]. For any subset E of a domain D and any $a \in D$, let

$$\mathrm{B}_a(E, D) = \{f(X) \in K[X] : f(aX + b) \in D[X] \text{ for all } b \in E\}$$

denote the *Bhargava ring of D over E at a*. Clearly one has

$$\mathrm{B}_a(E, D) = \bigcap_{b \in E} D\Big[\frac{X - b}{a}\Big]$$

for all $a \neq 0$, and $\mathrm{B}_0(E, D) = \mathrm{Int}(E, D)$. In fact, by the proof of [18, Proposition 1.1], if S is any system of representatives of E modulo aD, then

$$\mathrm{B}_a(E, D) = \bigcap_{b \in S} D\Big[\frac{X - b}{a}\Big].$$

Moreover, by the proof of [18, Theorem 1.4], one has

$$\mathrm{Int}(E, D) = \bigcup_{a \in D \setminus \{0\}} \mathrm{B}_a(E, D),$$

and in fact this union may be interpreted as a direct limit over the partially ordered set of principal ideals of D ordered by inclusion. This yields a potentially useful technique for proving $\mathrm{Int}(D)$ flat as a D-module: since flatness is preserved under direct limits, the question of flatness of $\mathrm{Int}(D)$ can be reduced to the question of flatness of the Bhargava rings $\mathrm{B}_a(D, D)$.

Proposition 6.4. *Let E be a subset of a domain D. Suppose that $\mathrm{B}_a(E, D)$ is flat as a D-module for every $a \in D \setminus \{0\}$. Then $\mathrm{Int}(E, D)$ is also flat as a D-module.*

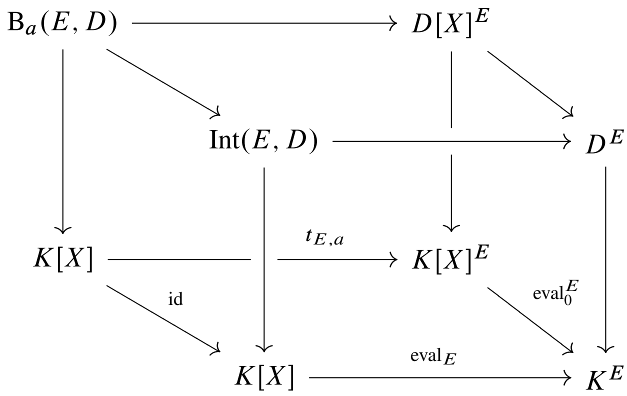
As with $\mathrm{Int}(E, D)$ in Proposition 6.3, we may characterize the Bhargava rings $\mathrm{B}_a(E, D)$ as pullbacks.

Proposition 6.5. *Let D be a domain with quotient field K, let E be a subset of K, and let $a \in D$. Let $t_{E,a} : K[X] \longrightarrow K[X]^E$ be the K-algebra homomorphism acting by $f(X) \longmapsto (f(aX + b))_{b \in E}$. The commutative square*

$$\begin{array}{ccc} \mathrm{B}_a(E, D) & \longrightarrow & D[X]^E \\ \downarrow & & \downarrow \\ K[X] & \xrightarrow{t_{E,a}} & K[X]^E \end{array}$$

is cartesian and co-cartesian.

Note that the pullback squares of Proposition 6.1 (for $\underline{X}$ a singleton) and Proposition 6.5 fit neatly into the following commutative diagram.



7 Lattices and factorial ideals

Let E be a subset of a domain D. The set $\mathrm{Int}_n(E, D)$ of polynomials in $\mathrm{Int}(E, D)$ of degree at most n is a D-submodule of $\mathrm{Int}(E, D)$. Since

$$\mathrm{Int}(E, D) = \bigcup_{n \geq 0} \mathrm{Int}_n(E, D),$$

the D-module $\mathrm{Int}(E, D)$ is a direct limit of the D-modules $\mathrm{Int}_n(E, D)$. The submodules $\mathrm{Int}_n(E, D)$ afford another technique for studying the module structure of $\mathrm{Int}(D)$. For example, we have the following result, whose proof is clear.

Proposition 7.1. *Let E be a subset of a domain D. If* $\mathrm{Int}_n(E, D)$ *is flat as a D-module for all n, then* $\mathrm{Int}(E, D)$ *is flat as a D-module. The converse is true if* $\mathrm{Int}_n(E, D)$ *is pure in* $\mathrm{Int}(E, D)$ *for all n.*

Note that $\mathrm{Int}_n(D)$ is projective for all n if D is a Dedekind domain, by [3, Corollary II.3.6], and [10, Theorem 3.4] suggests that the same might be true if D is a Krull domain.

The *rank* of a D-module M is the dimension of the K-vectorspace $K \otimes_D M$, where K is the quotient field of D. A torsion-free D-module M is a *D-lattice* if D is contained in a finitely generated D-submodule of $K \otimes_D M$, or, equivalently, if D is contained in a free D-module of the same finite rank.

For any n let $D[X]_n$ denote the free D-module consisting of the polynomials in $D[X]$ of degree at most n. The *n-th factorial ideal of E with respect to D* is the ideal

$$n!_E^D = (D[X]_n :_D \mathrm{Int}_n(E, D))$$

of D [6, Definition 1.2]. Note that $n!_{\mathbb{Z}}^{\mathbb{Z}} = n!\mathbb{Z}$ for all n, so factorial ideals generalize ordinary factorials. The factorial ideals $n!_E^D$ form a descending sequence of ideals of D with $0!_E^D = D$. Factorial ideals and D-lattices afford yet further techniques for studying integer-valued polynomial rings. Of particular interest in this regard is the following elementary result.

Proposition 7.2. *Let D be a domain, and let E be a subset of D of cardinality at least $n+1$. For any sequence $a_0, a_1, \ldots, a_n$ of $n+1$ distinct elements of E, one has $d_n \in n!_E^D$, or equivalently*

$$\mathrm{Int}_n(E, D) \subset (1/d_n)D[X]_n,$$

where

$$d_n = \prod_{0 \le j < i \le n} (a_i - a_j)$$

for all i. In particular the D-module $\mathrm{Int}_n(E, D)$ is a D-lattice of rank $n+1$.

Proof. This is a restatement of [3, Proposition I.3.1]. □

We end with the following result.

Proposition 7.3. *Let E be a subset of a domain D. The n-th factorial ideal $n!_E^D$ is divisorial for every positive integer n.*

Proof. Let $\mathcal{I}_n(E, D)$ denote the D-submodule of K generated by all of the coefficients of all of the polynomials in $\mathrm{Int}_n(E, D)$. By definition one clearly has $n!_E^D = \mathcal{I}_n(E, D)^{-1}$. Moreover, $\mathcal{I}_n(E, D)$ is a fractional ideal of D by Proposition 7.2 above. Since I^{-1} is a divisorial fractional ideal for any fractional ideal I of D, the proposition follows. □

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v-ordering sequences and countable sets

Youssef Fares

Abstract. The notion of v-ordering for a subset E of the domain V of a valuation v introduced by M. Bhargava turns out to be very useful for both the construction of regular bases of the V-module $\mathrm{Int}(E, V)$ of integer-valued polynomials on E and the construction of orthonormal bases of the ring $C(E, V)$ of continuous functions from E to V. The aim of this paper is to show that, when E is a countable and precompact subset of V, one can construct a v-ordering of E in which every element of E occurs exactly once. In order to do this, we use the notion of polynomial equivalence introduced by D. McQuillan.

Keywords. Integer-valued polynomial, valuation domain.

AMS classification. 13F20.

1 Polynomial equivalence

Let A be a domain with quotient field K and let E be a nonempty subset of A. Recall that $\mathrm{Int}(E, A)$ is the sub-A-algebra of $K[X]$ formed by all integer-valued polynomials on E, that is,

$$\mathrm{Int}(E, A) = \{f \in K[X] \mid \forall x \in E\,,\, f(x) \in A\}.$$

For every $k \in \mathbb{N}$, one denotes by $\mathrm{Int}_k(E, A)$ the sub-A-module of $K[X]$ formed by all integer-valued polynomials on E with degree $\leq k$, that is,

$$\mathrm{Int}_k(E, A) = \{f \in K[X] \mid \deg f \leq k, \forall x \in E\,,\, f(x) \in A\}.$$

The following notion of polynomial equivalence was introduced by McQuillan [7].

Definition 1.1. Let E and F be two subsets of A.

(1) The *polynomial closure* of E in A is the subset

$$\overline{E} = \{x \in A \mid \forall f \in \mathrm{Int}(E, A), f(x) \in A\}.$$

(2) The subset E is said to be *polynomially closed* in A if $\overline{E} = E$.

(3) The subsets E and F are said to be *polynomially equivalent* if $\mathrm{Int}(E, A) = \mathrm{Int}(F, A)$.

(4) The subset E is said to be *polynomially dense* in F if $E \subseteq F$ and if E and F are polynomially equivalent.

Remark 1.2. (1) Every finite subset of A is polynomially closed [3, Exercise IV.1].

(2) Every cofinite subset of A is polynomially dense in A [3, Proposition I.1.5].

We add the following definitions.

Definition 1.3. Let $n \in \mathbb{N}$.

(1) The *polynomial closure up to the order n* of E in A is the subset

$$\overline{E}^n = \{x \in A \mid \forall f \in \mathrm{Int}_n(E, A), f(x) \in A\}.$$

(2) The subsets E and F of V are said to be *polynomially equivalent up to the order n* if $\mathrm{Int}_n(E, A) = \mathrm{Int}_n(F, A)$, that is, if $\overline{E}^n = \overline{F}^n$.

Remark 1.4. Let E be a subset of A.

(1) Obviously, one has

$$\forall n \in \mathbb{N}, \quad \overline{E}^{n+1} \subseteq \overline{E}^n \quad \text{and} \quad \overline{E} = \bigcap_{n \in \mathbb{N}} \overline{E}^n.$$

(2) Consequently, for $m \leq n$, if E is polynomially closed up to the order n, then E is polynomially closed up to the order m.

(3) If E is finite of cardinality s, then, according to Lagrange's interpolation, E is polynomially closed up to the order $s + 1$.

2 Polynomial closure in a valuation domain

In the fundamental case of rings of integers of number fields and, more generally, of Dedekind domains, the previous notions behave well under localization. Thus, we will work with discrete valuation domains and, more generally, with rank-one valuation domains.

Notation. From now on, we will denote by V the ring of a rank-one valuation v, and by K the quotient field of V. Recall that in this case $v(K^*)$ is a subgroup of $\mathbb{R}$.

For every nonempty subset E of V, one has

$$\mathrm{Int}(E, V) = \{f \in K[X] \mid \forall x \in E, v(f(x)) \geq 0\}.$$

Since polynomial functions are continuous for the v-adic topology, we obviously have:

Proposition 2.1. *For every subset E of V, the topological closure of E in V is contained in the polynomial closure of E in V.*

Proposition 2.2. *Let E and F be two nonempty subsets of V and $n \in \mathbb{N}$. Then $\overline{E}^n = \overline{F}^n$ if and only if, for every $f \in K[X]$ of degree $\leq n$,*

$$\inf_{x \in E} v(f(x)) = \inf_{x \in F} v(f(x)).$$

Proof. Clearly, the condition is sufficient. Conversely, suppose that there exists $f \in K[X]$ of degree $\leq n$ such that $\inf_{x\in E} v(f(x)) > \inf_{x\in F} v(f(x))$. Then, there is $d \in K$ such that

$$\inf_{x\in E} v(f(x)) \geq v(d) > \inf_{x\in F} v(f(x)).$$

Then $\frac{f}{d} \in \mathrm{Int}(E, V)$ and $\frac{f}{d} \notin \mathrm{Int}(F, V)$. □

Remark 2.3. It is not difficult to extend the previous definitions to subsets of K. Recall that a subset E of K is fractional if there exists a nonzero $d \in V$ such that $dE \subseteq V$. Clearly,

Proposition 2.4. (1) *If $dE \subseteq V$, then $d\overline{E} \subseteq V$.*

(2) *If E is not fractional, then $\overline{E} = K$.*

Remark 2.5. (1) If $\{E_i\}_{i\in E}$ is a family of polynomially closed subsets of V, then $\bigcap_{i\in I} E_i$ is also polynomially closed. Indeed, for every $i \in I$, $\overline{\bigcap_{i\in I} E_i} \subseteq \overline{E}_i = E_i$, then $\overline{\bigcap_{i\in I} E_i} \subseteq \bigcap_{i\in I} E_i$, this implies the equality. In addition, $\overline{V} = V$ and $\overline{\varnothing} = \varnothing$. Thus, if a finite union of polynomially closed subsets is closed, then we could define a polynomial topology on K whose closed subsets were the polynomially closed subsets.

(2) If $\{E_i\}_{i\in E}$ is a family of polynomially closed up to the order n subsets of V, then $\bigcap_{i\in I} E_i$ is also polynomially closed up to the order n.

(3) We know that such a notion of polynomial topology does not hold in the global case (see [3, Remark IV.1.10]: $2\mathbb{Z}$ and $3\mathbb{Z}$ are polynomially closed in $\mathbb{Z}$, while $\overline{2\mathbb{Z} \cup 3\mathbb{Z}} = \mathbb{Z}$). But, the question remains to be answered in the local case.

3 $\boldsymbol{v}$-orderings

The notion of v-ordering has been introduced by M. Bhargava [1] for subsets of discrete valuation domains and generalized by J.-L. Chabert [4] to subsets of rank-one valuation domains. First, we recall some general results.

Definition 3.1. Let $N \in \mathbb{N} \cup \{\infty\}$. A sequence $(a_n)_{0\leq n\leq N}$ of elements of E is a *v-ordering of length N* of E if, for every $n \in \{0, \ldots, N\}$, one has

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = \inf_{x\in E} v\left(\prod_{k=0}^{n-1}(x - a_k)\right).$$

Remark 3.2. If either v is discrete or E is a precompact subset of V, then E always admits v-orderings.

The following proposition proved by Bhargava [1] for discrete valuations may be extended to rank-one valuations [4].

Proposition 3.3. *Suppose $N \leq \mathrm{card}(E) - 1$. Let $(a_n)_{0 \leq n \leq N}$ be a sequence of distinct elements of E. The sequence $(a_n)_{0 \leq n \leq N}$ is a v-ordering of E if and only if the polynomials $f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k}$ $(0 \leq n \leq N)$ form a basis of the V-module* $\mathrm{Int}_N(E, V)$.

Corollary 3.4. *Let $N \in \mathbb{N}$ and let $(a_n)_{0 \leq n \leq N}$ be a v-ordering of E. Then, the subsets $\{a_k \mid 0 \leq k \leq N\}$ and E are polynomially equivalent up to the order N. In particular, for every $P \in K[X]$ of degree $d \leq N$, one has*

$$\inf\{v(P(x)); x \in E\} = \inf\{v(P(a_k)); 0 \leq k \leq d\}.$$

For $n \in \mathbb{N}$, according to [1], the n-th factorial ideal of E in V is the following ideal:

$$(n!)_E^V = \{y \in V \mid y\mathrm{Int}(E, V) \subseteq V[X]\}.$$

Notation. Proposition 3.3 shows that if the sequence $(a_n)_{0 \leq n \leq N}$ is a v-ordering of E, then the product $\prod_{k=0}^{n-1}(a_n - a_k)$ generates the ideal $(n!)_E^V$, and hence its valuation does not depend on the choice of the v-ordering $(a_n)_{0 \leq n \leq N}$. One puts

$$w_E(n) = v(n!)_E^V = v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right).$$

Remark 3.5. (1) A sequence $(b_n)_{0 \leq n \leq N}$ of elements of E is a v-ordering of length N if, for every $0 \leq n \leq N$, one has

$$v\left(\prod_{k=0}^{n-1}(b_n - b_k)\right) = w_E(n).$$

(2) If $E \subseteq F$, then $w_E(n) \geq w_F(n)$ for every n, because $\mathrm{Int}_n(F, V) \subseteq \mathrm{Int}_n(E, V)$.

(3) If $E \subseteq F$ and $\overline{E} = \overline{F}$, then a sequence of elements of E is a v-ordering of E if and only if it is a v-ordering of F.

(4) If $F = \{as + b \mid s \in E\}$ where a and b are two elements of V, then

$$(n!)_F^V = a^n (n!)_E^V.$$

In particular, if a is invertible in V, then E and F have the same factorials.

(5) Note $\widehat{V}$ and $\widehat{E}$ the completions of V and E for the v-adic topology. If $(a_n)_{0 \leq n \leq N}$ is a v-ordering of E, then $(a_n)_{0 \leq n \leq N}$ is also a v-ordering of $\widehat{E}$ in $\widehat{V}$.

(6) When $V = \mathbb{Z}_p$ where p is a prime number and $v = v_p$ is the p-adic valuation, the sequence $(n)_{n \in \mathbb{N}}$ is a v_p-ordering of $\mathbb{Z}_p$. Thus, according to Legendre's formula,

$$w_{\mathbb{Z}_p}(n) = v_p(n!) = \sum_{k \geq 1} \left[\frac{n}{p^k}\right].$$

Pólya generalized this formula: if V is a discrete valuation domain with finite residue field of cardinality q, then

$$w_V(n) = v(n!_V) = \sum_{k \geq 1} \left[\frac{n}{q^k} \right].$$

Now, we give a technical result which will be useful for the proof of our main theorem.

Proposition 3.6. *Let E be an infinite subset of V. A sequence of elements of E which is a simple limit of a sequence of v-orderings of E is a v-ordering of E.*

Proof. Let $a = (a_k)_{k \in \mathbb{N}}$ be a sequence of elements of E and, for every $i \in \mathbb{N}$, let $a^i = (a_k^i)_{k \geq 0}$ be a v-ordering of E. Assume that, for every $k \in \mathbb{N}$, $a_k = \lim_{i \to +\infty} a_k^i$. Fix an integer n. By hypothesis, there exists an integer i such that $v(a_k^i - a_k) > w_E(n)$ for every $0 \leq k \leq n$. Then, for every $0 \leq k < n$, we have, $v(a_n - a_k) = v(a_n^i - a_k^i)$ because $(a_n - a_k) = (a_n - a_n^i) + (a_n^i - a_k^i) + (a_k^i - a_k)$ and $w_E(n) \geq v(a_n^i - a_k^i)$ for $0 \leq k < n$. Consequently,

$$v\left(\prod_{k=0}^{n-1} (a_n - a_k) \right) = v\left(\prod_{k=0}^{n-1} (a_n^i - a_k^i) \right) = w_E(n),$$

and the sequence $(a_n)_{n \in \mathbb{N}}$ is a v-ordering of E. □

4 v-orderings and isolated points

Definition 4.1. Let E be a subset of V and x an element of E.

(1) The element x is said to be *polynomially isolated* in E if

$$\mathrm{Int}(E, V) \neq \mathrm{Int}(E \setminus \{x\}, V).$$

(2) If x is *polynomially isolated* in E, we say that x is *polynomially isolated up to the order n* if n is the smallest integer such that

$$\mathrm{Int}_n(E, V) \neq \mathrm{Int}_n(E \setminus \{x\}, V).$$

Proposition 4.2. *Let $n \in \mathbb{N}$. An element x of E is polynomially isolated up to the order n if and only if every v-ordering of length n contains x and n is the smallest integer k such that every v-ordering of E of length k contains x.*

Proof. Assume that x is polynomially isolated in E up to the order n: n is the smallest integer k such that $\mathrm{Int}_k(E, V) \neq \mathrm{Int}_k(E \setminus \{x\}, V)$. If there exists a sequence $a_0, \ldots, a_k$ which is a v-ordering of E and which does not contain x, then $a_0, \ldots, a_k$ is also a v-ordering of $E \setminus \{x\}$. Hence, $\mathrm{Int}_k(E, V) = \mathrm{Int}_k(E \setminus \{x\}, V)$ and $k < n$. Thus, for $k \geq n$, every v-ordering of E of length k contains x.

Conversely, assume that k is such that every v-ordering of E of length k contains x, then, $\mathrm{Int}_k(E,V) \neq \mathrm{Int}_k(E \setminus \{x\}, V)$ because a v-ordering of $E \setminus \{x\}$ cannot be a v-ordering of E. Consequently, x is polynomially isolated in E up to some order n and $k \geq n$. □

Corollary 4.3. *Suppose that E admits a v-ordering. Let n be a positive integer. Then, every subset E of V contains at most n isolated points up to the order n.*

Recall that x is topologically isolated in E if

$$\sup_{x \in E, y \neq x} v(y-x) < \infty.$$

If x is not topologically isolated in E we have the following technical lemma that will be useful for the proof of our main result.

Lemma 4.4. *Let E be an infinite subset of V. Let x be a topologically non-isolated point of E and let $(a_n)_{n\in\mathbb{N}}$ be a v-ordering of E not containing x. Then, there exist infinitely many integers n such that $a_0, \ldots, a_{n-1}, x$ is a v-ordering of E (of length n).*

Proof. Let M be a real number. By hypothesis, there exists $k \in \mathbb{N}$ such that $v(a_k - x) > M$. Let n be the smallest integer k such that $v(a_k - x) > M$. Then, $v(x - a_i) = v(a_n - a_i)$ for every $0 \leq i \leq n-1$. Thus, $v(\prod_{i=0}^{n-1}(x - a_i)) = v(\prod_{i=0}^{n-1}(a_n - a_i)) = w_E(n)$. Consequently, $a_0, \ldots, a_{n-1}, x$ is a v-ordering of E of length n. □

From now on, we have to assume that E is a precompact subset of V, that is, its completion $\widehat{E}$ is compact. The following proposition is a consequence of the v-adic Stone–Weierstrass theorem ([1], [3]). One can find a direct proof in [5].

Proposition 4.5. *If E is a precompact infinite subset of V, then every infinite v-ordering of E is dense in E.*

If E is not precompact, a v-ordering is not necessarily dense in E as shown by the following counterexample.

Example 4.6. Let V be a discrete valuation domain with maximal ideal $\mathfrak{M}$ and with infinite residue field. Clearly, every sequence formed by elements of V that are non-congruent modulo $\mathfrak{M}$ is a v-ordering of V, while the sequence is not dense in V.

Proposition 4.7. *If E is precompact, then an element of E is topologically isolated in E if and only if it is polynomially isolated in E.*

Proof. According to Proposition 2.1, it is sufficient to prove that, if x is topologically isolated, then x is polynomially isolated. If E is a finite subset, the assertion is obvious. Thus, we assume that E is infinite and that $\sup_{x\in E, y\neq x} v(y-x) < \infty$. Since E is precompact, it follows from Remark 3.2, that there exists a v-ordering $(a_n)_{n\in}$ of E. According to Proposition 4.5, $\sup_{n\in\mathbb{N}} v(a_n - x) = \infty$. According to Proposition 4.2, there exists $n \in \mathbb{N}$ such that $a_n = x$. Thus, every v-ordering of E contains x. As a consequence, a v-ordering of $E \setminus \{x\}$ cannot be a v-ordering of E. In particular,

$$\mathrm{Int}(E, V) \neq \mathrm{Int}(E \setminus \{x\}, V). \qquad \square$$

Remark 4.8. In fact, the last proposition shows that the polynomial closure and the topological closure of a precompact subset are equal.

5 v-orderings of a countable set

Theorem 5.1. *Let V be a rank-one valuation domain and E be a countable and precompact subset of V. Then, it is possible to construct a v-ordering of E in which every element of E occurs exactly once.*

In other words, if $E = \{b_n \mid n \in \mathbb{N}\}$, then there is a one-to-one correspondence σ of $\mathbb{N}$ onto $\mathbb{N}$ such that $(b_{\sigma(n)})_{n\in\mathbb{N}}$ is a v-ordering of E.

Proof. If E is a finite subset, the proposition is obvious. Assume that E is infinite and let $E = \{b_n \mid n \in \mathbb{N}\}$.

Let $a^0 = (a_n^0)_{n\in\mathbb{N}}$ be a v-ordering of E. We know that the a_n are distinct.

(1) If the sequence $(a_n^0)_{n\in\mathbb{N}}$ runs over the whole subset E, then our aim is reached.

(2) If not, let r_1 be the smallest integer r such that $b_r \notin \{a_n^0 \mid n \in \mathbb{N}\}$. Then there are indices $k_0, k_1, \dots, k_{r-1}$ such that

$$b_0 = a_{k_0}^0,\ b_1 = a_{k_1}^0, \dots, b_{r_1-1} = a_{k_{r_1-1}}^0.$$

Since b_{r_1} does not belong to the v-ordering a^0, b_{r_1} is not polynomially isolated in E (Proposition 4.2), and hence b_{r_1} is not topologically isolated in E (Proposition 4.7). Then, thanks to Lemma 4.4, there exists u such that $u \geq \max(k_0, k_1, \dots, k_{r_1-1})$ and $a_0^0, \dots, a_u^0, b_{r_1}$ is a v-ordering of E. Let u_1 be the smallest such integer u.

Let $a^1 = (a_n^1)_{n\in\mathbb{N}}$ be a v-ordering of E whose first elements are $a_0^0, a_1^0, \dots, a_{u_1}^0$, b_{r_1}. If a^1 runs over the whole subset E, then our aim is reached.

(3) If not, we iterate the process. If the iteration is finite, our aim is reached.

(4) If not, the process leads to the construction of a sequence $(a^i)_{i\in\mathbb{N}}$ of v-orderings of E. Moreover, for every $j \in \mathbb{N}$, there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, the sequence $(a_n^i)_{n\in\mathbb{N}}$ contains $b_0 \cdots b_j$. By construction, for every $n \in \mathbb{N}$, the sequence $(a_n^i)_{i\in\mathbb{N}}$ is ultimately constant. We denote by a_n its limit (of course, a_n is an element of E). According to Proposition 3.6, the sequence $(a_n)_{n\in\mathbb{N}}$ is also a v-ordering of E. The construction shows that the sequence $(a_n)_{n\in\mathbb{N}}$ runs over the whole set E. $\square$

Remark 5.2. In her thesis, J. Yeramian proved that if A is a semi-local Dedekind domain with finite residue fields, then there exists a sequence which is simultaneously a v-ordering of A for all maximal ideals of A. Now, let E be a subset of such a domain A. Can we find a sequence of elements of E which is simultaneously a v-ordering of E?

The answer is negative in general as shown by the following example.

Example 5.3. Let $\mathbb{Z}_{\{2,3,5\}} = \{x \in \mathbb{Q} \mid v_p(x) \geq 0 \, \forall p \neq 2, 3, 5\}$. Then, $\mathbb{Z}_{\{2,3,5\}}$ is a semi-local Dedekind domain with maximal ideals $2\mathbb{Z}_{\{2,3,5\}}$, $3\mathbb{Z}_{\{2,3,5\}}$ and $5\mathbb{Z}_{\{2,3,5\}}$. Consider the subset $E = 30\mathbb{Z} \cup \{2, 3, 5\}$ of $\mathbb{Z}_{\{2,3,5\}}$. Then, the points 2, 3 and 5 are isolated up to the order 1 respectively for the topologies associated to the maximal ideals $3\mathbb{Z}_{\{2,3,5\}}, 5\mathbb{Z}_{\{2,3,5\}}$ and $2\mathbb{Z}_{\{2,3,5\}}$. It follows from Corollary 4.3 that E cannot admit a sequence which is simultaneously a v-ordering of E for each topology.

More generally, if $\mathfrak{M}_1, \ldots, \mathfrak{M}_r$ are the maximal ideals of a semi-local Dedekind domain A and if the subset E of A has more than r isolated points up to the order r, then such a simultaneous sequence does not exist.

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Mixed invertibility and Prüfer-like monoids and domains

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Abstract. We give a systematic theory of Prüfer-like domains using ideal systems on commutative cancellative monoids. Based on criteria for mixed invertibility of ideals, we unify and generalize characterizations of various classes of Prüfer-like monoids and domains and furnish them with new proofs. In particular, we generalize and extend criteria for v-domains recently proved by D. D. Anderson, D. F. Anderson, M. Fontana and M. Zafrullah.

Keywords. Prüfer domain, ideal system, star operation, v-domain, (generalized) GCD domain.

AMS classification. 20M12, 13A15, 13F05, 13G05, 20M14, 20M25.

1 Introduction

Prüfer domains and their various generalizations are topics of outstanding interest in non-Noetherian multiplicative ideal theory. For an overview of the more classical results we refer to [6], [9] and [15]. Among the various generalizations involving star operations studied in the literature we mention the following ones.

- Prüfer v-multiplication domains (PvMD's, first studied in [10] and called "pseudo-Prüfer domains" in [5, Ch. VII, § 2, Ex. 19]),
- general $*$-multiplication domains (investigated in [13] and in [12]),
- v-domains (called "regularly integrally closed domains" in [5, Ch. VII, §1, Ex. 30, 31], see [19] for an overview and the history of this concept),
- Generalized GCD-domains (GGCD domains, studied in [1]),
- Pseudo-Dedekind domains (introduced in [17] under the name "Generalized Dedekind domains" and thorough investigated in [3]),
- pre-Krull domains (investigated in [18]).

Several of these concepts have only recently successfully been generalized to the case of semistar operations (see [7] and [8]).

By the very definitions, the above-mentioned concepts can be defined in a purely multiplicative manner without referring to the ring addition, and thus they can be studied in the context of commutative cancellative monoids. In a systematic way, the ideal theory of commutative cancellative monoids was first developed by P. Lorenzen [16], and a thorough presentation of that theory in the language of ordered abelian groups was given by P. Jaffard [14]. A modern treatment of multiplicative ideal theory in the context of commutative monoids (including all above-mentioned generalizations) was

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given by the author in the monograph [11], which serves as the main reference for the present paper.

A first attempt to a general theory covering the various generalizations of Prüfer domains was made in [4] but not pursued further on. Only recently, these investigations were revived in [2] together with several completely new ideal-theoretic characterizations of v-domains. In this paper we continue these investigations. We show that the results of [2] and several of their refinements and generalizations remain valid in the context of commutative cancellative monoids, and we provide them with new (and simpler) proofs.

The paper is organized as follows. In Section 2 we fix our notations. Section 3 contains the basic result on mixed invertibility (Theorem 3.1) which is fundamental for the following investigations. In Section 4 we apply the concept of mixed invertibility to characterizations of Dedekind-like and Prüfer-like monoids and domains, and finally in Section 5 we continue the investigations of v-domains (resp. v-Prüfer monoids) started in [2].

2 Notations

For any set X, we denote by $|X| \in \mathbb{N}_0 \cup \{\infty\}$ its cardinality, by $\mathbb{P}(X)$ the set of all subsets and by $\mathbb{P}_{\mathsf{f}}(X)$ the set of all finite subsets of X.

Throughout this paper, let D be a commutative multiplicative monoid with unit element $1 \in D$ and a zero element $0 \in D$ (satisfying $0x = 0$ for all $x \in D$) such that $D^\bullet = D \setminus \{0\}$ is cancellative, and let $K = \mathsf{q}(D) = D^{\bullet -1}D$ be its total quotient monoid (then $K^\bullet$ is a quotient group of $D^\bullet$). The most important example we have in mind is when D is the multiplicative monoid of an integral domain (then K is the multiplicative monoid of its quotient field).

For subsets $X, Y \subset K$, we set $(X : Y) = \{z \in K \mid zY \subset X\}$, $X^{-1} = (D : X)$, and the set X is called D-fractional if $X^{-1} \cap D^\bullet \neq \emptyset$. We denote by $\mathcal{F}(D)$ the set of all D-fractional subsets of K.

Throughout, we use the language of ideal systems as developed in my book "Ideal Systems" [11], and all undefined notions are as there. For an ideal system r on D, let $\mathcal{F}_r(D) = \{X_r \mid X \in \mathcal{F}(D)\} = \{A \in \mathcal{F}(D) \mid A_r = A\}$ be the semigroup of all fractional r-ideals, equipped with the r-multiplication, defined by $(A, B) \mapsto (AB)_r$ and satisfying $(AB)_r = (A_rB)_r = (A_rB_r)_r$ for all $A, B \in \mathcal{F}(D)$. We denote by $\mathcal{F}_{r,\mathsf{f}}(D) = \{E_r \mid E \in \mathbb{P}_{\mathsf{f}}(K) \subset \mathcal{F}_r(D)$ the subsemigroup of all r-finite (that is, r-finitely generated) fractional r-ideals of D.

For any subset $\mathcal{X} \subset \mathbb{P}(K)$, we set $\mathcal{X}^\bullet = \mathcal{X} \setminus \{\{0\}\}$. In particular, if $\mathcal{J}$ is any set of ideals, then $\mathcal{J}^\bullet = \mathcal{J} \setminus \{\mathbf{0}\}$ (where $\mathbf{0} = \{0\}$ denotes the zero ideal). In this way we use the notions $\mathcal{F}(D)^\bullet$, $\mathcal{F}_r(D)^\bullet$, $\mathcal{F}_{r,\mathsf{f}}(D)^\bullet$ etc.

For an ideal system r on D, the associated finitary ideal system of r will be denoted by r_{f} (it is denoted by r_s in [11]). It is given by

$$X_{r_{\mathsf{f}}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r \quad \text{for every } X \in \mathcal{F}(D),$$

and it satisfies $\mathcal{F}_{r,\mathsf{f}}(D) = \mathcal{F}_{r_{\mathsf{f}},\mathsf{f}}(D)$. The ideal system r is called *finitary* if $r = r_{\mathsf{f}}$.

For any two ideal systems r and q on D we write $r \le q$ if $\mathcal{F}_q(D) \subset \mathcal{F}_r(D)$. Note that $r \le q$ holds if and only if $X_r \subset X_q$ (equivalently, $X_q = (X_r)_q$) for all $X \in \mathcal{F}(D)$.

We denote by $s = s(D)$ the system of semigroup ideals, given by $X_s = DX$ for all $X \in \mathcal{F}(D)$; by $v = v(D)$ the ideal system of multiples ("Vielfachenideale"), given by $X_v = (X^{-1})^{-1}$ for all $X \in \mathcal{F}(D)$, and by $t = t(D) = v_{\mathrm{f}}$ the associated finitary system. The systems s and t are finitary, the system v usually not. For every ideal system r on D we have $s \le r_{\mathrm{f}} \le r \le v$ and $r_{\mathrm{f}} \le t$. We shall frequently use that $\mathcal{F}_v(D) = \{A^{-1} \mid A \in \mathcal{F}(D)\}$ (see [11, Theorem 11.8]).

If D is an integral domain, then the (Dedekind) ideal system $d = d(D)$ of usual ring ideals is given by $X_d =_D \langle X \rangle$ for all $X \in \mathcal{F}(D)$ (that is, X_d is the fractional D-ideal generated by X). d is a finitary ideal system, and there is a one-to-one correspondence between ideal systems $r \ge d$ and star operations on D, given as follows:

If $* : \mathcal{F}_d(D)^\bullet \to \mathcal{F}_d(D)^\bullet$ is a star operation on D and $r^* : \mathcal{F}(D) \to \mathcal{F}(D)$ is defined by $X_{r^*} =_D \langle X \rangle^*$ for $X \in \mathcal{F}(D)^\bullet$ and $X_{r^*} = \{0\}$ if $X \subset \{0\}$, then r^* is an ideal system satisfying $r^* \ge d$. Conversely, if $r \ge d$ is an ideal system, and if we define $*_r$ by $J^{*_r} = J_r$ for all $J \in \mathcal{F}_d(D)^\bullet$, then $*_r$ is a star operation, and by the very definition we have $r^{*_r} = r$ and $*_{r^*} = *$.

Throughout this paper, we fix a (basic) ideal system δ on D and assume that all ideal systems r considered in this manuscript satisfy $r \ge \delta$. Of course, we may always assume that $\delta = s(D)$, but if D is an integral domain, it may also be convenient to assume that $\delta = d(D)$ in order to make the connection with star operations more apparent.

In any case, we denote by $\boldsymbol{F}(D) = \mathcal{F}_\delta(D)^\bullet$ the set of all non-zero fractional δ-ideals and by $\boldsymbol{f}(D) = \mathcal{F}_{\delta,\mathrm{f}}(D)$ the set of all δ-finite non-zero fractional δ-ideals of D. Then $\mathcal{F}_r(D)^\bullet = \{A_r \mid A \in \boldsymbol{F}(D)\}$ and $\mathcal{F}_{r,\mathrm{f}}(D)^\bullet = \{F_r \mid F \in \boldsymbol{f}(D)\}$ for every ideal system r on D.

3 Mixed invertibility

Mixed invertibility means, that we investigate the invertibility of ideals of one ideal system with respect to another ideal system. We start by recalling some basic facts concerning the concept invertibility in the theory of ideal systems. For details and proofs concerning the following remarks we refer to [11, Theorem 12.1].

Let r be an ideal system on D. A fractional ideal $A \in \boldsymbol{F}(D)$ is called *r-invertible* if $(AA^{-1})_r = D$ (equivalently, $(AB)_r = D$ for some $B \in \boldsymbol{F}(D)$). Hence a fractional ideal $A \in \boldsymbol{F}(D)$ is r-invertible if and only if A_r is r-invertible. By definition, a fractional r-ideal is r-invertible if and only if it is an invertible element of the semigroup $\mathcal{F}_r(D)$. If $A, B \in \boldsymbol{F}(D)$, then AB is r-invertible if and only if A and B are both r-invertible.

We denote by $\mathcal{F}_r(D)^\times$ the group of all r-invertible fractional r-ideals. If q is an ideal system such that $r \le q$, then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ is a subgroup (this holds in particular, if $q = v$). If r is finitary, then $\mathcal{F}_r(D)^\times = \mathcal{F}_{r,\mathrm{f}}(D)^\times$ (that is, if $A \in \mathcal{F}_r(D)$ is r-invertible, then both A and A^{-1} are r-finite). This may fail if r is not finitary;

then it may occur that $\mathcal{F}_{r,\mathrm{f}}(D)^\times \subsetneq \mathcal{F}_r(D)^\times \cap \mathcal{F}_{r,\mathrm{f}}(D)$ (it is well known that not every v-domain is a PvMD).

Theorem 3.1 (Mixed invertibility). *Let r, q and y be ideal systems on D, $q \le y$, and $B \in \boldsymbol{F}(D)$. Then the following assertions are equivalent:*

(a) *B_q is r-invertible.*

(b) *B^{-1} is r-invertible, and $B_q = B_v$.*

(c) *For every $A \in \boldsymbol{F}(D)$ such that $A_r \subset B_q$ there exists some $C \in \mathcal{F}_r(D)$ satisfying $A_r = (B_qC)_r$.*

(d) *$(A\colon B^{-1})_r = (AB_q)_r$ for all $A \in \boldsymbol{F}(D)$.*

(e) *$(A\colon B_q)_r = (AB^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

(f) *$[(A\colon B)_q]_r = (A_qB^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

(g) *$(A_q\colon B)_r = (A_qB^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

(h) *$(A_q\colon B^{-1})_r = (A_qB_q)_r$ for all $A \in \boldsymbol{F}(D)$.*

(i) *$(A_v\colon B^{-1}) = (A_vB_q)_r$ for all $A \in \boldsymbol{F}(D)$.*

(j) *$(A_r\colon B^{-1}) = (AB_q)_r$ for all $A \in \boldsymbol{F}(D)$.*

(k) *$(B_y\colon A)_r = (B_qA^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

(l) *$[(B\colon A)_y]_r = (B_qA^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

(m) *$(B_v\colon A^{-1}) = (B_qA_v)_r$ for all $A \in \boldsymbol{F}(D)$.*

Proof. (a) $\Rightarrow$ (b). If B_q is r-invertible, then $B_q = (B_q)_v = B_v$, and $(B_qB^{-1})_r = D$. Hence B^{-1} is r-invertible.

(b) $\Rightarrow$ (c). If $A \in \boldsymbol{F}(D)$ and $A_r \subset B_q$, then $C = (A_rB_q^{-1})_r \in \mathcal{F}_r(D)$, and $(B_qC)_r = (B_qB_q^{-1}A_r)_r = [(B_qB_q^{-1})_rA]_r = A_r = A$, since B^{-1} is r-invertible and thus $(B_qB_q^{-1})_r = (B_vB^{-1})_r = [(B^{-1})^{-1}B^{-1}]_r = D$.

(c) $\Rightarrow$ (a). If $a \in B_q^\bullet$, then $aD = (aD)_r \subset B_q$, and thus $aD = (B_qC)_r$ for some $C \in \mathcal{F}_r(D)$. Hence $D = [B_q(a^{-1}C)]_r$, and thus B_q is r-invertible.

(a) $\Rightarrow$ (d). Let $A \in \boldsymbol{F}(D)$. Since $AB_qB^{-1} \subset A(B_qB^{-1})_r = AD = A$, it follows that $AB_q \subset (A : B^{-1})$ and $(AB_q)_r \subset (A : B^{-1})_r$. To prove the reverse inclusion, let $x \in (A : B^{-1})_r$. Then $xB^{-1} \subset (A : B^{-1})_rB^{-1} \subset [(A : B^{-1})B^{-1}]_r \subset A_r$ and $x \in xD = (xB_qB^{-1})_r \subset (A_rB_q)_r = (AB_q)_r$.

(d) $\Rightarrow$ (j). For every $A \in \boldsymbol{F}(D)$, we may apply (d) with A_r instead of A and obtain $(A_r\colon B^{-1}) = (A_r\colon B^{-1})_r = (A_rB_q)_r = (AB_q)_r$.

(j) $\Rightarrow$ (i). For every $A \in \boldsymbol{F}(D)$, we may apply (j) with A_v instead of A and obtain $(A_v\colon B^{-1}) = ((A_v)_r\colon B^{-1}) = (A_vB_q)_r$.

(i) $\Rightarrow$ (a). With $A = B^{-1} = A_v$, (i) implies $D \supset (B_qB^{-1})_r = (B^{-1}\colon B^{-1}) \supset D$ and therefore $(B_qB^{-1})_r = D$.

(d) $\Rightarrow$ (h). For every $A \in \boldsymbol{F}(D)$, we apply (d) with A_q instead of A.

(h) $\Rightarrow$ (i). For every $A \in \boldsymbol{F}(D)$, we may apply (h) with A_v instead of A and obtain $(A_v : B^{-1}) = ((A_v)_q : B^{-1})_r = ((A_v)_q B_q)_r = (A_v B_q)_r$.

(b) $\Rightarrow$ (e). By (a) $\Rightarrow$ (d), applied with B^{-1} instead of B. In doing so observe that $(B^{-1})_q = B^{-1}$ and $(B^{-1})^{-1} = B_v = B_q$.

(e) $\Rightarrow$ (g). For every $A \in \boldsymbol{F}(D)$, we may apply (e) with A_q instead of A and obtain $(A_q : B)_r = (A_q : B_q)_r = (A_q B^{-1})_r$.

(a) $\Rightarrow$ (f). Let $A \in \boldsymbol{F}(D)$. Then $AB^{-1}B \subset A$ implies $AB^{-1} \subset (A : B)$ and thus $(A_q B^{-1})_r \subset [(A : B)_q]_r$. For the reverse inclusion, it suffices to show that $(A : B)_q \subset (A_q B^{-1})_r$. If $x \in (A : B)_q$, then $xB_q \subset (A : B)_q B_q \subset [(A : B)B]_q \subset A_q$, and consequently $x \in xD = (xB_q B^{-1})_r \subset (A_q B^{-1})_r$.

(f) $\Rightarrow$ (a) and (g) $\Rightarrow$ (a). In both cases, we set $A = B_q$, observe that $(B_q : B) \supset D$ and obtain $(B^{-1}B_q)_r \subset D$, whence $(B^{-1}B_q)_r = D$.

(a) $\Rightarrow$ (k) and (a) $\Rightarrow$ (l). Let $A \in \boldsymbol{F}(D)$. Since $BA^{-1}A \subset B$, it follows that $BA^{-1} \subset (B : A) \subset (B_y : A)$, hence $B_q A^{-1} \subset (BA^{-1})_q \subset (B : A)_q \subset (B : A)_y$ and $(BA^{-1})_q \subset (B_y : A)_q = (B_y : A)$. Thus we obtain $(B_q A^{-1})_r \subset (B_y : A)_r$ and $(B_q A^{-1})_r \subset [(B : A)_y]_r$.

For the reverse inclusions it suffices to show that $(B_y : A) \subset (B_q A^{-1})_r$ and $(B : A)_y \subset (B_q A^{-1})_r$. Thus assume that either $x \in (B_y : A)$ or $x \in (B : A)_y$. Since $(B : A)_y \subset (B_y : A)_y = (B_y : A)$, we obtain $xA \subset B_y$ in both cases. Now it follows that $xAB^{-1} \subset B_y B^{-1} \subset D$, hence $xB^{-1} \subset A^{-1}$ and $x \in (xB_q B^{-1})_r \subset (B_q A^{-1})_r$.

(k) $\Rightarrow$ (a) and (l) $\Rightarrow$ (a). With $A = B$ we obtain $(B_q B^{-1})_r = (B_y : B)_r \supset D$ from (k) and $(B_q B^{-1})_r = [(B : B)_y]_r \supset D$ from (l). Hence $(B_q B^{-1})_r = D$ follows in both cases.

(k) $\Rightarrow$ (m). Let $A \in \boldsymbol{F}(D)$. By the equivalence of (a) and (k) it follows that (k) holds with $y = v$. We apply (k) with $y = v$ and with A^{-1} instead of A. Then we obtain $(B_v : A^{-1}) = (B_v : A^{-1})_r = (B_q A_v)_r$.

(m) $\Rightarrow$ (a). With $A = B^{-1}$ we obtain $D \supset (B_q B^{-1})_r = (B_v : B_v) \supset D$ and thus $(B_q B^{-1})_r = D$. □

Remark 3.2. Let assumptions be as in Theorem 3.1, and assume moreover that $r \le q$. Then the conditions (f), (g), (h), (k) and (l) simplify by the relations $[(A : B)_q]_r = (A : B)_q$, $(A_q : B)_r = (A_q : B)$, $(A_q : B^{-1})_r = (A_q : B^{-1})$, $(B_y : A)_r = (B_y : A)$ and $[(B : A)_y]_r = (B : A)_y$.

Moreover, condition (g) is obviously equivalent to

(g)′ $(A_q : B_q) = (A_q B^{-1})_r$ for all $A \in \boldsymbol{F}(D)$ (compare [2, Remark 1.6]).

Corollary 3.3. *Let r, q and x be ideal systems on D, $x \ge r$ and $B \in \boldsymbol{F}(D)$. Then B_q is r-invertible if and only if $(A_x : B^{-1}) = (A_x B_q)_r$ for all $A \in \boldsymbol{F}(D)$.*

Proof. Let first B_q be r-invertible and $A \in \boldsymbol{F}(D)$. By Theorem 3.1 (j), applied with A_x instead of A, we obtain $(A_x : B^{-1}) = ((A_x)_r : B^{-1}) = (A_x B_q)_r$.

To prove the converse, we assume that $(A_x : B^{-1}) = (A_x B_q)_r$ for all $A \in \boldsymbol{F}(D)$. For any $A \in \boldsymbol{F}(D)$, we apply this relation with A_v instead of A, and then we obtain $(A_v : B^{-1}) = ((A_v)_x : B^{-1}) = ((A_v)_x B_q)_r = (A_v B_q)_r$. Hence B_q is r-invertible by Theorem 3.1 (i). □

Corollary 3.4. *Let r be an ideal system on D and $B \in \boldsymbol{F}(D)$. Then B_v is r-invertible if and only if $(AB)^{-1} = (A^{-1}B^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

Proof. Note that $(XY)^{-1} = (X^{-1} : Y)$ for all $X, Y \in \boldsymbol{F}(D)$ [11, Corollary 11.7 ii)].

Let first B_v be r-invertible and $A \in \boldsymbol{F}(D)$. By Theorem 3.1 (f), applied with $q = r$ and A^{-1} instead of A, we obtain

$$(A^{-1}B^{-1})_r = [(A^{-1})_v B^{-1}]_r = [(A^{-1} : B)_v]_r = (A^{-1} : B) = (AB)^{-1}.$$

Assume now that $(A^{-1}B^{-1})_r = (AB)^{-1}$ for all $A \in \boldsymbol{F}(D)$. For every $A \in \boldsymbol{F}(D)$, we apply this relation with A^{-1} instead of A and obtain

$$(A_v B^{-1})_r = [(A^{-1})^{-1} B^{-1}]_r = (A^{-1}B)^{-1} = ((A^{-1})^{-1} : B) = (A_v : B)_r.$$

Hence B_v is r-invertible by Theorem 3.1 (g), applied with $q = v$. □

4 (r, q)-Dedekind and (r, q)-Prüfer monoids

We use the notions of r-Prüfer monids and r-Dedekind monoids (resp. domains) as in [11, §17 and §23]. For any property **P** of monoids we say that an integral domain D is a **P**-domain if its multiplicative monoid is a **P**-monoid.

Definition 4.1. Let r and q be ideal systems on D such that $r \le q$.

(1) D is called an *(r, q)-Dedekind monoid* if $\mathcal{F}_q(D)^\bullet \subset \mathcal{F}_r(D)^\times$ [that is, every non-zero fractional q-ideal is r-invertible, or, equivalently, $(B_q B^{-1})_r = D$ for all $B \in \boldsymbol{F}(D)$].

(2) D is called an *(r, q)-Prüfer monoid* if $\mathcal{F}_{q,\mathrm{f}}(D)^\bullet \subset \mathcal{F}_r(D)^\times$ [that is, every non-zero fractional q-finite q-ideal is r-invertible, or, equivalently, $(F_q F^{-1})_r = D$ for all $F \in \boldsymbol{f}(D)$].

By definition, D is an r-Dedekind monoid [an r-Prüfer monoid] if and only if D is an (r, r)-Dedekind monoid [an (r, r)-Prüfer monoid].

A v-Dedekind monoid is a completely integrally closed monoid [11, Theorem 14.1], a t-Dedekind monoid is a Krull monoid [11, Theorem 23.4], and an (r, v)-Prüfer monoid is an r-GCD-monoid [11, Def. 17.6]. Consequently, a v-Dedekind domain is a completely integrally closed domain, a t-Dedekind domain is a Krull domain, and a d-Dedekind domain is just a Dedekind domain. A v-Prüfer domain is a v-domain (that is, a regularly integrally closed domain in the sense of [5, Ch. VII, §1, Ex. 30, 31]), a t-Prüfer domain is a PvMD (that is, a pseudo-Prüfer domain in the sense of [5, Ch. VII, §2, Ex. 19]), and a d-Prüfer domain is just a Prüfer domain. A (d, v)-Prüfer domain is a GGCD-domain (generalized GCD-domain, see [11, Def. 17.6]).

In [2], r-Dedekind domains are called r-CICDs (r-completely integrally closed domains) and (r, v)-Dedekind domains are called (r, v)-CICDs (note that [2, Proposition 1.1] follows from the equivalence of (a) and (c) in Theorem 3.1).

The definition of r-Dedekind domains given in [2] coincides with ours if r is finitary. In general, an r-Dedekind domain in the sense of [2] is an r_{f}-Dedekind domain according to our definition.

Lemma 4.2. *Let r, p, q be ideal systems on D such that $r \le p \le q$.*

If D is an (r, p)-Dedekind monoid, then D is an (r, q)-Dedekind monoid, and if D is an (r, q)-Dedekind monoid, then D is a (p, q)-Dedekind monoid. In particular, if D is an r-Dedekind monoid, then D is an (r, q)-Dedekind monoid, and if D is an (r, q)-Dedekind monoid, then D is a q-Dedekind monoid.

The same assertions hold true if "Dedekind" is replaced by "Prüfer". Moreover, if $r \le q_{\mathrm{f}}$, then D is an (r, q)-Prüfer monoid if and only if D is an (r, q_{f})-Prüfer monoid.

Proof. The statements concerning Dedekind-like monoids follow from the containments $\mathcal{F}_q(D)^\bullet \subset \mathcal{F}_p(D)^\bullet$ and $\mathcal{F}_r(D)^\times \subset \mathcal{F}_p(D)^\times$.

For the proof of the statements concerning Prüfer-like monoids, assume first that D is an (r, p)-Prüfer monoid, and let $F \in \boldsymbol{f}(D)$. Then $D = (F_p F^{-1})_r \subset (F_q F^{-1})_r \subset D$, hence $(F_q F^{-1})_r = D$, and D is an (r, q)-Prüfer monoid. If D is an (r, q)-Prüfer monoid and $F \in \boldsymbol{f}(D)$, then $D = (F_q F^{-1})_r \subset (F_q F^{-1})_p \subset D$ implies that also $(F_q F^{-1})_p = D$, and thus D is a (p, q)-Prüfer monoid.

The last statement follows since $\mathcal{F}_{q,\mathrm{f}}(D) = \mathcal{F}_{q_{\mathrm{f}},\mathrm{f}}(D)$. □

The statements of Theorem 3.1 provide a wealth of criteria for a monoid to be an (r, q)-Dedekind monoid or an (r, q)-Prüfer monoid. In the case of integral domains, most of them are already in [2] (in different arrangements and with different proofs). A detailed identification is left to the reader. The following two propositions highlight two special cases.

Proposition 4.3. *Let r and q be ideal systems on D such that $r \le q$. Then the following assertions are equivalent:*

(a) *D is an (r, q)-Dedekind monoid.*

(b) *D is an (r, v)-Dedekind monoid and $q = v$.*

(c) *For all $A, B \in \boldsymbol{F}(D)$ we have $(AB)^{-1} = (A^{-1}B^{-1})_r$, and $q = v$.*

(d) *D is a q-Dedekind monoid, and $(AB)_v = (A_v B_v)_r$ for all $A, B \in \boldsymbol{F}(D)$.*

Proof. (a) ⇒ (b). Since $\mathcal{F}_q(D) \subset \mathcal{F}_r(D)^\times \subset \mathcal{F}_v(D)$, it follows that $q = v$.

(b) ⇒ (c). If $A, B \in \boldsymbol{F}(D)$, then B_v is r-invertible, and thus Corollary 3.4 implies $(AB)^{-1} = (A^{-1}B^{-1})_r$.

(c) ⇒ (d). For every $B \in \boldsymbol{F}(D)$, Corollary 3.4 implies that $B_q = B_v$ is r-invertible, hence q-invertible, and thus D is a q-Dedekind monoid. For any $A, B \in \boldsymbol{F}(D)$, then we apply (c) twice and obtain

$$(AB)_v = ((AB)^{-1})^{-1} = ((A^{-1}B^{-1})_r)^{-1} = (A^{-1}B^{-1})^{-1} = (A_v B_v)_r.$$

(d) $\Rightarrow$ (a). If $A \in \mathcal{F}_q(D)^\bullet$, then A is q-invertible. Hence $D = (A^{-1}A)_q$, and since $A, A^{-1} \in \mathcal{F}_v(D)$, we obtain $D = (A^{-1}A)_v = (A^{-1}A)_r$. □

Proposition 4.4. *Let r be an ideal system on D. Then the following assertions are equivalent:*

(a) *D is an (r, v)-Prüfer monoid.*

(b) *$(AF)^{-1} = (A^{-1}F^{-1})_r$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*

(c) *D is a v-Prüfer monoid, and $(AF)_v = (A_vF_v)_r$ holds for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*

Proof. (a) $\Rightarrow$ (b). If $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$, then F_v is r-invertible. Hence Corollary 3.4 implies $(AF)^{-1} = (A^{-1}F^{-1})_r$.

(b) $\Rightarrow$ (c). If $F \in \boldsymbol{f}(D)$, then F_v is r-invertible and thus v-invertible by Corollary 3.4. Hence D is a v-Prüfer monoid. For $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$, we apply (b) twice and obtain $(AF)_v = ((AF)^{-1})^{-1} = ((A^{-1}F^{-1})_r)^{-1} = (A^{-1}F^{-1})^{-1} = (A_vF_v)_r$.

(c) $\Rightarrow$ (a). If $F \in \boldsymbol{f}(D)$, then F is v-invertible and $F^{-1} \in \boldsymbol{F}(D)$. Hence we obtain $D = (F^{-1}F)_v = (F^{-1}F_v)_r$, and therefore F_v is r-invertible. □

5 Characterization of r-Prüfer monoids

Most assertions of the following Theorem 5.1 is well known in the context of finitary ideal systems (see [11, §17] which is modeled after the antetype of [15, Theorem 6.6]). For star operations which are not necessarily of finite type such results was first proved in [2].

Theorem 5.1. *Let r and y be ideal systems on D such that $y \le r$. Then the following assertions are equivalent:*

(a) *D is an r-Prüfer monoid.*

(b) *For all $a, b \in D^\bullet$, the r-ideal $\{a, b\}_r$ is r-invertible.*

(c) *$[(A_y \cap B_y)(A \cup B)]_r = (AB)_r$ for all $A, B \in \boldsymbol{F}(D)$.*

(d) *$[(A_y \cap B_y)(A \cup B)]_r = (AB)_r$ for all $A, B \in \boldsymbol{f}(D)$.*

(e) *$[F(A_r \cap B_r)]_r = (FA)_r \cap (FB)_r$ for all $A, B, F \in \boldsymbol{f}(D)$.*

(f) *$[F(A_r \cap B_r)]_r = (FA)_r \cap (FB)_r$ for all $F \in \boldsymbol{f}(D)$ and $A, B \in \boldsymbol{F}(D)$.*

(g) *For all $I, J \in \mathcal{F}_r(D)^\times$ we have $I \cap J \in \mathcal{F}_r(D)^\times$ and $(I \cup J)_r \in \mathcal{F}_r(D)^\times$.*

(h) *For all $I, J \in \mathcal{F}_r(D)^\times$ we have $(I \cup J)_r \in \mathcal{F}_r(D)^\times$.*

(i) *For every family $(A_i)_{i \in I}$ in $\boldsymbol{F}(D)$ and all $F \in \boldsymbol{f}(D)$ we have*

$$\Big(\big(\bigcup_{i \in I} A_i\big)_y : F\Big)_r = \Big(\bigcup_{i \in I} ((A_i)_y : F)\Big)_r.$$

(j) $\big((A \cup B)_y : F\big)_r = [(A_y : F) \cup (B_y : F)]_r$ *for all* $A, B \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.

(k) $\big((A \cup B)_y : F\big)_r = [(A_y : F) \cup (B_y : F)]_r$ *for all* $A, B, F \in \boldsymbol{f}(D)$.

(l) $\big(A_y : (F_r \cap G_r)\big)_r = [(A_y : F_r) \cup (A_y : G_r)]_r$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F, G \in \boldsymbol{f}(D)$.

(m) $[(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r = D$ *for all* $a, b \in D^\bullet$.

Proof. (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d), (f) $\Rightarrow$ (e), (g) $\Rightarrow$ (h) and (i) $\Rightarrow$ (j) $\Rightarrow$ (k). Obvious.

(b) $\Leftrightarrow$ (m). Let $a, b \in D^\bullet$. Then $\{a, b\}^{-1} = a^{-1}D \cap b^{-1}D$ and therefore

$$\begin{aligned}(\{a,b\}\{a,b\}^{-1})_r &= (a\{a,b\}^{-1} \cup b\{a,b\}^{-1})_r \\ &= [a(a^{-1}D \cap b^{-1}D) \cup b(a^{-1}D \cap b^{-1}D)]_r \\ &= [(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r.\end{aligned}$$

Hence $\{a, b\}_r$ is r-invertible if and only if $[(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r = D$.

(b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (c). Let $A, B \in \boldsymbol{F}(D)$. Then obviously

$$(A_y \cap B_y)(A \cup B) \subset A_y B \cap AB_y \subset (AB)_r,$$

which implies $[(A_y \cap B_y)(A \cup B)]_r \subset (AB)_r$.

For the reverse inclusion, it suffices to prove that $AB \subset [(A_y \cap B_y)(A \cup B)]_r$. Thus let $a \in A$ and $b \in B$. Since $ab\{a, b\}^{-1} = ab(a^{-1}D \cap b^{-1}D) = aD \cap bD$, (b) implies that

$$\begin{aligned}ab \in [ab\{a,b\}\{a,b\}^{-1}]_r &= [a(aD \cap bD) \cup b(aD \cap bD)]_r \\ &= [aD \cup bD)(aD \cap bD)]_r \subset [(A_y \cap B_y)(A \cup B)]_r.\end{aligned}$$

By (d), it follows that

$$ab \in [(aD)(bD)]_r = [aD \cup bD)(aD \cap bD)]_r \subset [(A_y \cap B_y)(A \cup B)]_r,$$

and thus we obtain $AB \subset [(A_y \cap B_y)(A \cup B)]_r$ in both cases.

(c) $\Rightarrow$ (g). If $I, J \in \mathcal{F}_r(D)^\times$, then

$$[(I \cap J)(I \cup J)_r]_r = [(I \cap J)(I \cup J)]_r = (IJ)_r \in \mathcal{F}_r(D)^\times,$$

which implies $I \cap J \in \mathcal{F}_r(D)^\times$ and $(I \cup J)_r \in \mathcal{F}_r(D)^\times$.

(h) $\Rightarrow$ (a). We must prove that $E_r \in \mathcal{F}_r(D)^\times$ for every finite non-empty subset $E \subset D^\bullet$, and we proceed by induction on $|E|$. The assertion is obvious if $|E| = 1$. If $|E| \geq 1$, $E_r \in \mathcal{F}_r(D)^\times$ and $a \in D^\bullet$, then $(E \cup \{a\})_r = (E_r \cup aD)_r \in \mathcal{F}_r(D)^\times$, and thus the assertion follows by induction on $|E|$.

(a) $\Rightarrow$ (f). Let $F \in \boldsymbol{f}(D)$ and $A, B \in \boldsymbol{F}(D)$. Then

$$\begin{aligned}(FA)_r \cap (FB)_r &= (FF^{-1})_r[(FA)_r \cap (FB)_r] \\ &\subset \big(F[(F^{-1}FA)_r \cap (F^{-1}FB)_r]\big)_r = [F(A_r \cap B_r)]_r.\end{aligned}$$

Since $F(A_r \cap B_r) \subset FA_r \cap FB_r \subset (FA)_r \cap (FB)_r$, the reverse inclusion is obvious.

(e) ⇒ (d). As we have already proved the equivalence of (d) and (a), it suffices to show that (d) holds with $y = r$. Thus let $A, B \in \boldsymbol{f}(D)$, and set $F = (A \cup B)_\delta$. Then $F \in \boldsymbol{f}(D)$, and $[(A_r \cap B_r)(A \cup B)]_r = [(A_r \cap B_r)F]_r = (FA)_r \cap (FB)_r \supset (AB)_r$. On the other hand, we obviously have $(A_r \cap B_r)(A \cup B) \subset A_r B \cup AB_r \subset (AB)_r$, which implies the reverse inclusion.

(a) ⇒ (i). Let $(A_i)_{i \in I}$ be a family in $\boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$. Since F_r is r-invertible, Theorem 3.1 (f) (applied with $q = r$) implies

$$\Big(\big(\bigcup_{i \in I} A_i\big)_y : F\Big)_r = \Big(\big(\bigcup_{i \in I} A_i\big)_r F^{-1}\Big)_r = \Big(\bigcup_{i \in I} A_i F^{-1}\Big)_r \subset \Big(\bigcup_{i \in I} (A_i)_r F^{-1}\Big)_r = \Big(\bigcup_{i \in I} \big((A_i)_y : F\big)\Big)_r,$$

and the reverse inclusion is obvious.

(k) ⇒ (b). Let $a, b \in D^\bullet$ and apply (k) with $A = aD$, $B = bD$ and $F = \{a, b\}_\delta$. Then we obtain

$$\begin{aligned} D \subset (\{a,b\}_y : \{a,b\})_r &= [(aD : \{a,b\}) \cup (bD : \{a,b\})]_r \\ &= (a\{a,b\}^{-1} \cup b\{a,b\}^{-1})_r = (\{a,b\}\{a,b\}^{-1})_r \\ &= (\{a,b\}_r \{a,b\}^{-1})_r \subset D. \end{aligned}$$

Hence equality holds, and $\{a, b\}_r$ is r-invertible.

(a) ⇒ (l). Let $A \in \boldsymbol{F}(D)$ and $F, G \in \boldsymbol{f}(D)$. Then the fractional r-ideals F_r, G_r, F^{-1} and G^{-1} are r-invertible, and since we have already proved that (a) implies (g), it follows that $F_r \cap G_r$ and $(F^{-1} \cup G^{-1})_r$ are also r-invertible. Observe now that $F_r = (F^{-1})^{-1}$, $G_r = (G^{-1})^{-1}$ and

$$(F^{-1} \cup G^{-1})_r = [(F^{-1} \cup G^{-1})^{-1}]^{-1} = [(F^{-1})^{-1} \cap (G^{-1})^{-1}]^{-1} = (F_r \cap G_r)^{-1}.$$

We apply Theorem 3.1 (f) with $q = r$ and obtain

$$\begin{aligned} \big(A_y : (F_r \cap G_r)\big)_r &= [A_r (F_r \cap G_r)^{-1}]_r = [A_r (F^{-1} \cup G^{-1})_r]_r \\ &= [(A_r F^{-1})_r \cup (A_r G^{-1})_r]_r = [(A_y : F_r) \cup (A_y : G_r)]_r. \end{aligned}$$

(l) ⇒ (m). Let $a, b \in D^\bullet$, and apply 12. with $A = aD \cap bD$, $F = aD$ and $G = bD$. Then we obtain

$$\begin{aligned} D \subset ((aD \cap bD) : (aD \cap bD))_r &= \big[\big((aD \cap bD) : aD\big) \cup \big((aD \cap bD) : bD\big)\big]_r \\ &= [(D \cap a^{-1}bD) \cup (D \cap ab^{-1}D)]_r \subset D, \end{aligned}$$

and consequently $[(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r = D$. □

Remark 5.2. The presence of the ideal system y in Theorem 5.1 makes the criteria more flexible. The extremal cases $y = \delta$ and $y = r$ are the most interesting ones. Indeed, for $y = r$ the criteria become most transparent, for $y = d$ in the domain case they become comparable with criteria usually formulated in the literature, while the case $y = \delta$ is suitable for the monoid case.

Corollary 5.3. *Let r and q be ideal systems on D such that $r \le q$, and let D be an (r, q)-Prüfer monoid. Then the following assertions are equivalent:*

(a) *D is an r-Prüfer monoid.*

(b) $(A \cup B)_r = (A \cup B)_q$ *for all* $A, B \in \mathcal{F}_r(D)^\times$.

(c) $(A \cup B)_r = (A \cup B)_q$ *for all* $A, B \in \mathcal{F}_{r,\mathrm{f}}(D) \cap \mathcal{F}_r(D)^\times$.

(d) $\{a, b\}_r = \{a, b\}_q$ *for all* $a, b \in D^\bullet$.

Proof. (a) ⇒ (b). By Theorem 5.1 (h) we have $(A \cup B)_r \in \mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)$ and therefore $(A \cup B)_r = [(A \cup B)_r]_q = (A \cup B)_q$.

(b) ⇒ (c) ⇒ (d). Obvious.

(d) ⇒ (a). For all $a, b \in D^\bullet$ we have $\{a, b\}_r = \{a, b\}_q \in \mathcal{F}_{q,\mathrm{f}}(D)^\bullet \subset \mathcal{F}_r(D)^\times$, since D is an (r, q)-Prüfer monoid. Thus it follows that D is an r-Prüfer monoid by Theorem 5.1 (b). □

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Chain conditions in rings of the form $A + XB[X]$ and $A + XI[X]$

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Abstract. In this paper we study different chain conditions in pullbacks of the form $A + XB[X]$, $A + XB[[X]]$ where $A \subset B$ is an extension of commutative rings, $A + XI[X]$ and $A + XI[[X]]$ where I is a proper ideal of A. We give necessary and sufficient conditions for these rings to be Noetherian, to have Noetherian spectrum, to be Laskerian or to satisfy the ascending chain conditions for principal ideals when rings are supposed to be présimplifiable.

Keywords. Chain condition, Noetherian ring, Noetherian spectrum, ACCP, Laskerian ring, pullback.

AMS classification. 13E05, 13E99, 13B25, 13J05.

1 Introduction

All the rings considered below are commutative with unity, $\{X_1, \ldots, X_n\}$ is a finite non-empty set of analytically independent indeterminates over any ring. As usual if A is a commutative ring then $A[X]$ and $A[[X]]$ denote the rings of polynomials and of formal power series, respectively over A. In this paper we study different chain conditions (for ideals, for radical ideals, for principal ideals, Laskerian rings) in pullbacks of the form $A + XB[X]$ and $A + XB[[X]]$ where $A \subset B$ is an extension of commutative rings and $A + XI[X]$ and $A + XI[[X]]$ where A is a commutative ring and I is a proper ideal of A.

In the first section we give necessary and sufficient conditions for these rings to be Noetherian. We generalize some known results ([7] and [13]). We prove first that $A + XB[X]$ and $A + XB[[X]]$ are Noetherian if and only if they are nonnil-Noetherian if and only if A is Noetherian and B is a finitely generated A-module. Recall that a commutative ring A is said to be nonnil-Noetherian if each ideal of A which is not contained in the nilradical of A is finitely generated. For the rings of the form $A + XI[X]$ and $A + XI[[X]]$, using [14], we have the following:

Let A be a commutative ring with unity and I a proper ideal of A. The following statements are equivalent:

(1) A is Noetherian and $I^2 = I$.

(2) $A + (X_1, \ldots, X_n)I[X_1, \ldots, X_n]$ is Noetherian.

(3) $A + (X_1, \ldots, X_n)I[[X_1, \ldots, X_n]]$ is Noetherian.

In the second part of our paper, we characterize when rings of the form $A + XB[X]$ and $A + XI[X]$ have Noetherian spectrum (or equivalently satisfy the ascending chain condition for radical ideals) using results on pullbacks from [12]. We prove that $A +$

$XB[X]$ has Noetherian spectrum if and only if A and B do. And $A + XI[X]$ has Noetherian spectrum if and only if A has Noetherian spectrum.

In the third part, we generalize the ascending chain condition for principal ideals (ACCP) to présimplifiable rings. Recall that a ring is présimplifiable ([6]) if $\forall x, y \in A, (xy = x \Longrightarrow x = 0$ or $y \in \mathrm{U}(A))$ or equivalently $\mathrm{Z}(A) \subseteq 1 + \mathrm{U}(A)$, where $\mathrm{Z}(A)$ and $\mathrm{U}(A)$ denote the set of zero divisors and invertible elements of A respectively. It is known that for an integral domain A we have the equivalence: A satisfies ACCP if and only if $A[X]$ does if and only if $A[[X]]$ does. In [2] and [9], the authors gave necessary and sufficient conditions for the rings $A + XB[X]$ and $A + XB[[X]]$ to satisfy ACCP. We generalize some of these results under the assumption that rings are présimplifiable.

Finally, recall that a Laskerian ring is one for which every ideal is a finite intersection of primary ideals and a ring A is called a ZD ring (zero divisor ring) if for any ideal I of A the set $\{a \in A \mid \text{there exists } s \in A \backslash I \text{ such that } as \in I\}$ is a union of finitely many prime ideals. In [18], the authors proved that for a commutative ring A, the ring $A[X]$ is Laskerian if and only if it is ZD if and only if A is Noetherian. In the case of the power series ring Gilmer and Heinzer proved the equivalence $A[[X]]$ is Laskerian if and only if A is Noetherian but they gave an example of a non-Noetherian ring such that $A[[X]]$ is ZD.

In the last part we study Laskerian rings of the form $A + XB[X]$ and $A + XB[[X]]$. We generalize some results given by [1], [3] and [21]. We prove among others that if B is a finitely generated A-module then the following are equivalent:

(1) $A + XB[X]$ is ZD.

(2) $A + XB[X]$ is Laskerian.

(3) $A + XB[X]$ is strongly Laskerian.

(4) $A + XB[X]$ is Noetherian.

But in general we can have Laskerian rings of the form $A + XB[X]$ which are not Noetherian.

In this paper, R_n and S_n will denote the rings $A + (X_1, \ldots, X_n)B[X_1, \ldots, X_n]$, $A + (X_1, \ldots, X_n)B[[X_1, \ldots, X_n]]$ respectively. T_n and V_n will denote respectively the rings $A + (X_1, \ldots, X_n)I[X_1, \ldots, X_n]$ and $A + (X_1, \ldots, X_n)I[[X_1, \ldots, X_n]]$.

2 Noetherian rings of the form $A + XB[X]$, $A + XB[[X]]$, $A + XI[X]$ and $A + XI[[X]]$

Proposition 2.1. *Let $A \subset B$ be an extension of commutative rings with unity. The following statements are equivalent:*

(1) *The ring A is Noetherian and B is a finitely generated A-module.*

(2) *R_n is Noetherian.*

(3) *S_n is Noetherian.*

(4) *R_n is nonnil-Noetherian.*

(5) *S_n is nonnil-Noetherian.*

Proof. For the equivalence (1) $\iff$ (2) (resp. (1) $\iff$ (3)) we apply [7, Proposition 1] to the rings R_n (resp. S_n), $B[X_1, \dots, X_n]$ (resp. $B[[X_1, \dots, X_n]]$) and the common ideal $(X_1, \dots, X_n)B[X_1, \dots, X_n]$ (resp. $(X_1, \dots, X_n)B[[X_1, \dots, X_n]]$). Now we prove that 2 is equivalent to 4. If R_n is Noetherian, then it is nonnil-Noetherian. Conversely, let $p \in \operatorname{spec}(A)$. Then $p + (X_1, \dots, X_n)B[X_1, \dots, X_n] \in \operatorname{spec}(R_n)$, moreover, it's a nonnil ideal so it is finitely generated, which implies that A is Noetherian. The ideal $X_1B[X_1, \dots, X_n]$ of R is nonnil, so it is finitely generated, which implies that B is a finitely generated A-module. So R_n is Noetherian. The proof is the same for S_n. □

Example 2.2. Let $K \subset L$ be an extension of fields. It is known that the ring $K + (X_1, \dots, X_n)L[X_1, \dots, X_n]$ is Noetherian if and only if L/K is finite.

Remark 2.3. The ring R_n is an integral domain if and only if B is an integral domain. In this case $\operatorname{qf}(R_n) = \operatorname{qf}(B[X_1, \dots, X_n])$ as they have the common ideal $(X_1, \dots, X_n)B[X_1, \dots, X_n]$.

If $A \subset B$, the ring R_n is never principal. In fact it is never UFD because it is never completely integrally closed: if $b \in B \setminus A$, then $b \in \operatorname{qf}(R_n) \setminus R_n$ and for each $n \in \mathbb{N}$, $X_1 b^n \in R_n$.

Noetherian rings of the form $A + XI[X_1, \dots, X_n]$ are characterized in [14, Theorem 1]. The proof given there works also for power series rings. So we have the following result:

Proposition 2.4. *Let A be a commutative ring with unity and I a proper ideal of A. The following are equivalent:*

(1) *The ring A is Noetherian and $I^2 = I$.*

(2) *T_n is Noetherian.*

(3) *V_n is Noetherian.*

Corollary 2.5. *If the ideal $I \not\subseteq \operatorname{Nil}(A)$, then $A + XI[X]$ is nonnil-Noetherian if and only if it is Noetherian.*

Proof. Obviously, if $A + XI[X]$ is Noetherian then it is nonnil-Noetherian. Conversely, let $p \in \operatorname{spec}(A)$. Then $p + XI[X] \in \operatorname{spec}(A + XI[X])$, moreover it is a nonnil ideal, so it is finitely generated which implies that p is finitely generated and then A is Noetherian. The ideal $I[X]$ is a nonnil ideal of $A + XI[X]$, so it is finitely generated. Let $f_1, \dots, f_s \in I[X]$ such that $I[X] = f_1(A + XI[X]) + \cdots + f_s(A + XI[X])$. We reduce modulo I^2 and we compare the degrees, which implies that $I^2 = I$. Using the previous proposition we conclude that $A + XI[X]$ is Noetherian. □

Remark 2.6. The ring $A + XI[X]$ is an integral domain if and only if A is an integral domain. In this case $\operatorname{qf}(A + XI[X]) = \operatorname{qf}(A[X])$. If I is a proper ideal of A, then $A + XI[X]$ is never principal. In fact it is never UFD: let $a \in A \setminus I$. Then $aX \in \operatorname{qf}(A[X]) = \operatorname{qf}(A + XI[X])$. Let $b \in I \setminus (0)$. Then for each $n \in \mathbb{N}$, $b(aX)^n = ba^nX^n \in A + XI[X]$. So $A + XI[X]$ is never completely integrally closed.

3 Rings with Noetherian spectrum of the form $A + XB[X]$ and $A + XI[X]$

Recall that a ring A has Noetherian spectrum if it satisfies the ascending chain condition for radical ideals. It is known that a ring A has Noetherian spectrum if and only if $A[X]$ has Noetherian spectrum.

Proposition 3.1. *The ring R_n has Noetherian spectrum if and only if A and B have Noetherian spectrum.*

Proof. Suppose that R_n has Noetherian spectrum. As A is a quotient ring then the spectrum of A is Noetherian. If the spectrum of B is not Noetherian, then there exists a radical ideal I of B which is not the radical of a finitely generated ideal of B. Let $b_0 \in I$. Then $\sqrt{b_0 B} \subset I$, so there exists $b_1 \in I \setminus \sqrt{b_0 B}$. By induction, we construct a sequence $(b_i)_i$ of elements of I such that $\sqrt{b_0 B} \subset \sqrt{b_0 B + b_1 B} \subset \cdots \subset \sqrt{b_0 B + b_1 B + \cdots + b_i B} \subset \cdots$. We obtain the following sequence of radical ideals of R_n:

$$\sqrt{b_0 X_1 R_n} \subseteq \sqrt{b_0 X_1 R_n + b_1 X_1 R_n} \subseteq \cdots \subseteq \sqrt{b_0 X_1 R_n + b_1 X_1 R_n + \cdots + b_i X_1 R_n} \subseteq \cdots .$$

The sequence is strictly increasing, otherwise there exist $k, p \in \mathbb{N}$, such that $X_1^p b_{k+1}^p = X_1 b_0 f_0 + \cdots + X_1 b_k f_k$, with $f_0, \ldots, f_k$ in R_n, so $b_{k+1}^p = b_0 \alpha_0 + \cdots + b_k \alpha_k$, where $\alpha_i \in B$. So $b_{k+1} \in \sqrt{b_0, \ldots, b_k}$ in B, which is a contradiction. Conversely, consider the commutative diagram:

$$\begin{array}{ccc} R_n & \to & A \\ \downarrow & & \downarrow \\ B[X_1, \ldots, X_n] & \to & B[X_1, \ldots, X_n]/(X_1, \ldots, X_n)B[X_1, \ldots, X_n] \end{array}$$

Using [12, Corollary 1.6], $\operatorname{spec}(R_n)$ and $\operatorname{spec}(B)$ are Noetherian if and only if $\operatorname{spec}(A)$ and $\operatorname{spec}(B[X_1, \ldots, X_n])$ are Noetherian. So if A and B have Noetherian spectrum, then so are A and $B[X_1, \ldots, X_n]$ and then $\operatorname{spec}(R_n)$ is Noetherian. □

Example 3.2. Let $A \subset B$ be two Noetherian rings such that B is not a finitely generated A-module then R_n has Noetherian spectrum but is not Noetherian. For example suppose that $K \subset L$ is an extension of fields of infinite degree. Then the ring $K + (X_1, \ldots, X_n)L[X_1, \ldots, X_n]$ has Noetherian spectrum but is not Noetherian.

Remark 3.3. The equivalence is false in the case of formal power series. In fact, let $A = B$ be a non-discrete valuation domain of rank 1. Then A has Noetherian spectrum but in $A[[X]]$, there exists an infinite chain of prime ideals so $A[[X]]$ has not Noetherian spectrum.

We show that T_n has Noetherian spectrum if and only if spec(A) is Noetherian.

Proposition 3.4. *Let A be a commutative ring with unity and I a proper ideal of A. Then T_n has Noetherian spectrum if and only if* spec(A) *is Noetherian.*

Proof. If spec(T_n) is Noetherian, then so is spec(A). Conversely, consider the commutative diagram:

$$\begin{array}{ccc} T_n & \to & A \\ \downarrow & & \downarrow \\ A[X_1,\dots,X_n] & \to & A[X_1,\dots,X_n]/(X_1,\dots,X_n)I[X_1,\dots,X_n] \end{array}$$

By [12, Corollary 1.6], spec(T_n) and spec$(A[X_1,\dots,X_n]/(X_1,\dots,X_n)I[X_1,\dots,X_n])$ are Noetherian if and only if spec(A) and spec$(A[X_1,\dots,X_n])$ are Noetherian. So, if spec(A) (and then spec$(A[X_1,\dots,X_n])$) is Noetherian, then T_n has also Noetherian spectrum. □

Remark 3.5. Here also the equivalence is false in the case of formal power series. Take A to be a non-discrete valuation domain of rank 1 and I the unique maximal ideal of A, then I is not an SFT ideal and using [19] there exists an infinite chain of prime ideals in $A + XI[[X]]$ so spec$(A + XI[[X]])$ is not Noetherian, although spec(A) is Noetherian.

4 Ascending chain condition for principal ideals in rings of the form $A + XB[X]$ and $A + XI[X]$

First we recall from [9] some necessary and sufficient conditions for the rings R_n and S_n in order to satisfy the ascending chain condition for principal ideals, then we prove an analogous result for the rings T_n and V_n when the rings are supposed to be integral domains. In the second part we generalize these results under the assumption that rings are présimplifiable.

Proposition 4.1. *If $f = a_0 + g \in R_n$, then $f \in \mathrm{U}(R_n)$ if and only if $a_0 \in \mathrm{U}(A)$ and $g \in (X_1,\dots,X_n)\,\mathrm{Nil}(B)[X_1,\dots,X_n]$.*

Proof. If $a_0 \in \mathrm{U}(A)$ and $g \in \mathrm{Nil}(B)[X_1,\dots,X_n] \cap (X_1,\dots,X_n)B[X_1,\dots,X_n]$, then f is the sum of a nilpotent element and an invertible element so f is invertible in R_n. Conversely, we prove the result by induction on n. For $n = 1$, if $f = a_0 + a_1X + \cdots + a_nX^n \in \mathrm{U}(A + XB[X])$, then there exists $g = b_0 + b_1X + \cdots + b_mX^m \in A + XB[X]$ such that $fg = 1$, which implies that $a_0b_0 = 1$, so $a_0 \in \mathrm{U}(A)$. On the other hand, as $f \in \mathrm{U}(A + XB[X]) \subset \mathrm{U}(B[X])$, then $a_i \in \mathrm{Nil}(B)$, $\forall i \geq 1$. Let $n \geq 2$. Suppose that the result is true for $n - 1$, we show it for n. Let $f = f_0 + f_1X_1 + \cdots + f_sX_1^s$ with $f_0 \in A + (X_2,\dots,X_n)B[X_2,\dots,X_n]$ and $f_i \in (X_2,\dots,X_n)B[X_2,\dots,X_n]$ for $i \geq 1$. As $f \in \mathrm{U}(A + (X_1,\dots,X_n)B[X_1,\dots,X_n])$, then $f_0 \in \mathrm{U}(A + (X_2,\dots,X_n)B[X_2,\dots,X_n])$ and $\forall i \geq 1$, $f_i \in \mathrm{Nil}(B)[X_2,\dots,X_n]$.

By induction, we have $f_0 = a_0 + h$, where $h \in \mathrm{Nil}((X_2, \dots, X_n)B[X_2, \dots, X_n])$, so $a_0 \in \mathrm{U}(A)$ and $h \in \mathrm{Nil}(B)[X_2, \dots, X_n]$. Then, $g \in \mathrm{Nil}(B)[X_1, \dots, X_n]$. □

Recall the following results from [9]:

Lemma 4.2 ([9, Remark 1.1]). *Let A be an integral domain. Then A satisfies the ACCP if and only if for each sequence (a_n) of non-invertible elements of A, we have $\bigcap_{n \geq 1} a_1 \cdots a_n A = (0)$.*

Proposition 4.3 ([9, Proposition 1.2] and [9, Remark 1.4]). *Let $A \subset B$ be an extension of integral domains. The following are equivalent:*

(1) *$A + (X_1, \dots, X_n)B[X_1, \dots, X_n]$ satisfies the ACCP.*

(2) *$A + (X_1, \dots, X_n)B[[X_1, \dots, X_n]]$ satisfies the ACCP.*

(3) *For each sequence (a_j) of non-invertible elements of A, $\bigcap_{j \geq 1} a_1 \cdots a_j B = (0)$.*

Lemma 4.4. *The element $f = a_0 + g \in T_n$ is invertible in T_n if and only if a_0 is invertible in A and g is nilpotent.*

Proof. Let $f = a_0 + g \in \mathrm{U}(T_n) \subset \mathrm{U}(A[X_1, \dots, X_n])$. So a_0 is invertible in A and all the coefficients of g are nilpotent. Conversely, if $a_0 \in \mathrm{U}(A)$ and g is nilpotent, then f is invertible as it is the sum of an invertible and a nilpotent element. □

It is easy to prove the following lemma:

Lemma 4.5. *The element $f = a_0 + g \in V_n$ is invertible in V_n if and only if a_0 is invertible in A.*

Proposition 4.6. *Let A be an integral domain and I be a proper ideal of A. The following are equivalent:*

(1) *T_n satisfies the ACCP.*

(2) *V_n satisfies the ACCP.*

(3) *A satisfies the ACCP.*

Proof. The equivalence between (2) and (3) is proved in [19]. We show the equivalence between (1) and (3). Let $a_1 A \subseteq \cdots \subseteq a_n A \subseteq \cdots$ be a sequence of principal ideals of A. We obtain the sequence $a_1 T_n \subseteq \cdots \subseteq a_n T_n \subseteq \cdots$ in T_n. Since the ring T_n satisfies the ACCP, there exists $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, $a_k T_n = a_{k_0} T_n$. So, there exists $f_k \in \mathrm{U}(T_n) = \mathrm{U}(A)$ such that $a_k = f_k a_{k_0}$. So $a_k A = a_{k_0} A$ $\forall k \geq k_0$. Conversely, let $(f_k)_k$ be a sequence of non-invertible polynomials in T_n. We show that $\bigcap_{k \geq 1} f_1 \cdots f_k T_n = (0)$. If the sequence contains an infinite subsequence of non-constant polynomials, then $\bigcap_{k \geq 1} f_1 \cdots f_k T_n = (0)$. We can suppose that almost all the f_i are constant. If all the f_i are in A, so $\bigcap_{k \geq 1} f_1 \cdots f_k T_n = \bigcap_{k \geq 1} f_1 \cdots f_k A + (X_1, \dots, X_n)(\bigcap_{k \geq 1} f_1 \cdots f_k I)[X_1, \dots, X_n] = (0)$. In general, there exists $l \geq 1$ such that $\forall k > l$, $f_k \in A$. Let $g = f_1 \cdots f_l$. Then we have $\bigcap_{k \geq 1} f_1 \cdots f_k T_n = g \bigcap_{k \geq l+1} f_{l+1} \cdots f_k T_n = (0)$. □

Recall the definition of a “présimplifiable” ring [6]. A présimplifiable ring is a ring with zero divisors which is nearly an integral domain, in the sense that many domain properties are saved.

Proposition 4.7. *Let A be a commutative ring. The following are equivalent:*

(1) $\forall x, y \in A$, $(xy = x \Longrightarrow x = 0$ *or* $y \in \mathrm{U}(A))$.

(2) $\mathrm{Z}(A) \subseteq 1 + \mathrm{U}(A)$.

Definition 4.8. If the equivalent conditions of the preceding proposition are satisfied, A is said to be a “présimplifiable” ring.

Proposition 4.9 ([6]). *Let A be a commutative ring. Then $A[X_1, \dots, X_n]$ is présimplifiable if and only if* $\mathrm{Z}(A) = \mathrm{N}(A)$.

Remark 4.10. If A is présimplifiable then we don’t have $A[X]$ présimplifiable in general. We can take A a local ring (which implies that A is présimplifiable) such that $\mathrm{N}(A) \neq \mathrm{Z}(A)$. For example, let K be a field. In $K[[X, Y]]$, we take $I = \langle XY, Y^2 \rangle$ and $A = K[[X, Y]]/I$. Then A is local, so présimplifiable. But $\overline{X} \in \mathrm{Z}(A) \backslash \mathrm{N}(A)$. In fact $\overline{X} \notin \mathrm{N}(A)$ otherwise there exists $n \in \mathbb{N}$ such that $X^n \in I$, so there exist $f, g \in K[[X, Y]]$ such that $X^n = XYf + Y^2 g$. Take $Y = 0$, then $X^n = 0$, which is impossible. In the other hand $\overline{X} \in \mathrm{Z}(A)$, as $\overline{XY} = 0$ and $\overline{Y} \neq 0$.

Proposition 4.11. *A is présimplifiable if and only if $A[[X_1, \dots, X_n]]$ is présimplifiable.*

Proof. Using induction on n, we can prove the result for $n = 1$. Using [5], if $A[[X]]$ is présimplifiable, then so is A. Conversely, let $f = \sum_{i \geq 0} a_i X^i \in \mathrm{Z}(A[[X]])$; there exists $g \in A[[X]]\backslash(0)$ such that $fg = 0$. Let $g = \sum_{i \geq 0} b_i X^i$ and $k = \inf\{i \in \mathbb{N} \mid b_i \neq 0\}$. As $fg = 0$, then $a_0 b_k = 0$ with $b_k \neq 0$, so $a_0 \in \mathrm{Z}(A) \subset 1 + \mathrm{U}(A)$ because A is présimplifiable. So $f - 1 \in \mathrm{U}(A[[X]])$. □

Proposition 4.12. (1) $\mathrm{Nil}(R_n) = \mathrm{Nil}(A) + (X_1, \dots, X_n)\,\mathrm{Nil}(B)[X_1, \dots, X_n]$.

(2) *$f \in \mathrm{Z}(R_n)$ if and only if there exists $b \in B\backslash(0)$ such that $bf = 0$.*

Proof. (1) Let $f \in \mathrm{Nil}(R_n) \subseteq \mathrm{Nil}(B[X_1, \dots, X_n]) = \mathrm{Nil}(B)[X_1, \dots, X_n]$. So all the coefficients of f are nilpotent and $f \in \mathrm{Nil}(A) + (X_1, \dots, X_n)\,\mathrm{Nil}(B)[X_1, \dots, X_n]$. The other inclusion is clear.

(2) Let $f \in \mathrm{Z}(R_n) \subseteq \mathrm{Z}(B[X_1, \dots, X_n])$; there exists $b \in B\backslash(0)$ such that $bf = 0$. Conversely, let $f \in R_n$ such that there exist $b \in B\backslash(0)$ and $bf = 0$. Then $bX_1 f = 0$, and $f \in \mathrm{Z}(R_n)$. □

Proposition 4.13. *The ring R_n is présimplifiable if and only if* $\mathrm{Z}(B) = \mathrm{Nil}(B)$.

Proof. If $b \in \mathrm{Z}(B)$, then $bX_1 \in \mathrm{Z}(R_n) \subseteq 1 + \mathrm{U}(R_n)$. So $1 - bX_1 \in \mathrm{U}(R_n)$, which implies that $b \in \mathrm{Nil}(B)$. Conversely, let $f \in \mathrm{Z}(R_n)$; there exists $b \in B\backslash(0)$ such that $bf = 0$. So all the coefficients of f are in $\mathrm{Z}(B) \subseteq \mathrm{Nil}(B)$. So $f \in \mathrm{Nil}(R_n)$. And $f - 1 \in \mathrm{U}(R_n)$. □

Proposition 4.14. *S_n is présimplifiable if and only if* $\mathrm{Z}(B) \cap A \subseteq 1 + \mathrm{U}(A)$.

Proof. If $a \in \mathrm{Z}(B) \cap A$, then $a \in \mathrm{Z}(S_n) \subset 1 + \mathrm{U}(S_n)$. So $a \in 1 + \mathrm{U}(A)$. Conversely, let $f \in \mathrm{Z}(R)$. Then the constant term of f, $a_0 \in \mathrm{Z}(B) \cap A \subset 1 + \mathrm{U}(A)$. So $f - 1 \in \mathrm{U}(S_n)$. And $f \in 1 + \mathrm{U}(S_n)$. □

Lemma 4.15 ([24]). *The ring R is présimplifiable if and only if for every $a, b \in R\backslash(0)$ such that $(a) = (b)$ and $a = bc$, then $c \in \mathrm{U}(R)$.*

Proposition 4.16. *Let A be a présimplifiable ring. Then A satisfies ACCP if and only if $A[[X_1, \ldots, X_n]]$ satisfies ACCP.*

Proof. It is sufficient to prove the result for $n = 1$.

Let $(f_1) \subset (f_2) \subset \cdots$ be an increasing sequence of principal ideals of $A[[X]]$. We can simplify by a convenient power of X, and suppose that $f_i(0) \neq 0$ for each i. We obtain the increasing sequence of non-zero principal ideals of A: $(f_1(0)) \subset (f_2(0)) \subset \cdots$. As A satisfies ACCP, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $(f_k(0)) = (f_n(0))$. But for each $n \geq k$, there exists $g_n \in A[[X]]$ such that $f_k = g_n f_n$, so $f_k(0) = g_n(0) f_n(0)$. As A is présimplifiable, then $g_n(0) \in \mathrm{U}(A)$. So $g_n \in \mathrm{U}(A[[X]])$. And for each $n \geq k$, $(f_n) = (f_k)$.

Conversely, let $(a_1) \subset (a_2) \subset \cdots$ be an increasing sequence of principal ideals of A. We obtain the sequence $a_1 A[[X]] \subset a_2 A[[X]] \subset \cdots$ in $A[[X]]$. But $A[[X]]$ satisfies ACCP, so there exists $k \in N$ such that for each $n \geq k$, we have $a_n A[[X]] = a_k A[[X]]$, which implies that $a_n A = a_k A$. □

Remark 4.17. We use [16] to prove that if A is présimplifiable and satisfies ACCP, then $A[X]$ does not satisfy ACCP in general. First, we recall the counter-example of Heinzer and Lantz: Let K be a field and $(A_i)_{i \geq 1}$ indeterminates over K and $S = K[A_i, i \geq 1]/\langle A_n(A_{n-1} - A_n), n \geq 2\rangle$. Let a_n the image of A_n in S and $P = (a_1, a_2, \ldots)S$. Take $R = S_P$. Then S is a graded ring. If $f \in S$, then the order of f is the smallest degree of non-zero terms of f. If $f, g \in S$, then $\mathrm{ord}(fg) \geq \mathrm{ord}(f) + \mathrm{ord}(g)$. Each $f \notin (a_1, a_2, \ldots)S$ has a non-zero term of degree 0 which is invertible, so for such an element f and for each $g \in S$, $\mathrm{ord}(fg) = \mathrm{ord}(g)$. Moreover the elements of S of order 0 are invertible in R. We prove that R is présimplifiable, which is equivalent to: for each $a, b \in R\backslash(0)$ such that $(a) = (b)$ and $a = bc$, then $c \in \mathrm{U}(R)$. It is sufficient to take $f_1, f_2 \in S$ such that $f_1 R = f_2 R$ and $f_1 = \frac{g}{h} f_2$ where $\mathrm{ord}(g) \geq 0$ and $\mathrm{ord}(h) = 0$. We have to prove that $\frac{g}{h} \in \mathrm{U}(R)$. As $f_2 \in f_1 R$, then $f_2 = f_1 \frac{g'}{h'}$ where $\mathrm{ord}(h') = 0$. So $f_1 = \frac{g}{h}\frac{g'}{h'} f_1 = \frac{gg' f_1}{hh'}$. So there exists $u \in S\backslash P$ such that $uhh' f_1 = ugg' f_1$. Then $\mathrm{ord}(hh' f_1) = \mathrm{ord}(f_1) \geq \mathrm{ord}(g) + \mathrm{ord}(g') + \mathrm{ord}(f_1)$. We conclude that $\mathrm{ord}(g) = \mathrm{ord}(g') = 0$, and $g, g' \in \mathrm{U}(R)$. So, $\frac{g}{h} \in \mathrm{U}(R)$.

Proposition 4.18. *Let A be a présimplifiable ring and I a proper ideal of A. Then A satisfies ACCP if and only if $A + XI[[X]]$ does.*

Proof. ($\Longleftarrow$) Let $a_1 A \subset a_2 A \subset \cdots$ be an increasing sequence of principal ideals of A. We obtain the sequence $a_1(A + XI[[X]]) \subset a_2(A + XI[[X]]) \subset \cdots$ in $A + XI[[X]]$.

As $A + XI[[X]]$ satisfies ACCP, then there exists $n \in N$ such that for any $k \geq n$, $a_k(A + XI[[X]]) = a_n(A + XI[[X]])$ so $a_k A = a_n A$.

($\Longrightarrow$) Let $f_1(A + XI[[X]]) \subset f_2(A + XI[[X]]) \subset \cdots$ be an increasing sequence of principal ideals of $A + XI[[X]]$. We can simplify by a convenient power of X, and suppose that $f_i(0) \neq 0$ for each i. We obtain the increasing sequence of non-zero principal ideals of A : $(f_1(0)) \subset (f_2(0)) \subset \cdots$. As A satisfies ACCP, there exists $k \in \mathbb{N}$ such that for any $n \geq k$, $(f_k(0)) = (f_n(0))$. For each $n \geq k$, there exists $g_n \in A + XI[[X]]$ such that $f_k = g_n f_n$, so $f_k(0) = g_n(0) f_n(0)$. As A is présimplifiable, then $g_n(0) \in \mathrm{U}(A)$. So $g_n \in \mathrm{U}(A + XI[[X]])$ and for each $n \geq k$, $(f_n)_{A+XI[[X]]} = (f_k)_{A+XI[[X]]}$. □

Remark 4.19. If A is présimplifiable and satisfies ACCP, then $A + XI[X]$ does not satisfy ACCP in general. Take in the counter-example of Heinzer and Lantz $I = (S \backslash P)^{-1} P$ which is a proper ideal of R. Let $f_1 = 1 + a_1 X, \ldots, f_n = 1 + a_n X, \ldots$. Then $f_{n-1} = f_n((a_{n-1} - a_n)X + 1)$. So $f_{n-1}(R + XI[X]) \subset f_n(R + XI[X])$. These are proper inclusions. In fact, suppose that there exists $n \geq 2$ and $b(X) \in R + XI[X]$ such that $(1 + a_{n-1}X)b(X) = 1 + a_n X$. Let $b(X) = b_0 + b_1 X + \cdots$. Then $(1 + a_{n-1}X)(b_0 + b_1 X + \cdots) = 1 + a_n X$, which implies that $b_0 = 1$ and for $n \geq 1$, $b_m = (-1)^{m-1} a_{n-1}^{m-1}(a_n - a_{n-1})$, so each b_m is non-zero and $b(X)$ is not a polynomial which is impossible.

Lemma 4.20. *Let $A \subset B$ be an extension of commutative rings such that B is présimplifiable. If $\mathrm{U}(B) \cap A = \mathrm{U}(A)$, then A is présimplifiable.*

Proof. As $\mathrm{Z}(B) \subset 1 + \mathrm{U}(B)$, then $\mathrm{Z}(A) \subset \mathrm{Z}(B) \cap A \subset (1 + \mathrm{U}(B)) \cap A = 1 + \mathrm{U}(B) \cap A = 1 + \mathrm{U}(A)$. So A is présimplifiable. □

In the next proposition we prove using [2], that we have the same necessary and sufficient condition, as in the case of integral domains, for the ring $A + XB[[X]]$ to satisfy ACCP if we suppose that B is a présimplifiable ring.

Proposition 4.21. *Let $A \subset B$ be an extension of commutative rings such that B is présimplifiable. Then $A + XB[[X]]$ satisfies ACCP if and only if $\mathrm{U}(B) \cap A = \mathrm{U}(A)$ and for each sequence (b_n) of B such that $b_n/b_{n+1} \in A$, the sequence $b_1 B \subset b_2 B \subset \cdots$ is stationary.*

Proof. ($\Longrightarrow$) Let $a \in \mathrm{U}(B) \cap A$. Consider the sequence $(a^{-1}X)(A + XB[[X]]) \subset (a^{-2}X)(A + XB[[X]]) \subset \cdots$. There exists $n \in N$ such that $(a^{-n}X)(A + XB[[X]]) = (a^{-(n+1)}X)(A + XB[[X]])$. So $a^{-(n+1)}X = a^{-n}X(c + Xf)$, where $c \in A$ and $f \in B[[X]]$. We obtain $a^{-1} = c + Xf$, and $a^{-1} = c \in A$.

Let (b_n) be a sequence of B such that $b_n = b_{n+1}a_{n+1}$ for each $n \in N$. The element $Xb_n \in Xb_{n+1}(A + XB[[X]])$ as $Xb_n = Xb_{n+1}a_{n+1}$ with $a_{n+1} \in A + XB[[X]]$. The sequence $(Xb_n(A + XB[[X]]))$ is increasing so it stops. Then there exists $n \in N$ such that for each $k \geq n$, $Xb_n(A + XB[[X]]) = Xb_k(A + XB[[X]])$. So $Xb_k \in Xb_n(A + XB[[X]])$ and $b_k \in b_n(A + XB[[X]])$. Then $b_k \in b_n B$ and $b_n B = b_k B$ for each $k \geq n$.

($\Longleftarrow$) Let $f_1(A + XB[[X]]) \subset f_2(A + XB[[X]]) \subset \cdots$ be an increasing sequence of principal ideals of $A + XB[[X]]$. We can suppose that $b_n = f_n(0) \in B\backslash(0)$ and $b_n/b_{n+1} = a_{n+1} \in A$. So the sequence $b_1 B \subset b_2 B \subset \cdots$ stops. Let $n_0 \in N$ such that for each $n \geq n_0$, $b_n B = b_{n+1} B$. As $b_n = b_{n+1} a_{n+1}$ and B is présimplifiable, we have $a_{n+1} \in \mathrm{U}(B) \cap A = \mathrm{U}(A)$. So $f_n(A + XB[[X]]) = f_{n+1}(A + XB[[X]])$. □

5 Laskerian rings of the form $A + XB[X]$ and $A + XB[[X]]$

A ring A is Laskerian if every ideal of A is a finite intersection of primary ideals. Let A be a commutative ring and E an A-module, $\mathrm{Ass}(E)$ is the set of prime ideals of A which are associated to E in the weak Bourbaki sense that is $P \in \mathrm{Ass}(E)$ if and only if there exists $x \in E$ such that P is a minimal prime ideal over $\mathrm{ann}(x)$.

Proposition 5.1 ([23]). *A ring A is Laskerian if and only if for every cyclic A-module E,* $\mathrm{Ass}(E)$ *is finite and* $\forall P \in \mathrm{Ass}(E)$, $\exists x \in E$ *such that* $\mathrm{ann}(x)$ *is P-primary.*

Definition 5.2. A ring A is strongly Laskerian if it is Laskerian and for each ideal I of A, there exists $k \in \mathbb{N}$ such that $(\sqrt{I})^k \subseteq I$.

A submodule Q of M is primary if $Q \neq M$ and every zero divisor on M/Q is nilpotent that is if $x \in A$ and $m \in M$ are such that $xm \in Q$, then $m \in Q$ or there exists $k > 0$ such that $x^k M \subset Q$.

If Q is a primary submodule then $Q : M$ is a primary ideal. In fact, if $a, b \in A$, $ab \in Q : M$ and $a \notin Q : M$ then $abM \subset Q$ and $aM \not\subseteq Q$, so there exists $m \in M$ such that $am \notin Q$. But $abm \in Q$ and Q is primary so there exists k such that $b^k M \subset Q$ then $b^k \in Q : M$.

Proposition 5.3. *If M is a Laskerian A-module then for any submodule N of M,* $\mathrm{Ass}(M/N)$ *is finite* $(*)$ *(*$\mathrm{Ass}(M/N)$ *is the set of prime ideals of A which are minimal over $N : x$ where $x \in M\backslash N$). Moreover, $\forall p \in \mathrm{Ass}(M/N)$, there exists $x \in M\backslash N$ such that $N : x$ is P-primary* $(**)$.

Proof. Let N be a submodule of M. Consider a reduced decomposition of $N = Q_1 \cap \cdots \cap Q_n$ and let $P_i = r_M(Q_i) \in \mathrm{spec}(A)$. We prove that the P_i are independent of the decomposition, in fact we show that the P_i are exactly the prime ideals of A which are of the form $\sqrt{N : x}$ where $x \in M\backslash N$. Let $x \in M\backslash N$. Then $N : x = (\bigcap_{1\leq i\leq n} Q_i : x) = \bigcap_{1\leq i\leq n}(Q_i : x)$, so $\sqrt{N : x} = \bigcap_{1\leq i\leq n} \sqrt{Q_i : x}$. But if $x \in Q_i$, $Q_i : x = A$, otherwise $Q_i : x$ is P_i-primary. Let $x \in M\backslash N$ such that $\sqrt{N : x} \in \mathrm{spec}(A)$; there exists i such that $\sqrt{N : x} = P_i$. Conversely, we show that $\forall 1 \leq i \leq n$, there exists $x \in M\backslash N$ such that $P_i = \sqrt{N : x}$. We take $x \in \bigcap_{i\neq j} Q_j \backslash Q_i$ (the decomposition is reduced), then $\sqrt{N : x} = P_i$. So if M is a Laskerian A-module, then $\mathrm{Ass}(M/N)$ is finite. In fact, let $P \in \mathrm{Ass}(M/N)$; there exists $x \in M\backslash N$ such that P is a minimal prime ideal over $N : x$, so $N : x \subset P$, then $\sqrt{N : x} = \cap P_i \subset P$, so there exists j such that $P_j \subset P$ as P is minimal over $N : x$, we have $P = P_j$. Moreover, we have $\forall i$, $P_i = \sqrt{N : x}$ where $x \in \bigcap_{i\neq j} Q_j \backslash Q_i$, so $N : x$ is P_i-primary. □

Recall [22] that an A-module M satisfies accr if the ascending chain of residuals of the form $N : I \subset \cdots \subset N : I^n \subset \cdots$ terminates for every submodule N of M and every finitely generated ideal I of A and this is equivalent to: the ascending chain of submodules of the form $N : a \subset \cdots \subset N : a^n \subset \cdots$ terminates for every submodule N of M and every element a of A. A ring A satisfies accr if it does as a module over itself.

Proposition 5.4. *If $R = A + XB[X]$ (resp. $A + XB[[X]]$) is Laskerian, then A is Laskerian and B is Noetherian. Moreover if A is an integral domain then the A-module B satisfies the conditions $(*)$ and $(**)$ of the preceding proposition.*

Proof. We prove the result in the case of the polynomial ring. The proof is almost the same in the case of the formal power series ring. If R is Laskerian, then $A \simeq R/XB[X]$ is Laskerian. Moreover R Laskerian implies that R satisfies accr which implies that B is Noetherian. (Otherwise, let $J_1 \subset \cdots \subset J_n \subset \cdots$ be a strictly increasing chain of ideals of B, and I the ideal of R consisting of all polynomials of the form $\sum_{i=0}^{n} a_i X^i$, where $a_i \in J_i$ for every $i = 1, \ldots, n$. Then we obtain the following strictly increasing chain $J : X \subset \cdots \subset J : X^n \subset \cdots$ in R, which is impossible as R satisfies accr).

Suppose now that A is an integral domain. Let J be an A-submodule of B. Then $X(J + XB[X])$ is an ideal of R which is Laskerian, so $\mathrm{Ass}(R/X(J + XB[X])$ is finite. We are going to characterize $\mathrm{Ass}(R/X(J + XB[X]))$.

$P \in \mathrm{Ass}(R/X(J + XB[X]))$ if and only if there exists $f \in R\backslash X(J + XB[X])$ such that P is minimal over $X(J + XB[X]) : f$. Let $f = a_0 + a_1 X + \cdots + a_n X^n \in R$. Then $f \in R\backslash X(J + XB[X]) \Leftrightarrow a_0 \neq 0$ or $a_1 \notin J$. Let $g = b_0 + b_1 X + \cdots + b_s X^s \in X(J + XB[X]) : f$, so $fg \in X(J + XB[X])$, which is equivalent to $a_0 b_0 = 0$ and $a_0 b_1 + a_1 b_0 \in J$. We have two cases:

First case: $a_0 \neq 0$, as A is an integral domain, then $b_0 = 0$ and $a_0 b_1 \in J$, so $g \in XB[X]$. But $X(J + XB[X]) \subset X(J + XB[X]) : f \subset XB[X]$, and p is a minimal prime ideal over $X(J + XB[X]) : f$, then P contain X and so $P = XB[X]$.

Second case: $a_0 = 0$, so $a_1 b_0 \in J$ with $a_1 \notin J$. Let $J : a_1 = \{a \in A \mid a a_1 \in J\}$. Then $X(J + XB[X]) : f = J : a_1 + XB[X]$. So P is minimal over $X(J + XB[X]) : f \Longrightarrow P = p + XB[X]$ with p minimal over $J : a_1$. So $p \in \mathrm{Ass}(B/J) = \{p \in \mathrm{spec}(A) \mid \exists x \in B\backslash J$ and p is minimal over $J : x\}$. Conversely, let $p \in \mathrm{Ass}(B/J)$. Then there exists $a_1 \in B\backslash J$ such that p is minimal over $J : a_1$. So $P = p + XB[X]$ is minimal over $X(J + XB[X]) : f$ where $f = a_1 X$. So $\mathrm{Ass}(R/X(J + XB[X])) = \{XB[X], p + XB[X]$ with $p \in \mathrm{Ass}(B/J)\}$. Then $\mathrm{Ass}(B/J)$ is finite.

Let $p \in \mathrm{Ass}(B/J)$, then $P = p + XB[X] \in \mathrm{Ass}(R/X(J + XB[X]))$. As R is Laskerian, there exists $f \in R\backslash P$ such that the ideal $X(J + XB[X]) : f$ is P-primary. Let $f = a_0 + a_1 X + \cdots$. If $a_0 \neq 0$, then $\sqrt{X(J + XB[X]) : f} = XB[X]$ in this case $p = 0$, that is there exists $x \in B\backslash J$ such that $J : x = 0$ and $J : x$ is p primary.

If $a_0 = 0$, then $a_1 \notin J$ in this case $\sqrt{X(J + XB[X]) : f} = \sqrt{J : a_1 + XB[X]} = \sqrt{J : a_1} + XB[X]$. Moreover, $J : a_1 + XB[X]$ is P-primary implies that $J : a_1$ is p-primary. So, we have $\mathrm{Ass}(B/J)$ is finite for every submodule J of B and $\forall p \in \mathrm{Ass}(B/J)$ there exists $x \in B\backslash J$ such that $J : x$ is P-primary. □

Example 5.5. The ring $\mathbb{Z}+X\mathbb{Q}[X]$ is not Laskerian because the $\mathbb{Z}$-module $\mathbb{Q}$ does not verify the condition $(*)$ of Proposition 5.3. For instance, let $J=\mathbb{Z}$, the $\mathbb{Z}$-submodule of $\mathbb{Q}$, and $x=\frac{1}{p}$ with p prime, $x\notin J$. We have $J:x=\{a\in\mathbb{Z}$ such that $\frac{a}{p}\in\mathbb{Z}\}=p\mathbb{Z}$. So $\{p\mathbb{Z},\ p$ prime integer$\}\subset \mathrm{Ass}(\mathbb{Q}/\mathbb{Z})$, so $\mathrm{Ass}(\mathbb{Q}/\mathbb{Z})$ is infinite, so $\mathbb{Z}+X\mathbb{Q}[X]$ is not Laskerian.

Remark 5.6. (1) In [1], the author shows that if $R=\bigoplus_{i=0}^{\infty}R_i$ is a graded ring with $R_0=K$, a field and if $k\subset K$ is a subfield of K and $A=\{r\in R\mid r_0\in k\}$, then R is Laskerian (resp. strongly Laskerian) if and only if A is Laskerian (resp. strongly Laskerian). In particular, as $K[X]$ is strongly Laskerian, then for any extension of fields $k\subset K$, the ring $k+XK[X]$ is strongly Laskerian.

So, the conditions B is a finitely generated A-module or B is a Noetherian A-module are not necessary to have $A+XB[X]$ Laskerian.

(2) If A is Laskerian and B is Noetherian then $\mathrm{spec}(A)$ and $\mathrm{spec}(B)$ are Noetherian so $\mathrm{spec}(A+XB[X])$ is also Noetherian. So for each ideal I of R, $\{\sqrt{I:f},\ f\in R\}$ satisfies the ascending chain condition. Then using [17, Lemma 3.2], we can deduce that each element of $\mathrm{Ass}(A+XB[X]/I)$ is of the form $\sqrt{I:f}$ for some $f\in R\backslash I$, that is each associated of I in the weak Bourbaki sense is an associated of I in the Zariski–Samuel sense.

We can prove directly that if we have an extension of fields $A\subset B$, then $A+XB[X]$ is Laskerian.

Proposition 5.7. *Let $A\subset B$ be an extension of integral domains. Then $A+XB[X]$ (resp. $A+XB[[X]]$) is of Krull dimension 1 if and only if $A\subset B$ is an extension of fields. In this case $A+XB[X]$ and $A+XB[[X]]$ are Laskerian.*

Proof. Recall that for a domain of Krull dimension 1, the following are equivalent: R Laskerian, R is a ZD ring and $\mathrm{spec}(R)$ is Noetherian. First, we prove the result in the polynomial case. If $A\subset B$ is an extension of integral domains, then $\mathrm{ht}_{A+XB[X]}(XB[X])\geq 1$ and $\max(\dim A+\mathrm{ht}_{A+XB[X]}(XB[X]),\dim B[X])\leq \dim(A+XB[X])\leq \dim A+\dim B[X]$. So we have $\dim(A+XB[X])=1\iff A\subset B$ is an extension of fields and in this case, using the paragraph 2, we have $\mathrm{spec}(A+XB[X])$ is Noetherian so $A+XB[X]$ is Laskerian.

For the case of power series, using [8, Theorem 11], we have the inequalities:

$$\begin{aligned}1+\max(\dim(B[[X]][X^{-1}]),\dim(A)+\lambda_{(A,B)})&\leq \dim(A+XB[[X]])\\ &\leq \dim(B[[X]][X^{-1}])+\dim(A).\end{aligned}$$

So $\dim(A+XB[[X]])=1$ if and only if $\dim A=\dim B=0$, so $A\subset B$ is an extension of fields. But in this case using [12, Corollary 1.6,], we can deduce that $\mathrm{spec}(A+XB[[X]])$ is Noetherian and then $A+XB[[X]]$ is Laskerian. □

Recall that a ring A is called a ZD ring (zero divisor ring) if for any ideal I of A the set $\{a\in A\mid$ there exists $s\in A\backslash I$ such that $as\in I\}$ is a union of finitely many prime ideals.

Proposition 5.8. *If $A + XB[X]$ is a ZD ring, then B is Noetherian and A is ZD.*

Proof. The proof is similar to [18]. As $A \simeq (A + XB[X])/XB[X]$, then A is ZD. We prove that B is Noetherian. Otherwise there exists an increasing chain of ideals of $B : (b_1) \subset (b_1, b_2) \subset \cdots$. Let $f_0 = X$, $f_1 = 1 + X, \dots, f_i = 1 + f_0 f_1 \cdots f_{i-1} \cdots$, and $\forall i \geq 1$, $g_i = Xf_i \in A + XB[X]$. Let I be the ideal of $A + XB[X]$ generated by the elements $(b_1 g_1, b_2 g_1 g_2, \dots, b_n g_1 \cdots g_n, \dots)$. We show that $\forall n \in \mathbb{N}$, $g_n \in \mathrm{Z}(A + XB[X]/I)$. We have $b_n g_1 \cdots g_{n-1} g_n \in I$ and $b_n g_1 \cdots g_{n-1} \notin I$. Otherwise the element $b_n g_1 \cdots g_{n-1} \in \langle b_1 g_1, b_2 g_1 g_2, \dots, b_n g_1 \cdots g_n, \dots \rangle$. Consider the ring $\overline{B} = B/\langle b_1, \dots, b_{n-1} \rangle$, then in $\overline{B}[X]$, we obtain $\overline{b_n}\,\overline{g_1} \cdots \overline{g_{n-1}} \in \langle \overline{b_n}\,\overline{g_1} \cdots \overline{g_n}, \dots \rangle$. As the element X^{n-1} is regular, then we obtain $\overline{b_n f_1 \cdots f_{n-1}} \in \langle \overline{b_n X f_1 \cdots f_n}, \dots \rangle$, but $\overline{f_1 \cdots f_{n-1}}$ is also regular so $\overline{b_n} \in \langle \overline{b_n X f_n}, \dots \rangle$. Then $\overline{b_n} = \overline{0}$ and $b_n \in \langle b_1, \dots, b_{n-1} \rangle$, which is impossible. □

Proposition 5.9. *Let $A \subset B$ be an extension of rings and B a finitely generated A-module. Then the following are equivalent:*

(1) *$A + XB[X]$ is ZD.*

(2) *$A + XB[X]$ is Laskerian.*

(3) *$A + XB[X]$ is strongly Laskerian.*

(4) *$A + XB[X]$ is Noetherian.*

Proof. If $A + XB[X]$ is Noetherian, then it is ZD. Conversely, if $A + XB[X]$ is ZD, then B is Noetherian and as B is a finitely generated A-module, we have by [10] that A is Noetherian so $A + XB[X]$ is Noetherian. □

Proposition 5.10. *If $A + XB[X]$ (resp. $A + XB[[X]]$) is Laskerian, then the A-module B satisfies accr.*

Proof. As $A + XB[X]$ is Laskerian, then it satisfies accr. Let J be an A-submodule of B and $a \in A$. Then $I = X(J + XB[X])$ is an ideal of $A + XB[X]$. The chain $I : a \subset \cdots \subset I : a^n \subset \cdots$ terminates, which implies that the chain $(J :_B a^n)$ terminates. In fact, let $n \in \mathbb{N}$ such that $I : a^n = I : a^{n+1}$. If $b \in J :_B a^{n+1}$, then $ba^{n+1} \in J$, so $(Xb)a^{n+1} \in I$, and $(Xb)a^n \in I$. Then $ba^n \in J$ that is $b \in J : a^n$. □

Proposition 5.11. *If $A \subset B$ is an extension of commutative rings such that B is an A-module satisfying accr, then $\mathrm{U}(B) \cap A = \mathrm{U}(A)$. So if B is a field then A is a field.*

Proof. Let B be an A-module satisfying accr. Let $a \in \mathrm{U}(B) \cap A$, the chain $A : a \subseteq A : a^2 \subseteq \cdots$ terminates, so there exists $n \in \mathbb{N}$ such that $A : a^n = A : a^{n+1}$. But $\frac{1}{a^{n+1}} \in B$ and $\frac{1}{a^{n+1}} a^{n+1} \in A$, so $\frac{1}{a^{n+1}} \in A : a^n$, and $\frac{1}{a} \in A$, so $a \in \mathrm{U}(A)$. □

Remark 5.12. If $A + XB[X]$ (resp. $A + XB[[X]]$) is Laskerian and $B = L$ is a field, then A is a field [3, Theorem 6].

Corollary 5.13. *If $A + XB[X]$ (resp. $A + XB[[X]]$) is Laskerian then* $\{m \cap A \mid m \in \mathrm{Max}(B)\} \subset \mathrm{Max}(A)$.

Example 5.14. If A is a valuation domain and B is an overring of A, then $A + XB[X]$ (resp. $A + XB[[X]]$) is Laskerian if and only if $B = A$ and A is a discrete valuation domain of rank 1. In fact, if $B = A$ and A is a discrete valuation domain of rank 1, then $A + XB[X] = A[X]$ is Noetherian so Laskerian. Conversely if $A + XB[X]$ is Laskerian and A is a valuation domain, then A is a valuation domain of rank 1 (the only valuation domains which are Laskerian), moreover if B is an overring of A, then $B = \mathrm{qf}(A)$ or $B = A$, but if $B = \mathrm{qf}(A)$ and as $A + XB[X]$ is Laskerian, then A must be a field which is not possible as A is of rank 1.

Lemma 5.15. *Let $P \in \mathrm{spec}(R)$ such that $X \notin P$ and Q the unique prime ideal of $B[X]$ (resp. $B[[X]]$) such that $Q \cap R = P$. Let I be an ideal of R. Then $P \in \mathrm{Ass}(R/I)$ if and only if $Q \in \mathrm{Ass}(B[X]/IB[X])$ (resp. $\mathrm{Ass}(B[[X]]/IB[[X]])$).*

Proof. Apply [17, Proposition 1.2]. □

Lemma 5.16. *Suppose that A is Laskerian and B is Noetherian. Let I be an ideal of $R = A + XB[X]$. Then the set of prime ideals P of R which are associated to R/I and do not contain X is finite and for such prime ideal P there exists $f \in R \backslash I$ such that $I : f$ is P-primary.*

Proof. Let P be a prime ideal of R associated to R/I and which does not contain X. Using the preceding lemma, $P \in \mathrm{Ass}(R/I)$ if and only if $Q \in \mathrm{Ass}(B[X]/IB[X])$. As B is Noetherian, then $B[X]$ is Laskerian. So $\mathrm{Ass}(B[X]/IB[X])$ is finite, and the number of such prime ideal P is finite. Now we prove that there exists $f \in R\backslash I$ such that $I : f$ is P-primary. In fact there exists $f \in B[X]\backslash IB[X]$ such that $IB[X]$ is Q-primary. We prove that $I : Xf$ is P-primary. We have $\sqrt{I : Xf} = P$. In fact, let $g \in I : Xf$. Then $Xfg \in I \subset IB[X]$, and $Xg \in IB[X] : f \subset Q$, but $X \notin Q$, so $g \in Q \cap R = P$, and $I : Xf \subset P$. Conversely, let $g \in P \subset Q = \sqrt{IB[X] : f}$. Then there exists $n \in \mathbb{N}$ such that $g^n \in IB[X] : f$, so $g^n f \in IB[X]$, and $g^n Xf \in I$ that is $g^n \in I : Xf$ and $g \in \sqrt{I : Xf}$. We prove that $I : Xf$ is primary. Let $g, h \in R$ such that $gh \in I : Xf$ and $g \notin P$. Then $Xgh \in IB[X] : f$ and $Xg \notin Q$, so $h \in IB[X] : f$, which implies that $hf \in IB[X]$ so $Xhf \in I$ and $h \in I : Xf$. □

Lemma 5.17. *If I is a radical ideal of R containing X, then $I = (I \cap A) + XB[X]$.*

Proof. We prove that if I is a radical ideal containing X, then it contains $XB[X]$. Let $f \in B[X]$. Then $(Xf)^2 = X(Xf^2) \in I$, so $Xf \in \sqrt{I} = I$. □

Proposition 5.18. *If $A = k$ is a field and B is Noetherian, then $k + XB[X]$ is Laskerian.*

Proof. We use a similar proof as [21]. In fact, let I be an ideal of R. We show that $\mathrm{Ass}(R/I)$ is finite and $\forall P \in \mathrm{Ass}(R/I)$, $\exists f \in R\backslash I$ such that $I : f$ is P-primary. Let $P \in \mathrm{Ass}(R/I)$. For the P which does not contain X this is done. If

$X \in P$ then $P = p + XB[X]$ with $p \in \text{spec}(A)$. As A is a field then $p = (0)$, so $P = XB[X] \in \max(R)$. We prove that there exists $g \in R$ such that $P = XB[X]$ is minimal over $I : f$. As $P = XB[X] \in \text{Ass}(R/I)$, there exists $f \in R \backslash I$ such that $XB[X]$ is minimal over $I : f$.

First case. $I : f$ is not contained in other prime ideal, so $\sqrt{I : f} = XB[X] \in \max(R)$. So $I : f$ is primary.

Second case. $I : f$ is contained in other prime ideals minimal over $I : f$. Let $P_1, \ldots, P_k$ the prime minimal over $I : f$ other than P (as $\text{Ass}(R/I)$ is finite, there is a finite number of such ideals). We have $\sqrt{I : f} = (\bigcap_{1 \le i \le k} P_i) \cap P$ and $X \in P \backslash (\bigcap_{1 \le i \le k} P_i)$. As $P_i \not\subseteq P$, there exists $f_i \in P_i \backslash P$. Let $z = f_1 \cdots f_n \notin P$. We show that there exists $n \in \mathbb{N}$ such that $I : z^n f$ is P-primary. We have $Xz \in \sqrt{I : f} = (\bigcap_{1 \le i \le k} P_i) \cap P$. So there exists $n \in \mathbb{N}$ such that $X^n z^n f \in I$, and $X^n \in I : z^n f$, so $X \in \sqrt{I : z^n f}$ which is a radical ideal of R, containing X, so it is of the form $J + XB[X]$ where J is an ideal of A which is a field, so $J = (0)$. Then $\sqrt{I : z^n f} = XB[X]$ and $I : z^n f$ is P-primary. □

Example 5.19. If $K \subset L$ is an extension of fields then $k + XL[X][Y_1, \ldots, Y_n]$ is Laskerian.

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On (n, d)-perfect rings

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Abstract. In this paper, we introduce the concept of (n, d)-perfect rings which is – in some way – a generalization of the notion of S-rings. We give some basic results on these rings and investigate the correlation between the $A(n)$ and (n, d)-perfect properties. We also study the (n, d)-perfect property in some pullback constructions.

Keywords. (n, d)-perfect ring, $A(n)$ ring, n-presented, homological dimension, pullback.

AMS classification. 13D02, 13D05,13D07, 13B02.

Dedicated to Alain Bouvier

1 Introduction

The object of this paper is to introduce a doubly filtered set of classes of rings which may serve to shed further light on the structures of non-Noetherian rings. Throughout this work, all rings are commutative with identity element and all modules are unitary. By a "local" ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal.

Let R be a ring and let M be an R-module. As usual, we use $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the usual projective and flat dimensions of M, respectively. The classical global and weak global dimension of R are denoted by $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$, respectively. If R is an integral domain, we denote its quotient field by $\mathrm{qf}(R)$.

An R-module M is n-presented if there is an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of R-modules in which each F_i is a free finitely generated R-module. In particular, 0-presented and 1-presented R-modules are respectively finitely generated and finitely presented R-modules. We recall that a coherent ring is a ring such that each finitely generated ideal is finitely presented. As in [2, 15], we set $\lambda_R(M) = \sup\{n \mid M$ is n-presented$\}$ except that we set $\lambda_R(M) = -1$ if M is not finitely generated. Note that $\lambda_R(M) \geq n$ is a way to express the fact that M is n-presented.

This paper introduces and studies a new class of rings; namely, (n, d)-perfect rings which generalize the notion of $A(n)$ ring introduced by Cox and Pendleton in [4]. The second section presents the definition of this new class of rings along with (mostly well-known) basic results. The third section establishes its relationship with the $A(n)$ property. The fourth section investigates the transfer of the (n, d)-perfect property in pullback constructions. General background material can be found in Rotman [14] and Glaz [10].

2 Definition and basic results

In this section we introduce and study the concept of (n,d)-perfect rings which are defined as follows.

Definition 2.1. Let n and d be nonnegative integers. A ring R is said to be an (n,d)-perfect ring, if every n-presented module with flat dimension at most d, has projective dimension at most d.

We illustrate this notion with the following example. First it is well known that if a flat R-module M is finitely presented, or finitely generated with R either a semilocal ring or an integral domain, then M is projective [7, Theorem 2]. A ring R is called an S-ring if every finitely generated flat R-module is projective [13].

Example 2.2. (1) R is an S-ring if and only if R is a $(0,0)$-perfect ring.

(2) If R is a semilocal ring, then R is an (n,n)-perfect ring for every $n \geq 0$.

(3) If R is a domain, then R is an (n,n)-perfect ring for every $n \geq 0$.

(4) If R is an (n,d)-perfect ring, then R is an (n',d)-perfect ring for every $n' \geq n$.

(5) For every $n > d$, R is an (n,d)-perfect ring.

(6) If R is a perfect ring, then R is (n,d)-perfect for every $n \geq 0$ an $d \geq 0$.

Proof. Obvious. □

The following proposition gives two results concerning Noetherian rings and coherent rings.

Proposition 2.3. (1) *If R is a Noetherian ring, then R is an (n,d)-perfect ring for every $n \geq 0$ and $d \geq 0$.*

(2) *If R is a coherent ring, then R is an (n,d)-perfect ring for every $n \geq 1$ and $d \geq 0$.*

Proof. Obvious. □

Furthermore, we construct an example of a ring which is a $(0,1)$-perfect ring and not a $(0,0)$-perfect ring (Example 2.4). Also we exhibit an example of a $(1,1)$-perfect ring which is not a $(0,1)$-perfect ring (Example 2.5).

Example 2.4. Let R be a hereditary and von Neumann regular ring which is not semisimple. Then R is a $(0,1)$-perfect ring which is not a $(0,0)$-perfect ring.

Proof. The ring R is a $(0,1)$-perfect ring since R is a hereditary ring. If R is a $(0,0)$-perfect ring, then every finitely generated R-module is projective (since R is a von Neumann regular ring), hence we have a contradiction with the fact that R is not semisimple. □

Example 2.5. Let R be a non-Noetherian Prüfer domain. Then R is a $(1,1)$-perfect domain which is not a $(0,1)$-perfect domain.

Proof. The ring R is a $(1,1)$-perfect ring since R is a domain. On the other hand, we show that R is not a $(0,1)$-perfect ring. Let I be a not finitely generated ideal of R (since R is not Noetherian). Then I is not projective. Of course I is flat since $\mathrm{wdim}(R) \leq 1$. Thus R/I is 0-presented with $\mathrm{fd}_R(R/I) \leq 1$ and $\mathrm{pd}_R(R/I) \geq 2$, as desired. □

Next we give a homological characterization of an (n,d)-perfect ring.

Theorem 2.6. *Let R be a commutative ring. Then the following statements are equivalent:*

(1) *R is an (n,d)-perfect ring.*

(2) $\mathrm{Ext}_R^{d+1}(M,N) = 0$ *for all R-modules M, N such that $\lambda_R(M) \geq n$,* $\mathrm{fd}_R(M) \leq d$ *and* $\mathrm{fd}_R(N) \leq d$.

(3) $\mathrm{Ext}_R^{d+1}(M,N) = 0$ *for all R-modules M, N such that $\lambda_R(M) \geq n$, $\lambda_R(N) \geq n-(d+1)$,* $\mathrm{fd}_R(M) \leq d$ *and* $\mathrm{fd}_R(N) \leq d$.

The proof of this theorem involves the following lemmas.

Lemma 2.7. *Let R be a ring, and let M be an n-presented flat R-module, where $n \geq 0$. Then M is projective if and only if* $\mathrm{Ext}_R^1(M,N) = 0$ *for all R-modules N such that $\lambda_R(N) \geq n-1$ and N is a flat R-module.*

Proof. Necessity is clear. To prove sufficiency, let $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ be an exact sequence with F a finitely generated free R-module. Then K is an $(n-1)$-presented flat R-module, hence by hypothesis $\mathrm{Ext}_R^1(M,K) = 0$. It follows that the exact sequence splits, making M a direct summand of F. Therefore M is a projective R-module. □

Lemma 2.8. *Let R be a ring and M an n-presented R-module such that* $\mathrm{fd}_R(M) \leq d$. *Then* $\mathrm{pd}_R(M) \leq d$ *if and only if* $\mathrm{Ext}_R^{d+1}(M,N) = 0$ *for all R-modules N such that $\lambda_R(N) \geq n-(d+1)$ and* $\mathrm{fd}_R(N) \leq d$.

Proof. This follows from Lemma 2.7 by dimension shifting. □

Proof of Theorem 2.6. (1) ⇒ (2). Let M be an R-module such that $\lambda_R(M) \geq n$ and $\mathrm{fd}_R(M) \leq d$. Then, $\mathrm{pd}_R(M) \leq d$ since R is an (n,d)-perfect ring. Therefore, $\mathrm{Ext}_R^{d+1}(M,N) = 0$ for any R-module N.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). Let M be an R-module such that $\lambda_R(M) \geq n$ and $\mathrm{fd}_R(M) \leq d$. Hence by (3), $\mathrm{Ext}_R^{d+1}(M,N) = 0$ for any R-module N such that $\lambda_R(N) \geq n-(d+1)$ and $\mathrm{fd}_R(N) \leq d$. Then, by Lemma 2.8, $\mathrm{pd}_R(M) \leq d$. Therefore, R is an (n,d)-perfect ring. □

Next we prove that the (n,d)-perfect property descends into a faithfully flat ring homomorphism.

Theorem 2.9. *Let $R \longrightarrow S$ be a ring homomorphism making S a faithfully flat R-module. If S is an (n,d)-perfect ring, then R is an (n,d)-perfect ring.*

Proof. Let M be an n-presented R-module with $\mathrm{fd}_R(M) \leq d$. Our aim is to show that $\mathrm{pd}_R(M) \leq d$. We have $\lambda_S(M \otimes_R S) \geq n$ and $\mathrm{fd}_S(M \otimes_R S) \leq d$ since S is a flat R-module, so $\mathrm{pd}_S(M \otimes_R S) \leq d$ since S is an (n,d)-perfect ring.

Let $0 \longrightarrow P \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be an exact sequence of R-modules, where F_i is a free R-module for each i and P is a flat R-module. Thus $P \otimes_R S$ is a projective S-module. By [11, Example 3.1.4, page 82], P is a projective R-module. □

We use this result to study the (n,d)-perfect property in some particular rings.

Corollary 2.10. (1) *Let $A \subseteq B$ be two rings such that B is a flat A-module. Let $S = A + XB[X]$, where X is an indeterminate over B. If S is an (n,d)-perfect ring, then so is A.*

(2) *Let R be a ring and X an indeterminate over R. If $R[X]$ is an (n,d)-perfect ring, then so is R.*

Proof. (1) The ring B is a flat A-module and $XB[X] \cong B[X]$ thus $S = A + XB[X]$ is a faithfully flat A-module. By Theorem 2.9 the ring A is (n,d)-perfect since S is an (n,d)-perfect ring.

(2) Obvious via (1). □

We close this section by establishing the transfer of the (n,d)-perfect property to finite direct products.

Theorem 2.11. *Let $(R_i)_{i=1,\ldots,m}$ be a family of rings. Then $\prod_{i=1}^{m} R_i$ is an (n,d)-perfect ring if and only if R_i is an (n,d)-perfect ring for each $i = 1,\ldots,m$.*

The proof of this theorem involves the following results.

Lemma 2.12 ([12, Lemma 2.5]). *Let $(R_i)_{i=1,2}$ be a family of rings and E_i be an R_i-module for $i = 1,2$. We have*

(1) $\mathrm{pd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\mathrm{pd}_{R_1}(E_1), \mathrm{pd}_{R_2}(E_2)\}$.

(2) $\lambda_{R_1 \times R_2}(E_1 \times E_2) = \inf\{\lambda_{R_1}(E_1), \lambda_{R_2}(E_2)\}$.

Lemma 2.13. *Let $(R_i)_{i=1,2}$ be a family of rings and E_i be an R_i-module for $i = 1,2$. We have* $\mathrm{fd}_{R_1 \times R_2}(E_1 \times E_2) = \sup\{\mathrm{fd}_{R_1}(E_1), \mathrm{fd}_{R_2}(E_2)\}$.

Proof. This proof is analogous to the proof of Lemma 2.12 (1). □

Proof of Theorem 2.11. We use induction on m, it suffices to prove the assertion for $m = 2$. Let R_1 and R_2 be two rings such that $R_1 \times R_2$ is an (n,d)-perfect ring. Let E_1 be an R_1-module such that $\mathrm{fd}_{R_1}(E_1) \leq d$, $\lambda_{R_1}(E_1) \geq n$ and let E_2 be an R_2-module such that $\mathrm{fd}_{R_2}(E_2) \leq d$, $\lambda_{R_2}(E_2) \geq n$. By Lemma 2.13, $\mathrm{fd}_{R_1 \times R_2}(E_1 \times$

$E_2) = \sup\{\mathrm{fd}_{R_1}(E_1), \mathrm{fd}_{R_2}(E_2)\}$. So $\lambda_{R_1\times R_2}(E_1 \times E_2) = \inf\{\lambda_{R_1}(E_1), \lambda_{R_2}(E_2)\}$ by Lemma 2.12 (2). Thus $\lambda_{R_1\times R_2}(E_1 \times E_2) \geq n$ and $\mathrm{fd}_{R_1\times R_2}(E_1 \times E_2) \leq d$. So $\mathrm{pd}_{R_1\times R_2}(E_1 \times E_2) \leq d$ since $R_1 \times R_2$ is an (n,d)-perfect ring. By Lemma 2.12 (1) $\mathrm{pd}_{R_1\times R_2}(E_1 \times E_2) = \sup\{\mathrm{pd}_{R_1}(E_1), \mathrm{pd}_{R_2}(E_2)\}$. Thus $\mathrm{pd}_{R_1}(E_1) \leq n$ and $\mathrm{pd}_{R_2}(E_2) \leq n$. Therefore R_1 is an (n,d)-perfect ring and R_2 is an (n,d)-perfect ring.

Conversely, let R_1 and R_2 be two (n,d)-perfect rings and let $E_1\times E_2$ be an $R_1\times R_2$-module where E_i is an R_i-module for each $i = 1,2$, such that $\mathrm{fd}_{R_1\times R_2}(E_1 \times E_2) \leq d$ and $\lambda_{R_1\times R_2}(E_1 \times E_2) \geq n$. By Lemma 2.12, $\lambda_{R_1}(E_1) \geq n$, $\lambda_{R_2}(E_2) \geq n$ and by Lemma 2.13, $\mathrm{fd}_{R_1}(E_1) \leq d$, $\mathrm{fd}_{R_2}(E_2) \leq d$, then $\mathrm{pd}_{R_1}(E_1) \leq d$ and $\mathrm{pd}_{R_2}(E_2) \leq d$, since R_1 and R_2 are (n,d)-perfect rings. By Lemma 2.12, $\mathrm{pd}_{R_1\times R_2}(E_1 \times E_2) \leq d$. Therefore $R_1 \times R_2$ is an (n,d)-perfect ring. □

3 Relationship between the $A(n)$ and (n,d)-perfect properties

The purpose of the present section is to investigate the correlation between $A(n)$ rings and (n,d)-perfect rings. First, we recall the definition of an $A(n)$ ring introduced in [4].

Definition 3.1 ([4, page 139]). Let n be a nonnegative integer. A ring R is said to be an $A(n)$ ring if given any exact sequence $0 \longrightarrow M \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$ of finitely generated R-modules with M flat and E_i free for each i, then M is projective.

Next we state the main theorem of this section.

Theorem 3.2. *A ring R is an $A(n)$ ring if and only if R is an (n,n)-perfect ring.*

Proof. Assume that R is an $A(n)$ ring and let M be an R-module such that $\lambda_R(M) \geq n$ and $\mathrm{fd}_R(M) \leq n$. Then there exists an exact sequence $0 \longrightarrow P \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ of finitely generated R-modules with P flat and F_i free for each i. So P is projective. Therefore R is an (n,n)-perfect ring.

Conversely, assume that R is an (n,n)-perfect ring. Let $0 \longrightarrow M \longrightarrow F_1 \longrightarrow \cdots \xrightarrow{u_n} F_n$ be an exact sequence of finitely generated R-modules with M flat and F_i free for each i. We show that M is projective. The exact sequence

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow \cdots \xrightarrow{u_n} F_n \longrightarrow \operatorname{Coker} u_n \longrightarrow 0$$

shows that $\lambda_R(\operatorname{Coker} u_n) \geq n$ and $\mathrm{fd}_R(\operatorname{Coker} u_n) \leq n$. Hence, $\mathrm{pd}_R(\operatorname{Coker} u_n) \leq n$ since R is an (n,n)-perfect ring and so M is projective. Therefore R is an $A(n)$ ring. □

Theorem 3.2 combined with the results obtained by Cox and Pendleton in [4] may be used to state several corollaries.

Corollary 3.3. *Let $\varphi : R \hookrightarrow T$ be an injective ring homomorphism.*

(1) *If T is a $(0,0)$-perfect ring, then so is R.*

(2) *If T is an (n,n)-perfect ring ($n \geq 1$) and T is a flat R-module, then R is an (n,n)-perfect ring.*

Proof. By [4, Theorem 2.4] and Theorem 3.2. □

Corollary 3.4. *A ring R is a $(1,1)$-perfect ring if and only if each pure ideal of R which is the annihilator of a finitely generated ideal of R is generated by an idempotent.*

Proof. By [4, Theorem 3.8] and Theorem 3.2. □

The next example gives a $(1,1)$-perfect ring R and a multiplicative set S of R such that $S^{-1}R$ is not a $(1,1)$-perfect ring.

Example 3.5 ([4, Example 5.17]). Let $R = \mathbb{Z}[f, x_1, x_2, \dots]$, with defining relations $fx_i(1-x_j) = 0$, $1 \leq i < j$, and $2fx_i = 0$, $1 \leq i$. Put $S = \{f^n \mid n \geq 1\}$. Then R is a $(1,1)$-perfect ring, but $S^{-1}R$ is not a $(1,1)$-perfect ring.

4 Transfer of the (n,d)-perfect property in pullbacks

Pullbacks occupy an important niche in homological algebra because they produce interesting examples (see for instance [10, Section 1, Chapter 5], [8, Appendix 2] and [9, pages 582–584]).

The following theorem is the main result of this section.

Theorem 4.1. *Let $A \hookrightarrow B$ be an injective flat ring homomorphism and let Q be a pure ideal of A such that $QB = Q$ and $\lambda_A(Q) \geq n-1$.*

(1) *Assume that B is an (n,d)-perfect ring. Then A/Q is an (n,d)-perfect ring if and only if A is an (n,d)-perfect ring.*

(2) *Assume that $B = S^{-1}A$, where S is a multiplicative set of A. Then A is an (n,d)-perfect ring if and only if B and A/Q are (n,d)-perfect rings.*

Before proving this theorem, we establish the following lemmas.

At the start, we recall the notion of flat epimorphism of rings, which is defined as follows: Let $\Phi : A \to B$ be a ring homomorphism. B (or Φ) is called a flat epimorphism of A, if B is a flat A-module and Φ is an epimorphism, that is, for any two ring homomorphisms $B \overset{f}{\underset{g}{\rightrightarrows}} C$, satisfying $f \circ \Phi = g \circ \Phi$, we have $f = g$; see [10, pages 13–14]. For example $S^{-1}A$ is a flat epimorphism of A for every multiplicative set S of A. Also, the quotient ring A/I is a flat epimorphism of A for every pure ideal I of A, that is, A/I is a flat A-module [10, Theorem 1.2.15].

Lemma 4.2. *Let A and B be two rings such that $\Phi : A \to B$ is a flat epimorphism of A and $\lambda_A(B) \geq n$. If A is an (n,d)-perfect ring, then B is an (n,d)-perfect ring. In particular, if A is an (n,d)-perfect ring, then so is the quotient ring A/I for every pure ideal I of A such that $\lambda_A(I) \geq n-1$ ($n \geq 0$).*

Proof. Let M be a B-module such that $\lambda_B(M) \geq n$ and $\mathrm{fd}_B(M) \leq d$. Our aim is to show that $\mathrm{pd}_B(M) \leq d$. By hypothesis we have $\mathrm{Tor}_k^A(M, B) = 0$ for all $k > 0$. By [3, Proposition 4.1.3], we have for any B-module N

$$\mathrm{Ext}_A^{d+1}(M, N \otimes_A B) \cong \mathrm{Ext}_B^{d+1}(M \otimes_A B, N \otimes_A B). \qquad (*)$$

From [10, Theorem 1.2.19] we get $M \otimes_A B \cong M$ and $N \otimes_A B \cong N$. On the other hand, we have $\mathrm{fd}_A(M) \leq \mathrm{fd}_B(M) \leq d$ [3, Exercise 10, p. 123] and $\lambda_A(M) \geq n$ [5, Lemma 2.6]. Thus $\mathrm{pd}_A(M) \leq d$ since A is an (n, d)-perfect ring. Hence, $(*)$ implies $\mathrm{Ext}_B^{d+1}(M, N) = 0$, therefore $\mathrm{pd}_B(M) \leq d$. □

The Example 3.5 shows that Lemma 4.2 is not true in general without assuming that $\lambda_A(B) \geq n$.

Lemma 4.3. *Let $A \hookrightarrow B$ be an injective flat ring homomorphism and let Q be a pure ideal of A such that $QB = Q$. Let E be an A-module. Then:*

(1) $\lambda_A(E) \geq n \Leftrightarrow \lambda_B(E \otimes_A B) \geq n$ *and* $\lambda_{A/Q}(E \otimes_A A/Q) \geq n$.

(2) $\mathrm{fd}_A(E) \leq d \Leftrightarrow \mathrm{fd}_B(E \otimes_A B) \leq d$ *and* $\mathrm{fd}_{A/Q}(E \otimes_A A/Q) \leq d$.

(3) $\mathrm{pd}_A(E) \leq d \Leftrightarrow \mathrm{pd}_B(E \otimes_A B) \leq d$ *and* $\mathrm{pd}_{A/Q}(E \otimes_A A/Q) \leq d$.

Proof. Similar to the proof of [6, Lemma 2.4]. □

Proof of Theorem 4.1. (1) If A is an (n, d)-perfect ring since Q is an $(n-1)$-presented pure ideal of A, by Lemma 4.2 A/Q is an (n, d)-perfect ring. Conversely, assume that B and A/Q are (n, d)-perfect rings. Let M be an A-module such that $\lambda_A(M) \geq n$ and $\mathrm{fd}_A(M) \leq d$. Then $\lambda_B(M \otimes_A B) \geq n$ and $\mathrm{fd}_B(M \otimes_A B) \leq d$. So $\mathrm{pd}_B(M \otimes_A B) \leq d$ since B is an (n, d)-perfect ring. Also $\lambda_{A/Q}(M \otimes_A A/Q) \geq n$ and $\mathrm{fd}_{A/Q}(M \otimes_A A/Q) \leq d$. So $\mathrm{pd}_{A/Q}(M \otimes_A A/Q) \leq d$ since A/Q is an (n, d)-perfect rings. By Lemma 4.3. $\mathrm{pd}_A(M) \leq n$. Therefore A is an (n, d)-perfect ring.

(2) Follows from Lemma 4.2, Lemma 4.3 and (1). □

Corollary 4.4. *Let D be an integral domain, $K = \mathrm{qf}(D)$ and let $n \geq 2$, $m \geq 0$ and $d \geq 0$ be positive integers. Consider the quotient ring $S = K[X]/(X^n - X) = K + \overline{X}K[\overline{X}] = K + I$ with $I = \overline{X}K[\overline{X}]$. Set $R = D + I$. Then R is an (m, d)-perfect ring if and only if D is an (m, d)-perfect ring.*

Proof. First we show that I is a pure ideal of R. Let $u := \overline{X}^i(a_0 + a_1\overline{X} + \cdots + a_{n-1}\overline{X}^{n-1})$ be an element of I, where $a_i \in K$ for $1 \leq i \leq n-1$, and $a_0 \neq 0$. Hence $u(1-\overline{X}^{n-1}) = 0$ $(*)$ since $\overline{X}^i(1-\overline{X}^{n-1}) = \overline{X}^i - \overline{X}^{n+(i-1)} = \overline{X}^i - \overline{X}^n\overline{X}^{i-1} = \overline{X}^i - \overline{X}^i = \overline{0}$. Therefore, I is a pure ideal of R by [10, Theorem 1.2.15] since $\overline{X}^{n-1} \in I$. Our aim is to show that $\lambda_R(I) = \infty$. We have $R\overline{X}^{n-1} = (D + \overline{X}K[\overline{X}])\overline{X}^{n-1} = D\overline{X}^{n-1} + \overline{X}K[\overline{X}] = D\overline{X}^{n-1} + I = I$ since $\overline{X}^n = \overline{X}$. We claim that $\mathrm{Ann}_R(I) = R(1-\overline{X}^{n-1})$. Indeed, by $(*)$ $R(1-\overline{X}^{n-1}) \subseteq \mathrm{Ann}_R(I)$. Conversely, let $v := d + a_1\overline{X} + \cdots + a_{n-1}\overline{X}^{n-1} \in \mathrm{Ann}_R(I)$, where $d \in D$ and $a_i \in K$. Hence $0 = (d + a_1\overline{X} + \cdots + a_{n-1}\overline{X}^{n-1})\overline{X}^{n-1} = a_1\overline{X} + \cdots + (d + a_{n-1})\overline{X}^{n-1}$ and so $a_1 = a_2 = \cdots = a_{n-2} = 0$

and $d + a_{n-1} = 0$. This means, $v = d(1-\overline{X}^{n-1}) \in R(1-\overline{X}^{n-1})$ as desired. Also, the same proof as above shows that $\mathrm{Ann}_R(R(1-\overline{X}^{n-1})) = I$. Therefore, $\lambda_R(I) = \infty$. On the other hand, S is an Artinian ring and hence a perfect ring [1, Corollary 28.8]. So S is an (m, d)-perfect ring. We conclude via Theorem 4.1. □

From this corollary we deduce easily the following example.

Example 4.5. Let D be an integral domain such that $\mathrm{gldim}(D) = d$. Let $K = \mathrm{qf}(D)$ and $n \geq 2$. Consider the quotient ring $S = K[X]/(X^n - X) = K + \overline{X}K[\overline{X}] = K + I$ with $I = \overline{X}K[\overline{X}]$. Set $R = D + I$. Then R is a $(1, d)$-perfect ring.

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t-class semigroups of Noetherian domains

S. Kabbaj and A. Mimouni

Abstract. The t-class semigroup of an integral domain R, denoted $\mathcal{S}_t(R)$, is the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals with the operation induced by ideal t-multiplication. This paper investigates ring-theoretic properties of a Noetherian domain that reflect reciprocally in the Clifford or Boolean property of its t-class semigroup.

Keywords. Class semigroup, t-class semigroup, t-ideal, t-closure, Clifford semigroup, Clifford t-regular, Boole t-regular, t-stable domain, Noetherian domain, strong Mori domain.

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1 Introduction

Let R be an integral domain. The class semigroup of R, denoted $\mathcal{S}(R)$, is the semigroup of nonzero fractional ideals modulo its subsemigroup of nonzero principal ideals [3], [19]. We define the t-class semigroup of R, denoted $\mathcal{S}_t(R)$, to be the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals, that is, the semigroup of the isomorphy classes of the t-ideals of R with the operation induced by t-multiplication. Notice that $\mathcal{S}_t(R)$ stands as the t-analogue of $\mathcal{S}(R)$, whereas the class group $\mathrm{Cl}(R)$ is the t-analogue of the Picard group $\mathrm{Pic}(R)$. In general, we have

$$\mathrm{Pic}(R) \subseteq \mathrm{Cl}(R) \subseteq \mathcal{S}_t(R) \subseteq \mathcal{S}(R)$$

where the first and third containments turn into equality if R is a Prüfer domain and the second does so if R is a Krull domain.

A commutative semigroup S is said to be Clifford if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x = ax^2$. A Clifford semigroup S has the ability to stand as a disjoint union of subgroups G_e, where e ranges over the set of idempotent elements of S, and G_e is the largest subgroup of S with identity equal to e (cf. [7]). The semigroup S is said to be Boolean if for each $x \in S$, $x = x^2$. A domain R is said to be *Clifford* (resp., *Boole*) *t-regular* if $S_t(R)$ is a Clifford (resp., Boolean) semigroup.

This paper investigates the t-class semigroups of Noetherian domains. Precisely, we study conditions under which t-stability characterizes t-regularity. Our first result, Theorem 2.2, compares Clifford t-regularity to various forms of stability. Unlike regularity, Clifford (or even Boole) t-regularity over Noetherian domains does not force the

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t-dimension to be one (Example 2.4). However, Noetherian strong t-stable domains happen to have t-dimension 1. Indeed, the main result, Theorem 2.6, asserts that "*R is strongly t-stable if and only if R is Boole t-regular and* t-dim$(R) = 1$." This result is not valid for Clifford t-regularity as shown by Example 2.9. We however extend this result to the Noetherian-like larger class of strong Mori domains (Theorem 2.10).

All rings considered in this paper are integral domains. Throughout, we shall use qf(R) to denote the quotient field of a domain R, $\overline{I}$ to denote the isomorphy class of a t-ideal I of R in $S_t(R)$, and $\mathrm{Max}_t(R)$ to denote the set of maximal t-ideals of R.

2 Main results

We recall that for a nonzero fractional ideal I of R, $I_v := (I^{-1})^{-1}$, $I_t := \bigcup J_v$ where J ranges over the set of finitely generated subideals of I, and $I_w := \bigcup(I : J)$ where the union is taken over all finitely generated ideals J of R with $J^{-1} = R$. The ideal I is said to be divisorial or a v-ideal if $I = I_v$, a t-ideal if $I = I_t$, and a w-ideal if $I = I_w$. A domain R is called *strong Mori* if R satisfies the ascending chain condition on w-ideals [5]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. Suitable background on strong Mori domains is [5]. Finally, recall that the *t-dimension* of R, abbreviated t-dim(R), is by definition equal to the length of the longest chain of t-prime ideals of R.

The following lemma displays necessary and sufficient conditions for t-regularity. We often will be appealing to this lemma without explicit mention.

Lemma 2.1 ([9, Lemma 2.1]). *Let R be a domain. We have*

(1) *R is Clifford t-regular if and only if, for each t-ideal I of R,* $I = (I^2(I : I^2))_t$.

(2) *R is Boole t-regular if and only if, for each t-ideal I of R,* $I = c(I^2)_t$ *for some* $c \neq 0 \in \mathrm{qf}(R)$. □

An ideal I of a domain R is said to be *L-stable* (here L stands for Lipman) if $R^I := \bigcup_{n\geq 1}(I^n : I^n) = (I : I)$, and R is called L-stable if every nonzero ideal is L-stable. Lipman introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings in order to give a characterization of Arf rings; in this context, L-stability coincides with Boole regularity [12].

Next, we state our first theorem of this section.

Theorem 2.2. *Let R be a Noetherian domain and consider the following statements:*

(1) *R is Clifford t-regular.*

(2) *Each t-ideal I of R is t-invertible in* $(I : I)$.

(3) *Each t-ideal is L-stable.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *Moreover, if* t-dim$(R) = 1$, *then* (3) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2). Let I be a t-ideal of a domain A. Then for each ideal J of A, $(I : J) = (I : J_t)$. Indeed, since $J \subseteq J_t$, then $(I : J_t) \subseteq (I : J)$. Conversely, let $x \in (I : J)$. Then $xJ \subseteq I$ implies that $xJ_t = (xJ)_t \subseteq I_t = I$, as claimed. So $x \in (I : J_t)$ and therefore $(I : J) \subseteq (I : J_t)$. Now, let I be a t-ideal of R, $B = (I : I)$ and $J = I(B : I)$. Since $\overline{I}$ is regular in $\mathcal{S}_t(R)$, then $I = (I^2(I : I^2))_t = (IJ)_t$. By the claim, $B = (I : I) = (I : (IJ)_t) = (I : IJ) = ((I : I) : J) = (B : J)$. Since B is Noetherian, then $(I(B : I))_{t_1} = J_{t_1} = J_{v_1} = B$, where t_1- and v_1 denote the t- and v-operations with respect to B. Hence I is t-invertible as an ideal of $(I : I)$.

(2) $\Longrightarrow$ (3). Let $n \geq 1$, and $x \in (I^n : I^n)$. Then $xI^n \subseteq I^n$ implies that $xI^n(B : I) \subseteq I^n(B : I)$. So $x(I^{n-1})_{t_1} = x(I^n(B : I))_{t_1} \subseteq (I^n(B : I))_{t_1} = (I^{n-1})_{t_1}$. Now, we iterate this process by composing the two sides by $(B : I)$, applying the t-operation with respect to B and using the fact that I is t-invertible in B, we obtain that $x \in (I : I)$. Hence I is L-stable.

(3) $\Longrightarrow$ (1). Assume that $t\text{-dim}(R) = 1$. Let I be a t-ideal of R and $J = (I^2(I : I^2))_t = (I^2(I : I^2))_v$ (since R is Noetherian, and so a *TV*-domain). We wish to show that $I = J$. By [10, Proposition 2.8.(3)], it suffices to show that $IR_M = JR_M$ for each t-maximal ideal M of R. Let M be a t-maximal ideal of R. If $I \nsubseteq M$, then $J \nsubseteq M$. So $IR_M = JR_M = R_M$. Assume that $I \subseteq M$. Since $t\text{-dim}(R) = 1$, then $\dim(R)_M = 1$. Since IR_M is L-stable, then by [12, Lemma 1.11] there exists a nonzero element x of R_M such that $I^2R_M = xIR_M$. Hence $(IR_M : I^2R_M) = (IR_M : xIR_M) = x^{-1}(IR_M : IR_M)$. So $I^2R_M(IR_M : I^2R_M) = xIR_Mx^{-1}(IR_M : IR_M) = IR_M$. Now, by [10, Lemma 5.11], $JR_M = ((I^2(I : I^2))_v)R_M = (I^2(I : I^2))R_M)_v = (I^2R_M(IR_M : I^2R_M))_v = (IR_M)_v = I_vR_M = I_tR_M = IR_M$. □

According to [2, Theorem 2.1] or [8, Corollary 4.3], a Noetherian domain R is Clifford regular if and only if R is stable if and only if R is L-stable and $\dim(R) = 1$. Unlike Clifford regularity, Clifford (or even Boole) t-regularity does not force a Noetherian domain R to be of t-dimension one. In order to illustrate this fact, we first establish the transfer of Boole t-regularity to pullbacks issued from local Noetherian domains.

Proposition 2.3. *Let (T, M) be a local Noetherian domain with residue field K and $\phi : T \longrightarrow K$ the canonical surjection. Let k be a proper subfield of K and $R := \phi^{-1}(k)$ the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Then R is Boole t-regular if and only if so is T.

Proof. By [4, Theorem 4] (or [6, Theorem 4.12]) R is a Noetherian local domain with maximal ideal M. Assume that R is Boole t-regular. Let J be a t-ideal of T. If $J(T : J) = T$, then $J = aT$ for some $a \in J$ (since T is local). Then $J^2 = aJ$ and so $(J^2)_{t_1} = aJ$, where t_1 is the t-operation with respect to T (note that $t_1 = v_1$ since T

is Noetherian), as desired. Assume that $J(T:J) \subsetneq T$. Since T is local with maximal ideal M, then $J(T:J) \subseteq M$. Hence $J^{-1} = (R:J) \subseteq (T:J) \subseteq (M:J) \subseteq J^{-1}$ and therefore $J^{-1} = (T:J)$. So $(T:J^2) = ((T:J):J) = ((R:J):J) = (R:J^2)$. Now, since R is Boole t-regular, then there exists $0 \neq c \in \mathrm{qf}(R)$ such that $(J^2)_t = ((J_t)^2)_t = cJ_t$. Then $(T:J^2) = (R:J^2) = (R:(J^2)_t) = (R:cJ_t) = c^{-1}(R:J_t) = c^{-1}(R:J) = c^{-1}(T:J)$. Hence $(J^2)_{t_1} = (J^2)_{v_1} = cJ_{v_1} = cJ_{t_1} = cJ$, as desired. It follows that T is Boole t-regular.

Conversely, assume that T is Boole t-regular and let I be a t-ideal of R. If $II^{-1} = R$, then $I = aR$ for some $a \in I$. So $I^2 = aI$, as desired. Assume that $II^{-1} \subsetneq R$. Then $II^{-1} \subseteq M$. So $T \subseteq (M:M) = M^{-1} \subseteq (II^{-1})^{-1} = (I_v : I_v) = (I:I)$. Hence I is an ideal of T. If $I(T:I) = T$, then $I = aT$ for some $a \in I$ and so $I^2 = aI$, as desired. Assume that $I(T:I) \subsetneq T$. Then $I(T:I) \subseteq M$, and so $I^{-1} \subseteq (T:I) \subseteq (M:I) \subseteq I^{-1}$. Hence $I^{-1} = (T:I)$. So $(T:I^2) = ((T:I):I) = ((R:I):I) = (R:I^2)$. But since T is Boole t-regular, then there exists $0 \neq c \in qf(T) = \mathrm{qf}(R)$ such that $(I^2)_{t_1} = ((I_{t_1})^2)_{t_1} = cI_{t_1}$. Then $(R:I^2) = (T:I^2) = (T:(I^2)_{t_1}) = (T:cI_{t_1}) = c^{-1}(T:I_{t_1}) = c^{-1}(T:I) = c^{-1}(R:I)$. Hence $(I^2)_t = (I^2)_v = cI_v = cI_t = cI$, as desired. It follows that R is Boole t-regular. □

Now we are able to build an example of a Boole t-regular Noetherian domain with t-dimension $\gneqq 1$.

Example 2.4. Let K be a field, X and Y two indeterminates over K, and k a proper subfield of K. Let $T := K[[X,Y]] = K + M$ and $R := k + M$ where $M := (X,Y)$. Since T is a UFD, then T is Boole t-regular [9, Proposition 2.2]. Further, R is a Boole t-regular Noetherian domain by Proposition 2.3. Now M is a v-ideal of R, so that t-$\dim(R) = \dim(R) = 2$.

Recall that an ideal I of a domain R is said to be *stable* (resp., *strongly stable*) if I is invertible (resp., principal) in its endomorphism ring $(I:I)$, and R is called a stable (resp., strongly stable) domain provided each nonzero ideal of R is stable (resp., strongly stable). Sally and Vasconcelos [17] used this concept to settle Bass' conjecture on one-dimensional Noetherian rings with finite integral closure. Recall that a stable domain is L-stable [1, Lemma 2.1]. For recent developments on stability, we refer the reader to [1] and [14, 15, 16]. By analogy, we define the following concepts:

Definition 2.5. A domain R is *t-stable* if each t-ideal of R is stable, and R is *strongly t-stable* if each t-ideal of R is strongly stable.

Strong t-stability is a natural stability condition that best suits Boolean t-regularity. Our next theorem is a satisfactory t-analogue for Boolean regularity [8, Theorem 4.2].

Theorem 2.6. *Let R be a Noetherian domain. The following conditions are equivalent:*

(1) *R is strongly t-stable;*

(2) *R is Boole t-regular and* t-$\dim(R) = 1$.

The proof relies on the following lemmas.

Lemma 2.7. *Let R be a t-stable Noetherian domain. Then t-dim$(R) = 1$.*

Proof. Assume t-dim$(R) \geq 2$. Let $(0) \subset P_1 \subset P_2$ be a chain of t-prime ideals of R and $T := (P_2 : P_2)$. Since R is Noetherian, then so is T (as $(R : T) \neq 0$) and $T \subseteq \overline{R} = R'$, where $\overline{R}$ and R' denote respectively the complete integral closure and the integral closure of R. Let Q be any minimal prime over P_2 in T and let M be a maximal ideal of T such that $Q \subseteq M$. Then QT_M is minimal over P_2T_M which is principal by t-stability. By the principal ideal theorem, $\mathrm{ht}(Q) = \mathrm{ht}(QT_M) = 1$. By the Going-Up theorem, there is a height-two prime ideal Q_2 of T contracting to P_2 in R. Further, there is a minimal prime ideal Q of P_2 such that $P_2 \subseteq Q \subsetneqq Q_2$. Hence $Q \cap R = Q_2 \cap R = P_2$, which is absurd since the extension $R \subset T$ is INC. Therefore t-dim$(R) = 1$. □

Lemma 2.8. *Let R be a one-dimensional Noetherian domain. If R is Boole t-regular, then R is strongly t-stable.*

Proof. Let I be a nonzero t-ideal of R. Set $T := (I : I)$ and $J := I(T : I)$. Since R is Boole t-regular, then there is $0 \neq c \in \mathrm{qf}(R)$ such that $(I^2)_t = cI$. Then $(T : I) = ((I : I) : I) = (I : I^2) = (I : (I^2)_t) = (I : cI) = c^{-1}(I : I) = c^{-1}T$. So $J = I(T : I) = c^{-1}I$. Since J is a trace ideal of T, then $(T : J) = (J : J) = (c^{-1}I : c^{-1}I) = (I : I) = T$. Hence $J_{v_1} = T$, where v_1 is the v-operation with respect to T. Since R is one-dimensional Noetherian domain, then so is T ([11, Theorem 93]). Now, if J is a proper ideal of T, then $J \subseteq N$ for some maximal ideal N of T. Hence $T = J_{v_1} \subseteq N_{v_1} \subseteq T$ and therefore $N_{v_1} = T$. Since $\dim(T) = 1$, then each nonzero prime ideal of T is t-prime and since T is Noetherian, then $t_1 = v_1$. So $N = N_{v_1} = T$, a contradiction. Hence $J = T$ and therefore $I = cJ = cT$ is strongly t-stable, as desired. □

Proof of Theorem 2.6. (1) $\Longrightarrow$ (2). Clearly R is Boole t-regular and, by Lemma 2.7, t-dim$(R) = 1$.

(2) $\Longrightarrow$ (1). Let I be a nonzero t-ideal of R. Set $T := (I : I)$ and $J := I(T : I)$. Since R is Boole t-regular, then there is $0 \neq c \in \mathrm{qf}(R)$ such that $(I^2)_t = cI$. Then $(T : I) = ((I : I) : I) = (I : I^2) = (I : (I^2)_t) = (I : cI) = c^{-1}(I : I) = c^{-1}T$. So $J = I(T : I) = c^{-1}I$. It suffices to show that $J = T$. Since $T = (I : I) = (II^{-1})^{-1}$, then T is a divisorial (fractional) ideal of R, and since $J = c^{-1}I$, then J is a divisorial (fractional) ideal of R too. Now, for each t-maximal ideal M of R, since R_M is a one-dimensional Noetherian domain which is Boole t-regular, by Lemma 2.8, R_M is strongly t-stable. If $I \nsubseteq M$, then $T_M = (I : I)_M = (IR_M : IR_M) = R_M$ and $J_M = I(T : I)_M = R_M$. Assume that $I \subseteq M$. Then IR_M is a t-ideal of R_M. Since R_M is strongly t-stable, then $IR_M = aR_M$ for some nonzero $a \in I$. Hence $T_M = (I : I)R_M = (IR_M : IR_M) = R_M$. Then $J_M = I_M(T_M : I_M) = R_M = T_M$. Hence $J = J_t = \bigcap_{M \in \mathrm{Max}_t(R)} J_M = \bigcap_{M \in \mathrm{Max}_t(R)} T_M = T_t = T$. It follows that $I = cJ = cT$ and therefore R is strongly t-stable. □

An analogue of Theorem 2.6 does not hold for Clifford t-regularity, as shown by the next example.

Example 2.9. There exists a Noetherian Clifford t-regular domain with $t\text{-dim}(R) = 1$ such that R is not t-stable. Indeed, let us first recall that a domain R is said to be pseudo-Dedekind if every v-ideal is invertible [10]. In [18], P. Samuel gave an example of a Noetherian UFD domain R for which $R[[X]]$ is not a UFD. In [10], Kang noted that $R[[X]]$ is a Noetherian Krull domain which is not pseudo-Dedekind; otherwise, $\text{Cl}(R[[X]]) = \text{Cl}(R) = 0$ forces $R[[X]]$ to be a UFD, absurd. Moreover, $R[[X]]$ is a Clifford t-regular domain by [9, Proposition 2.2] and clearly $R[[X]]$ has t-dimension 1 (since Krull). But for $R[[X]]$ not being a pseudo-Dedekind domain translates into the existence of a v-ideal of $R[[X]]$ that is not invertible, as desired.

We recall that a domain R is called strong Mori if it satisfies the ascending chain condition on w-ideals. Noetherian domains are strong Mori. Next we wish to extend Theorem 2.6 to the larger class of strong Mori domains.

Theorem 2.10. *Let R be a strong Mori domain. Then the following conditions are equivalent:*

(1) *R is strongly t-stable;*

(2) *R is Boole t-regular and $t\text{-dim}(R) = 1$.*

Proof. We recall first the following useful facts:

Fact 1 ([10, Lemma 5.11]). Let I be a finitely generated ideal of a Mori domain R and S a multiplicatively closed subset of R. Then $(I_S)_v = (I_v)_S$. In particular, if I is a t-ideal (i.e., v-ideal) of R, then I is v-finite, that is, $I = A_v$ for some finitely generated subideal A of I. Hence $(I_S)_v = ((A_v)_S)_v = ((A_S)_v)_v = (A_S)_v = (A_v)_S = I_S$ and therefore I_S is a v-ideal of R_S.

Fact 2. For each v-ideal I of R and each multiplicatively closed subset S of R, $(I : I)_S = (I_S : I_S)$. Indeed, set $I = A_v$ for some finitely generated subideal A of I and let $x \in (I_S : I_S)$. Then $xA \subseteq xA_v = xI \subseteq xI_S \subseteq I_S$. Since A is finitely generated, then there exists $\mu \in S$ such that $x\mu A \subseteq I$. So $x\mu I = x\mu A_v \subseteq I_v = I$. Hence $x\mu \in (I : I)$ and then $x \in (I : I)_S$. It follows that $(I : I)_S = (I_S : I_S)$.

(1) $\Longrightarrow$ (2). Clearly R is Boole t-regular. Let M be a maximal t-ideal of R. Then R_M is a Noetherian domain ([5, Theorem 1.9]) which is strongly t-stable. By Theorem 2.6, $t\text{-dim}(R_M) = 1$. Since MR_M is a t-maximal ideal of R_M (Fact 1), then $\text{ht}(M) = \text{ht}(MR_M) = 1$. Therefore $t\text{-dim}(R) = 1$.

(2) $\Longrightarrow$ (1). Let I be a nonzero t-ideal of R. Set $T := (I : I)$ and $J := I(T : I)$. Since R is Boole t-regular, then $(I^2)_t = cI$ for some nonzero $c \in \text{qf}(R)$. So $J = c^{-1}I$. Since J and T are (fractional) t-ideals of R, to show that $J = T$, it suffices to show it t-locally. Let M be a t-maximal ideal of R. Since R_M is one-dimensional Noetherian domain which is Boole t-regular, by Theorem 2.6, R_M is strongly t-stable. By Fact 1, I_M is a t-ideal of R_M. So $I_M = a(I_M : I_M)$. Now, by Fact 2, $T_M = (I : I)_M = (I_M : I_M)$ and then $I_M = aT_M$. Hence $J_M = I_M(T_M : I_M) = T_M$, as desired. □

We close the paper with the following discussion about the limits as well as possible extensions of the above results.

Remark 2.11. (1) Unlike Clifford regularity, Clifford (or even Boole) t-regularity does not force a strong Mori domain to be Noetherian. Indeed, it suffices to consider a UFD domain which is not Noetherian.

(2) Example 2.4 provides a Noetherian Boole t-regular domain of t-dimension two. We do not know whether the assumption "$t\text{-dim}(R) = 1$" in Theorem 2.2 can be omitted.

(3) Following [8, Proposition 2.3], the complete integral closure $\overline{R}$ of a Noetherian Boole regular domain R is a PID. We do not know if $\overline{R}$ is a UFD in the case of Boole t-regularity. However, it's the case if the conductor $(R : \overline{R}) \neq 0$. Indeed, it's clear that $\overline{R}$ is a Krull domain. But $(R : \overline{R}) \neq 0$ forces $\overline{R}$ to be Boole t-regular, when R is Boole t-regular, and by [9, Proposition 2.2], $\overline{R}$ is a UFD.

(4) The Noetherian domain provided in Example 2.4 is not strongly t-discrete since its maximal ideal is t-idempotent. We do not know if the assumption "R strongly t-discrete, i.e., R has no t-idempotent t-prime ideals" forces a Clifford t-regular Noetherian domain to be of t-dimension one.

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Gorenstein dimensions in trivial ring extensions

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Abstract. In this paper, we show that the Gorenstein global dimension of trivial ring extensions is often infinite. Also we study the transfer of Gorenstein properties between a ring and its trivial ring extensions. We conclude with an example showing that, in general, the transfer of the notion of Gorenstein projective module does not carry up to pullback constructions.

Keywords. (Gorenstein) projective dimension, (Gorenstein) injective dimension, (Gorenstein) flat dimension, trivial ring extension, global dimension, weak global dimension, quasi-Frobenius ring, perfect ring.

AMS classification. 13D05, 13B02.

Dedicated to Alain Bouvier

1 Introduction

Throughout this work, all rings are commutative with identity element and all modules are unital. Let R be a ring and M an R-module. We use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote the usual projective, injective and flat dimensions of M, respectively. It is convenient to use "local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal.

In 1967–69, Auslander and Bridger [1, 2] introduced the concept of G-dimension for finitely generated modules over Noetherian rings. Several decades later, Enochs, Jenda and Torrecillas [10, 11, 12] extended this notion by introducing three homological dimensions called Gorenstein projective, injective, and flat dimensions, which have all been studied extensively by their founders and also by Avramov, Christensen, Foxby, Frankild, Holm, Martsinkovsky, and Xu among others [3, 8, 9, 14, 16, 22]. For a ring R, the Gorenstein projective, injective and flat dimension of an R-module M denoted $\mathrm{Gpd}_R(M)$, $\mathrm{Gid}_R(M)$ and $\mathrm{Gfd}_R(M)$, respectively, is defined in terms of resolutions of Gorenstein projective, injective and flat modules, respectively (see [16]). The Gorenstein projective dimension is a refinement of projective dimension to the effect that $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ and equality holds when $\mathrm{pd}_R(M)$ is finite.

Recently, in [5], the authors introduced three classes of modules called strongly Gorenstein projective, injective and flat modules. These modules allowed for nice characterizations of Gorenstein projective and injective modules [5, Theorem 2.7], similar to the characterization of projective modules via the free modules. In [6], the authors started the study of Gorenstein homological dimensions of a ring R; namely, the Gorenstein global dimension of R, denoted G-gldim(R), and the Gorenstein weak (global) dimension of R, denoted G-wgldim(R), and defined as follows: G-gldim(R) $=$

$\sup\{\mathrm{Gpd}_R(M) \mid M\ R\text{-module}\} = \sup\{\mathrm{Gid}_R(M) \mid M\ R\text{-module}\}$ [6, Theorem 3.2] and $\text{G-wgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M\ R\text{-module}\}$. They proved that, for any ring R, $\text{G-wgldim}(R) \leq \text{G-gldim}(R)$ [6, Theorems 4.2] and that the Gorenstein weak and global dimensions are refinements of the classical ones, i.e., $\text{G-gldim}(R) \leq \mathrm{gldim}(R)$ and $\text{G-wgldim}(R) \leq \mathrm{wgldim}(R)$ with quality holding if the weak global dimension of R is finite [6, Propositions 3.11 and 4.5].

This paper studies the Gorenstein dimensions in trivial ring extensions. Let A be a ring and E an A-module. The trivial ring extension of A by E is the ring $R := A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$ [17, 18]. Specifically, we investigate the possible transfer of Gorenstein properties between a ring A and its trivial ring extensions. Section 2 deals with the descent and ascent of the (strongly) Gorenstein properties between A-modules and R-modules, where R is a trivial ring extension of A (Theorem 2.1, Corollary 2.3 and Proposition 2.4). The last part of this section is dedicated to the Gorenstein global dimension (Theorem 2.5). In Section 3, we compute $\text{G-gldim}(A \ltimes E)$ when (A, m) is a local ring with $mE = 0$ (Theorem 3.1) as well as $\text{G-gldim}(D \ltimes E)$ when D is an integral domain and E is an $\mathrm{qf}(D)$-vector space (Theorem 3.5). The last theorem gives rise to an example showing that, in general, the notion of Gorenstein projective module does not carry up to pullback constructions (Example 3.10).

2 Transfer of Gorenstein properties to trivial ring extensions

Throughout this section, we adopt the following notation: A is a ring, E an A-module and $R = A \ltimes E$, the trivial ring extension of A by E. We study the transfer of (strongly) Gorenstein projective and injective notions between A and R. We start this section with the following theorem which handles the transfer of strongly Gorenstein properties between A-modules and R-modules.

Theorem 2.1. *Let M be an A-module. Then:*

(1) (a) *Suppose that $\mathrm{pd}_A(E) < \infty$. If M is a strongly Gorenstein projective A-module, then $M \otimes_A R$ is a strongly Gorenstein projective R-module.*

(b) *Conversely, suppose that E is a flat A-module. If $M \otimes_A R$ is a strongly Gorenstein projective R-module, then M is a strongly Gorenstein projective A-module.*

(2) *Suppose that $\mathrm{Ext}_A^p(R, M) = 0$ for all $p \geq 1$ and $\mathrm{fd}_A(R) < \infty$. If M is a strongly Gorenstein injective A-module, then $\mathrm{Hom}_A(R, M)$ is a strongly Gorenstein injective R-module.*

Proof. (1) (a) Suppose that M is a strongly Gorenstein projective A-module. Then there is an exact sequence of A-modules:

$$0 \to M \to P \to M \to 0 \qquad (\star)$$

where P is projective [5, Proposition 2.9]. It is known that $R = A \oplus_A E$ and since $\mathrm{pd}_A(E) < \infty$ we have $\mathrm{pd}_A(R) < \infty$ and from the exact sequence $(\star)$, $\mathrm{Tor}_A^i(M, R) =$

$0, \forall i \geq 1$. Then the sequence $0 \to M \otimes_A R \to P \otimes_A R \to M \otimes_A R \to 0$ is exact. Note that $P \otimes_A R$ is a projective R-module. On the other hand, for any R-module projective Q, $\mathrm{pd}_A(Q) < \infty$ [7, Exercise 5, page 360]. Then, since M is strongly Gorenstein projective, $\mathrm{Ext}_R(M \otimes_A R, Q) = \mathrm{Ext}_A(M, Q) = 0$ [7, page 118]. Therefore $M \otimes_A R$ is a strongly Gorenstein projective R-module [5, Proposition 2.9].

(b) If E is a flat A-module, then $R = A \ltimes E$ is a faithfully flat A-module. Suppose that $M \otimes_A R$ is strongly Gorenstein projective; combining [5, Remark 2.8] and [5, Proposition 2.9], there is an exact sequence of R-modules:

$$0 \to M \otimes_A R \to F \to M \otimes_A R \to 0 \qquad (\star\star)$$

where $F = R^{(J)}$ is a free R-module. Then the sequence $(\star\star)$ is equivalent to the exact sequence:

$$0 \to M \otimes_A R \to A^{(J)} \otimes_A R \to M \otimes_A R \to 0.$$

Since R is a faithfully flat A-module, the sequence of A-module $0 \to M \to A^{(J)} \to M \to 0$ is exact. On the other hand, let P be a projective A-module. Then $P \otimes_A R$ is a projective R-module and $\mathrm{Ext}_A^k(M, P \otimes_A R) = \mathrm{Ext}_R^k(M \otimes_A R, P \otimes_A R) = 0$, since $\mathrm{Tor}_i^A(M, R) = 0$ and by [7, Proposition 4.1.3, page 118]. But $0 = \mathrm{Ext}_A^k(M, P \otimes_A R) \cong \mathrm{Ext}_A^k(M, P) \oplus_A \mathrm{Ext}_A^k(M, P \otimes_A E)$, then $\mathrm{Ext}_A^k(M, P) = 0$. Therefore M is a strongly Gorenstein projective A-module.

(2) If M is a strongly Gorenstein injective A-module, there exists an exact sequence of A-modules:

$$0 \to M \to I \to M \to 0$$

where I is an injective A-module. Since $\mathrm{Ext}_A(R, M) = 0$, the sequence

$$0 \to \mathrm{Hom}_A(R, M) \to \mathrm{Hom}_A(R, I) \to \mathrm{Hom}_A(R, M) \to 0$$

is exact. Note that $\mathrm{Hom}_A(R, I)$ is an injective R-module. On the other hand, for any injective R-module J, we have $\mathrm{id}_A(J) < \infty$ (since $\mathrm{fd}_A(R) < \infty$ and by [7, Exercise 5, page 360]) and $\mathrm{Ext}_R^i(J, \mathrm{Hom}_A(R, M)) \cong \mathrm{Ext}_A^i(J, M) = 0$ [7, Proposition 4.1.4, page 118]. Therefore $\mathrm{Hom}_A(R, M)$ is a strongly Gorenstein injective R-module. □

Remark 2.2. The statements (1)(a) and (b) in Theorem 2.1 hold for any homomorphism from A to R of finite projective dimension in (a) and faithfully flat in (b), respectively. But here we restrain our study to trivial ring extensions.

Corollary 2.3. *Let M be an A-module. Then:*

(1) *Suppose that* $\mathrm{pd}_A(E) < \infty$. *If M is a Gorenstein projective A-module, then $M \otimes_A R$ is a Gorenstein projective R-module.*

(2) *Suppose that* $\mathrm{Ext}_A^p(R, M) = 0$ *for all $p \geq 1$ and* $\mathrm{fd}_A(R) < \infty$. *If M is a Gorenstein injective A-module, then* $\mathrm{Hom}_A(R, M)$ *is a Gorenstein injective R-module.*

Next we compare the Gorenstein projective (resp., injective) dimension of an A-module M and the Gorenstein projective (resp., injective) dimension of $M \otimes_A R$ (resp., $\mathrm{Hom}_A(R, M)$) as an R-module.

Proposition 2.4. *Let M be an A-module. Then:*

(1) *Suppose that* $\mathrm{Gpd}_A(M)$ *is finite and* $\mathrm{Tor}_A^k(M, R) = 0$, $\forall\, k \geq 1$. *Then:*

$$\mathrm{Gpd}_A(M) \leq \mathrm{Gpd}_R(M \otimes_A R).$$

(2) *Suppose that* $\mathrm{Gid}_A(M)$ *is finite and* $\mathrm{Ext}_A^k(R, M) = 0$, $\forall\, k \geq 1$. *Then:*

$$\mathrm{Gid}_A(M) \leq \mathrm{Gid}_R(\mathrm{Hom}_A(R, M)).$$

Proof. (1) By hypothesis $\mathrm{Tor}_A^k(M, R) = 0$ for all $k \geq 1$. So, by [7, Proposition 4.1.3, page 118], for any A-module P and all $n \geq 1$ we have

$$\mathrm{Ext}_A^k(M, P \otimes_A R) \cong \mathrm{Ext}_R^k(M \otimes_A R, P \otimes_A R).$$

Suppose that $\mathrm{Gpd}_R(M \otimes_A R) \leq d$ for some integer $d \geq 0$. Let P be a projective A-module. Then by [16, Theorem 2.20], $0 = \mathrm{Ext}_R^{d+1}(M \otimes_A R, P \otimes_A R) \cong \mathrm{Ext}_A^{d+1}(M, P \otimes_A R)$. But we have $0 = \mathrm{Ext}_A^{d+1}(M, P \otimes_A R) \cong \mathrm{Ext}_A^{d+1}(M, P) \oplus \mathrm{Ext}_A^{d+1}(M, P \otimes_A E)$. So $\mathrm{Ext}_A^{d+1}(M, P) = 0$ for any projective A-module P. Therefore $\mathrm{Gpd}_A(M) \leq d$.

(2) The proof is essentially dual to (1). Here we use [7, Proposition 4.1.4, page 118] instead of [7, Proposition 4.1.3, page 118]. □

The following theorem gives a relation between G-gldim(A) and G-gldim(R).

Theorem 2.5. *Suppose that* G-gldim(A) *is finite and* $\mathrm{fd}_A(E) = r$*, for some integer* $r \geq 0$. *Then:*

$$\text{G-gldim}(A) \leq \text{G-gldim}(R) + r.$$

Proof. Let M be an A-module and let

$$P_r \overset{f_r}{\to} P_{r-1} \overset{f_{r-1}}{\to} \cdots \to P_0 \overset{f_0}{\to} M \to 0 \qquad \text{(i)}$$

be an exact sequence of A-modules where each P_i is projective. Since $R = A \oplus_A E$ as an A-modules, $\mathrm{fd}_A(E) = \mathrm{fd}_A(R) = r$. Then for all $k \geq 1$ we have

$$\mathrm{Tor}_A^k(\mathrm{Im}\, f_r, R) \cong \mathrm{Tor}_A^{k+r}(M, R) = 0 \qquad \text{(ii)}$$

If G-gldim$(R) \leq n$, then $\mathrm{Gpd}_R(\mathrm{Im}\, f_r \otimes_A R) \leq n$, and by Proposition 2.4 we have $\mathrm{Gpd}_A(\mathrm{Im}\, f_r) \leq n$. From (i), we have $0 = \mathrm{Ext}_A^{n+1}(\mathrm{Im}\, f_r, P) \cong \mathrm{Ext}_A^{r+n+1}(M, P)$, for every projective A-module P. Therefore $\mathrm{Gpd}_A(M) \leq n + r$ and so G-gldim$(A) \leq n + r$ [6, Lemma 3.3]. □

3 Gorenstein global dimension of some trivial ring extensions

In this section, we study the Gorenstein global dimension of particular trivial ring extensions. We start by investigating the Gorenstein global dimension of $R = A \ltimes E$, where (A, m) is a local ring with maximal ideal m and E is an A-module such that $mE = 0$. Recall that a Noetherian ring R is quasi-Frobenius if $\mathrm{id}_R(R) = 0$ and a ring R is perfect if all flat R-modules are projective [21].

Next we announce the first main result of this section.

Theorem 3.1. *Let (A, m) be a local ring with maximal ideal m and E an A-module such that $mE = 0$. Let $R = A \ltimes E$. Then:*

(1) *If A is a Noetherian ring which is not a field and E is a finitely generated A-module (i.e., R is Noetherian), then* $\text{G-gldim}\,(R) = \infty$.

(2) *If A is a perfect ring, then* $\text{G-gldim}\,(R) =$ *either* ∞ *or* 0. *Moreover, in the case* $\text{G-gldim}\,(R) = 0$, *necessarily $A = K$ is a field and E is a K-vector space with* $\dim_k E = 1$ *(i.e., $R = K \ltimes K$).*

To prove this theorem, we need the following lemmas.

Lemma 3.2 ([6, Lemma 3.4]). *Let R be a ring with* $\text{G-gldim}(R) < \infty$ *and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $\text{G-gldim}(R) \leq n$;

(2) $\mathrm{pd}_R(I) \leq n$, *for all injective R-modules I.*

The next lemma gives a characterization of quasi-Frobenius rings.

Lemma 3.3 ([20, Theorem 1.50]). *For a ring R, the following statements are equivalent:*

(1) *R is quasi-Frobenius;*

(2) *R is Noetherian and* $\mathrm{Ann}_R(\mathrm{Ann}_R(I)) = I$ *for any ideal I of R, where* $\mathrm{Ann}_R(I)$ *denotes the annihilator of I in R.*

Recall that the finitistic Gorenstein projective dimension of a ring R, denoted by $\mathrm{FGPD}(R)$, is defined in [16] as follows:

$$\mathrm{FGPD}(R) := \{\mathrm{Gpd}_R(M) \mid M \ R\text{-module and } \mathrm{Gpd}_R(M) < \infty\}.$$

Proof of Theorem 3.1. (1) Suppose that $\text{G-gldim}(R) = n < \infty$ for some positive integer n. If $n \geq 1$, let I be an injective R-module. By [6, Lemma 3.4], $\mathrm{pd}_R(I) \leq n$. Then there is an exact sequence of R-modules

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow I \longrightarrow 0$$

with P_i projective and hence free (R is local). Since A is local and $mE = 0$, every finitely generated ideal of R has a nonzero annihilator. From [15, Corollary 3.3.18], coker($P_n \longrightarrow P_{n-1}$) is flat. Then $\mathrm{fd}_R(I) \leq (n-1)$. Therefore from [6, Theorem 4.11] and [16, Theorem 3.14] we obtain

$$\text{G-wgldim}(R) \leq n-1 = \text{G-gldim}(R) - 1. \qquad (\star)$$

On the other hand, R is Noetherian by [13, Theorem 25.1], and from [6, Corollary 2.3] we get

$$\text{G-wgldim}(R) = \text{G-gldim}(R). \qquad (\star\star)$$

So from $(\star)$ and $(\star\star)$ we conclude that G-gldim$(R) = \infty$.

Now if G-gldim$(R) = 0$, then R is quasi-Frobenius. First we claim that A is a quasi-Frobenius ring. Since A is Noetherian and by Lemma 3.3 we must prove only that $\mathrm{Ann}_A(\mathrm{Ann}_A(I)) = I$ for any ideal I of A. Let I be an ideal of A. Since R is quasi-Frobenius it is easy to see that $\mathrm{Ann}_R(\mathrm{Ann}_R(I \ltimes E)) = \mathrm{Ann}_A(\mathrm{Ann}_A(I)) \ltimes E = I \ltimes E$. Hence $I = \mathrm{Ann}_A(\mathrm{Ann}_A(I))$ and A is quasi-Frobenius; thus G-gldim$(A) = 0$. On the other hand, since R is quasi-Frobenius, R is self-injective. Then $\mathrm{Ext}^i_R(A, R) = 0$ for any integer $i \geq 1$ and so $\mathrm{id}_A(m \oplus_A E) = \mathrm{id}_R(R) = 0$ by [13, Lemma 4.35]. Hence, $m \oplus_A E$ is a projective A-module by Lemma 3.2; in particular E is a projective A-module and so E is free since A is local. Contradiction since $mE = 0$ and $m \neq 0$. Therefore, we conclude that G-gldim$(R) = \infty$.

(2) First, suppose that G-gldim$(R) < \infty$. Note that since A is perfect, R is perfect too by [13, Proposition 1.15]. Combining [4, Corollary 7.12] and [16, Theorem 2.28] we conclude that FGPD$(R) = FPD(R) = 0$ and so G-gldim$(R) =$ FGPD$(R) = 0$. Then from Lemma 3.2 and [20, Theorem 7.56] R is quasi-Frobenius. In particular R is Noetherian and by (1) $A = K$ is a field. Now we claim that $\dim_K E = 1$. Assume that $\dim_K E \geq 2$ and let $E' \subsetneq E$ be a proper submodule of E. Obviously $0 \ltimes E \subseteq \mathrm{Ann}_R(\mathrm{Ann}_R(0 \ltimes E')) \neq 0 \ltimes E'$, this is a contradiction since R is quasi-Frobenius and by Lemma 3.3. Therefore $\dim_K E = 1$ and $E \cong K$. Then $R = K \ltimes K$. □

Example 3.4. Let K be a field, let $X_1, X_2, \ldots, X_n$ be n indeterminates over K, $A = K[[X_1, \ldots, X_n]]$ the power series ring in n variables over K, and $R := A \ltimes K$. Then, G-gldim$(R) = \infty$.

Next we announce the second main theorem of this section.

Theorem 3.5. *Let D be an integral domain which is not a field, K its quotient field, E a K-vector space, and $R := D \ltimes E$. Then* G-gldim$(R) = \infty$

To prove this theorem we need the following lemmas.

Lemma 3.6 ([6, Remarks 3.10]). *For a ring R, if* G-gldim(R) *is finite, then*

$$\begin{aligned} \text{G-gldim}(R) &= \sup\{\mathrm{Gpd}_R(R/I) \mid I \text{ ideal of } R\} \\ &= \sup\{\mathrm{Gpd}_R(M) \mid M \text{ finitely generated } R\text{-module}\}. \end{aligned}$$

Lemma 3.7 ([20, Corollary 1.38]). *Let A be a ring. If A is self-injective (i.e.,* $\mathrm{id}_A(A) = 0$*), then* $\mathrm{Ann}_A(\mathrm{Ann}_A(I)) = I$ *for any finitely generated ideal I of A.*

Proof of Theorem 3.5. First we claim that $\frac{R}{0\ltimes E}$ is not a Gorenstein projective R-module. For this, let $0 \neq a \in D$ a non-invertible element, then $R(0,a) = 0 \ltimes Da$ is an ideal of R. Clearly, $0 \ltimes Da \subsetneq 0 \ltimes E \subseteq \mathrm{Ann}_R(\mathrm{Ann}_R(0 \ltimes Da))$, and by Lemma 3.7, $\mathrm{id}_R(R) \neq 0 = \mathrm{id}_D(E)$, then $\mathrm{Ext}^i_R(\frac{R}{0\ltimes E}, R) \cong \mathrm{Ext}^i_R(D, R) \neq 0$, for some $i \geq 1$ [13, Proposition 4.35]. So $\frac{R}{0\ltimes E}$ is not a Gorenstein projective R-module [16, Proposition 2.3]. Now we claim that $0 \ltimes E$ is not a Gorenstein projective R-module. Suppose the opposite, i.e., that $0 \ltimes E$ is a Gorenstein projective R-module. Then there is an exact sequence of R-modules

$$0 \longrightarrow 0 \ltimes E \longrightarrow F \longrightarrow G \longrightarrow 0 \tag{1}$$

where $F \cong R^I$ is a free R-module and G is Gorenstein projective by [16, Proposition 2.4]. Consider the pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0\ltimes E & \longrightarrow & R & \longrightarrow & \frac{R}{0\ltimes E} & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0\ltimes E & \longrightarrow & R^I & \longrightarrow & C' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R^{I'} & = & R^{I'} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Combining the exact sequence (1) and the short exact sequence in the pushout $0 \to 0 \ltimes E \to R^I \to C' \to 0$, yields $C' \cong G$ is Gorenstein projective. Then from the short exact sequence $0 \to \frac{R}{0\ltimes E} \to C' \to R^{I'} \to 0$, we get $\frac{R}{0\ltimes E}$ is Gorenstein projective [16, Theorem 2.5]. But this contradicts the fact that $\frac{R}{0\ltimes E}$ is not Gorenstein projective in the first part of the proof. Then $0 \ltimes E$ is not Gorenstein projective. On the other hand, from the short exact sequence $0 \longrightarrow (0 \ltimes E)^J \longrightarrow R^J \longrightarrow 0 \ltimes E \longrightarrow 0$ we obtain $\mathrm{Gpd}_R(0 \ltimes E) = \infty$ [16, Proposition 2.18]. Therefore $\text{G-gldim}(R) = \infty$. □

Note that the condition "D is not a field" in Theorem 3.5 is necessary. For, the next corollary shows that for any field K, $\text{G-gldim}(K \ltimes K) = 0$. However [19, Lemma 2.2] asserts that $\mathrm{gldim}(K \ltimes K) = \infty$.

Corollary 3.8. *Let K be a field. Then:*

(1) $\text{G-gldim}(K \ltimes K) = 0$.

(2) $\text{G-gldim}(K \ltimes K^n) = \infty$, *for any* $n \geq 2$.

Example 3.9. Let $R := \mathbb{Z} \ltimes \mathbb{Q}$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ the field of rational numbers. Then $\text{G-gldim}(R) = \infty$.

Next we exhibit an example showing that, in general, the transfer of the notion of Gorenstein projective module does not carry up to pullback constructions.

Example 3.10. Let (D, m) be a discrete valuation domain and $K = \mathrm{qf}(D)$. Consider the following pullback:

$$\begin{array}{ccc} R = D \ltimes K & \longrightarrow & T = K \ltimes K \\ \downarrow & & \downarrow \\ D \cong \frac{R}{0 \ltimes K} & \longrightarrow & K \end{array}$$

Let $0 \neq a \in m$ and $I = 0 \ltimes Da$. Consider the following short exact sequence of R-modules

$$0 \longrightarrow 0 \ltimes K \longrightarrow R \xrightarrow{u} 0 \ltimes Da \longrightarrow 0$$

where $u(b,e) = (b,e)(0,a) = (0,ba)$. Similar arguments used in the proof of Theorem 3.5 yield $0 \ltimes Da$ is not Gorenstein projective, $I \otimes_R T \cong 0 \ltimes K$ is a Gorenstein projective ideal of T, and $I \otimes_R \frac{R}{0 \ltimes K} \cong \frac{R}{0 \ltimes K}$ is a free $\frac{R}{0 \ltimes K}$-module, then Gorenstein projective.

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Note on Prüfer $\star$-multiplication domains and class groups

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Abstract. D. F. Anderson, M. Fontana and M. Zafrullah [2] investigated Prüfer $\star$-multiplication domains and class groups. We study [2] and give semigroup analogs for its theorems. Thus, let D be an integral domain and let S be a grading monoid (or a g-monoid). We show that there is a canonical monomorphism from the v-class group $\mathrm{Cl}^v(D)$ to $\mathrm{Cl}^v(D[S])$, we show that $\mathrm{Cl}^v(S) \cong \mathrm{Cl}^v(S[X])$, and we compute $\mathrm{Cl}^v(V[X])$ for one-dimensional valuation semigroups V. We also show that S is a Prüfer v-multiplication semigroup if and only if $e(fg)^v = (e(f) + e(g))^v$ for every $f, g \in D[S] - \{0\}$.

Keywords. Semistar operation, commutative semigroup, class group.

AMS classification. 13A15.

1 Introduction

Let D be an integral domain with quotient field K. Let $\bar{\mathrm{F}}(D)$ be the set of non-zero submodules of K, let $\mathrm{F}(D)$ be the set of non-zero fractional ideals I of D, i.e., $I \in \bar{\mathrm{F}}(D)$ and $dI \subset D$ for some $d \in D - \{0\}$, and let $\mathrm{f}(D)$ be the set of non-zero finitely generated D-submodules of K. A semistar operation on D is a mapping $\star : \bar{\mathrm{F}}(D) \longrightarrow \bar{\mathrm{F}}(D)$, $I \longmapsto I^\star$, such that the following properties hold for every $x \in K - \{0\}$ and for every $I, J \in \bar{\mathrm{F}}(D)$: $(xI)^\star = xI^\star; I \subset I^\star; (I^\star)^\star = I^\star; I \subset J$ implies $I^\star \subset J^\star$. For subsets I, J of K, the subset $\{x \in K \mid xI \subset J\}$ of K is denoted by $(J : I)$, the subset $(D : I)$ of K is denoted by I^{-1}, and $(I^{-1})^{-1}$ is denoted by I^v. For every $I \in \mathrm{F}(D)$, I^v is the intersection of principal fractional ideals of D containing I (cf. [8, Theorem 34.1]). The mapping $I \longmapsto I^v$ (resp., $I \longmapsto I$) from $\bar{\mathrm{F}}(D)$ to $\bar{\mathrm{F}}(D)$ is a semistar operation, and is called the v-semistar operation (resp., the d-semistar operation). A semistar operation $\star$ on D is said of finite type if, for every $I \in \bar{\mathrm{F}}(D)$, $I^\star = \cup\{J^\star \mid J \in \mathrm{f}(D) \text{ with } J \subset I\}$. A semistar operation $\star$ on D is said stable if $(I \cap J)^\star = I^\star \cap J^\star$ for every $I, J \in \bar{\mathrm{F}}(D)$.

D is called a $\mathrm{P}\star\mathrm{MD}_1$ (or, a Prüfer $\star$-multiplication domain$_1$) if, for every $I \in \mathrm{f}(D)$, there is $J \in \mathrm{f}(D)$ with $IJ \subset D$ such that $(IJ)^\star = D^\star$ (cf. M. Fontana, P. Jara and E. Santos [5]). D is called a $\mathrm{P}\star\mathrm{MD}_2$ if, for every $I \in \mathrm{f}(D)$, there is $J \in \mathrm{f}(D)$ such that $(IJ)^\star = D^\star$. Clearly, $\mathrm{P}\star\mathrm{MD}_1$ implies $\mathrm{P}\star\mathrm{MD}_2$. An element $I \in \mathrm{F}(D)$ is called $\star$-invertible$_1$ if $(IJ)^\star = D^\star$ for some $J \in \mathrm{F}(D)$ with $IJ \subset D$. I is called $\star$-invertible$_2$ if $(IJ)^\star = D^\star$ for some $J \in \mathrm{F}(D)$. Set $\mathrm{Inv}^\star(D)_1 = \{I^\star \mid I \in \mathrm{F}(D)$ which is $\star$-invertible$_1\}$, and set $\mathrm{Inv}^\star(D)_2 = \{I^\star \mid I \in \mathrm{F}(D)$ which is $\star$-invertible$_2\}$. $\mathrm{Inv}^\star(D)_1$ (resp., $\mathrm{Inv}^\star(D)_2$) is a group under the multiplication $(I^\star, J^\star) \longmapsto (I^\star J^\star)^\star$. Set $\frac{\mathrm{Inv}^\star(D)_1}{\mathrm{Prin}(D)} = \mathrm{Cl}^\star(D)_1$, and set $\frac{\mathrm{Inv}^\star(D)_2}{\mathrm{Prin}(D)} = \mathrm{Cl}^\star(D)_2$, where $\mathrm{Prin}(D) = \{(a)^\star \mid a \in$

$K-\{0\}\}$ (cf. D. F. Anderson [1], M. Fontana and G. Picozza [7]). $\mathrm{Cl}^\star(D)_1$ and $\mathrm{Cl}^\star(D)_2$ are called $\star$-class groups of D.

If $D^\star = D$, we have that P$\star$MD$_1$ = P$\star$MD$_2$, $\star$-invertible$_1$ = $\star$-invertible$_2$, $\mathrm{Inv}^\star(D)_1 = \mathrm{Inv}^\star(D)_2$ and $\mathrm{Cl}^\star(D)_1 = \mathrm{Cl}^\star(D)_2$. If $D^\star = D$, we denote P$\star$MD$_1$ (resp., $\star$-invertible$_1$, $\mathrm{Inv}^\star(D)_1$, $\mathrm{Cl}^\star(D)_1$) by P$\star$MD (resp., $\star$-invertible, $\mathrm{Inv}^\star(D)$, $\mathrm{Cl}^\star(D)$). An element of $\mathrm{Inv}^d(D)$ is called invertible, $\mathrm{Cl}^d(D)$ is called the Picard group of D, and $\mathrm{Cl}^d(D)$ is denoted also by $\mathrm{Pic}(D)$.

A maximal member in the set $\{I \mid I$ is a non-zero ideal of D with $I \not\ni 1$ such that $I^\star \cap D = I\}$ is called a quasi-$\star$-maximal ideal of D. The set of quasi-$\star$-maximal ideals of D is denoted by $\mathrm{QMax}^\star(D)$.

For every $f \in K[X] - \{0\}$, $c_D(f)$ (or, $c(f)$) denotes the D-submodule of K generated by the coefficients of f.

D. F. Anderson, M. Fontana and M. Zafrullah [2] investigated P$\star$MD$_1$, P$\star$MD$_2$, and $\mathrm{Cl}^v(D)$. The theorems in [2] are the following,

Theorem 1.1 ([2, Theorem 1.1]). *Let D be an integral domain with quotient field K, let X be an indeterminate over K, and let $\star$ be a stable and finite type semistar operation on D. Then the following are equivalent:*

(1) $c_D(fg)^\star = (c_D(f)c_D(g))^\star$ *for every* $f, g \in K[X] - \{0\}$.

(2) D_P *is a valuation domain for every* $P \in \mathrm{QMax}^\star(D)$.

(3) D *is a* P$\star$MD$_1$.

Theorem 1.2 ([2, Theorem 2.7]). *Let Γ be a subgroup of the additive group $\mathbb{R}$ of real numbers, let V be a one-dimensional valuation domain with value group Γ.*

(1) *If V is discrete, then* $\mathrm{Pic}(V) = 0$ *and* $\mathrm{Cl}^v(V) = 0$.

(2) *If V is not discrete, then* $\mathrm{Pic}(V) = 0$ *and* $\mathrm{Cl}^v(V) \cong \mathbb{R}/\Gamma$.

Theorem 1.3 ([2, Theorem 3.5]). *Let V be a valuation domain with maximal ideal M, and let $P \subsetneqq M$ be a prime ideal of V. Then* $\mathrm{Cl}^v(V) \cong \mathrm{Cl}^v(V/P)$.

In this paper, we are interested in semigroup versions of Theorems 1.1, 1.2 and 1.3, and will translate them to semigroups, and will answer the question ([2]): Find analogs of Theorems 1.2 and 1.3 for valuation semigroups. This paper consists of three sections. Section 1 is an introduction, Section 2 contains results for g-monoids, and Section 3 contains the proofs for our results.

2 Semigroups

A submonoid S of a torsion-free abelian additive group is called a grading monoid (or, a g-monoid). Throughout the paper, we assume that $S \supsetneqq \{0\}$. The set $\mathbb{Z}_0$ of non-negative integers is a g-monoid. For the general theory of g-monoids, we refer to [3], [9], [11] and [12]. Thus, let S be a g-monoid. Then the additive group $\{s - s' \mid s, s' \in S\}$ is called the quotient group of S, and is denoted by $\mathrm{q}(S)$. A non-empty subset I of S is called an ideal of S if $S + I \subset I$. An ideal $P \subsetneqq S$ is called a prime ideal of S if, for every $a, b \in S, a + b \in P$ implies $a \in P$ or $b \in P$. An ideal $Q \subsetneqq S$ of S is called a primary ideal of S if, for every $a, b \in S, a + b \in Q$ and $b \notin Q$ implies $na \in Q$ for some positive integer n. We see easily that if S is not a group, then S has a unique maximal ideal. Let P be a prime ideal of S. Then the g-monoid $\{s - t \mid s \in S, t \in S - P\}$ is denoted by S_P. Assume that there are prime ideals $P_1, \ldots, P_n$ of S such that $P_1 \subsetneqq \cdots \subsetneqq P_n$ and that there are not prime ideals $Q_1, \ldots, Q_{n+1}$ such that $Q_1 \subsetneqq \cdots \subsetneqq Q_{n+1}$, then n is called the dimension of S.

Set $G = \mathrm{q}(S)$. Let $\bar{\mathrm{F}}(S)$ be the set of non-empty subsets I of G such that $S + I \subset I$. An element I of $\bar{\mathrm{F}}(S)$ is called a fractional ideal of S if $s + I \subset S$ for some $s \in S$. Let $\mathrm{F}(S)$ be the set of fractional ideals of S, and let $\mathrm{f}(S)$ be the set of finitely generated fractional ideals of S. If a mapping $I \longmapsto I^\star$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ satisfies the following conditions, then $\star$ is called a semistar operation on S: For every $a \in G$ and for every $I, J \in \bar{\mathrm{F}}(S)$, we have $(a + I)^\star = a + I^\star$, $I \subset I^\star$, $(I^\star)^\star = I^\star$, $I \subset J$ implies $I^\star \subset J^\star$. For subsets I, J of G, the subset $\{x \in G \mid x + I \subset J\}$ of G is denoted by $(J : I)$, the subset $(S : I)$ of G is denoted by I^{-1}, and $(I^{-1})^{-1}$ is denoted by I^v (we let $\varnothing^{-1} = G$). For every $I \in \mathrm{F}(S)$, I^v is the intersection of principal fractional ideals of S containing I. The mapping $I \longmapsto I^v$ (resp., $I \longmapsto I$) from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation, and is called the v-semistar operation (resp., the d-semistar operation). A semistar operation $\star$ is said of finite type if, for every $I \in \bar{\mathrm{F}}(S)$, $I^\star = \cup\{J^\star \mid J \in \mathrm{f}(S) \text{ with } J \subset I\}$. $\star$ is said stable if $(I \cap J)^\star = I^\star \cap J^\star$ for every $I, J \in \bar{\mathrm{F}}(S)$.

S is called a P$\star$MS$_1$ (or, a Prüfer $\star$-multiplication semigroup$_1$) if, for every $I \in \mathrm{f}(S)$, there is a $J \in \mathrm{f}(S)$ with $I + J \subset S$ such that $(I + J)^\star = S^\star$. S is called a P$\star$MS$_2$ if, for every $I \in \mathrm{f}(S)$, there is a $J \in \mathrm{f}(S)$ such that $(I + J)^\star = S^\star$. An element $I \in \mathrm{F}(S)$ is called $\star$-invertible$_1$ if $(I + J)^\star = S^\star$ for some $J \in \mathrm{F}(S)$ with $I + J \subset S$. I is called $\star$-invertible$_2$ if $(I + J)^\star = S^\star$ for some $J \in \mathrm{F}(S)$. Set $\mathrm{Inv}^\star(D)_1 = \{I^\star \mid I \in \mathrm{F}(S) \text{ which is } \star\text{-invertible}_1\}$, and set $\mathrm{Inv}^\star(S)_2 = \{I^\star \mid I \in \mathrm{F}(S) \text{ which is } \star\text{-invertible}_2\}$. $\mathrm{Inv}^\star(S)_1$ (resp., $\mathrm{Inv}^\star(S)_2$) is a group under the multiplication: $(I^\star, J^\star) \longmapsto (I^\star + J^\star)^\star$. Set $\frac{\mathrm{Inv}^\star(S)_1}{\mathrm{Prin}(S)} = \mathrm{Cl}^\star(S)_1$, and set $\frac{\mathrm{Inv}^\star(S)_2}{\mathrm{Prin}(S)} = \mathrm{Cl}^\star(S)_2$, where $\mathrm{Prin}(S) = \{(a)^\star \mid a \in G\}$. $\mathrm{Cl}^\star(S)_1$ and $\mathrm{Cl}^\star(S)_2$ are called $\star$-class groups of S.

If $S^\star = S$, we have P$\star$MS$_1$ = P$\star$MS$_2$, $\star$-invertible$_1$ = $\star$-invertible$_2$, $\mathrm{Inv}^\star(S)_1 = \mathrm{Inv}^\star(S)_2$ and $\mathrm{Cl}^\star(S)_1 = \mathrm{Cl}^\star(S)_2$. If $S^\star = S$, we denote P$\star$MS$_1$ (resp., $\star$-invertible$_1$, $\mathrm{Inv}^\star(S)_1$, $\mathrm{Cl}^\star(S)_1$) by P$\star$MS (resp., $\star$-invertible, $\mathrm{Inv}^\star(S)$, $\mathrm{Cl}^\star(S)$). An element of $\mathrm{Inv}^d(S)$ is called invertible, and $\mathrm{Cl}^d(S)$ is denoted also by $\mathrm{Pic}(S)$.

For totally ordered abelian additive groups and valuations on fields, we refer to [8]. Thus, the cardinality of the set of convex subgroups $\subsetneqq \Gamma$ is called the rank of Γ, and is denoted by $\mathrm{rank}(\Gamma)$. If H_1 and H_2 are distinct convex subgroups of Γ such that there is no convex subgroup of Γ properly between H_1 and H_2, then H_1 and H_2 are called adjacent. Γ is called discrete if, for every pair H_1, H_2 of adjacent convex subgroups of Γ (say, $H_1 \subsetneqq H_2$), the ordered factor group H_2/H_1 is order isomorphic with $\mathbb{Z}$. If $\mathrm{rank}(\Gamma) = 1$ and Γ is discrete, then Γ is order isomorphic with $\mathbb{Z}$. If $\mathrm{rank}(\Gamma) = 1$ and Γ is not discrete, then Γ is order isomorphic with a dense subgroup of the additive group $\mathbb{R}$ of the real numbers (cf. [8, p. 193]). Let w be a valuation on a field K, let Γ be the value group of w, and let V be the valuation domain belonging to w. The rank(Γ) is called the rank of w (resp., V) and is denoted by rank(w) (resp., rank(V)). Then rank(V) equals the dimension dim(V) of V. If Γ is discrete, then w (resp., V) is called discrete. w (resp., V) is discrete if and only if every primary ideal of V is a power of its radical ([8, III, Exercises 22]).

Let G be a torsion-free abelian additive group, and let Γ be a totally ordered abelian additive group. A mapping w from G onto Γ is called a valuation if, for every $a, b \in G$, $w(a + b) = w(a) + w(b)$. Γ is called the value group of w, the subset $V = \{a \in G \mid w(a) \geq 0\}$ of G is called the valuation semigroup of (or, belonging to) w, and w is called a valuation on G belonging to V.

Let w be a valuation on G with value group Γ, and let V be the valuation semigroup of w. If H is a convex subgroup $\subsetneqq \Gamma$, then the set $\{a \in V \mid a \notin H\}$ is a prime ideal of V, and thus there is a canonical bijection from the set of convex subgroups $\subsetneqq \Gamma$ onto the set Spec(V) of prime ideals of V. rank(Γ) is called the rank of w (resp., V) and is denoted by rank(w) (resp., rank(V)). Then rank(V) equals the dimension dim(V) of V. If Γ is discrete, then w (resp., V) is called discrete. w (resp., V) is discrete if and only if every primary ideal of V is a multiple of its radical.

Let S be a g-monoid with quotient group G. Let P be a prime ideal of S. Then $S - P$ is a g-monoid. We denote $S - P$ by S/P. Then q(S/P) is canonically regarded as a subgroup of G.

Let V be a valuation semigroup, let $G = \mathrm{q}(V)$, let w be a valuation on G belonging to V, and let Γ be the value group of w. Let P be a prime ideal of V, let H be the convex subgroup of Γ associated to P. Then the restriction $\bar{w}$ of w to q(V/P) is a valuation on q(V/P), H is the value group of $\bar{w}$, and V/P is the valuation semigroup of $\bar{w}$. If I is an ideal of V with $I \supsetneqq P$, then $I - P$ is an ideal of V/P, and is denoted by I/P. If J is an ideal of V/P, there is a unique ideal $I \supsetneqq P$ of V such that $J = I/P$.

Let D be an integral domain with quotient field K, and let S be a g-monoid with quotient group G. Then the semigroup ring of S over D is denoted by $D[S]$ (or, by $D[X; S]$). $D[X; S]$ is the ring of elements $\sum_{\text{finite}} a_i X^{s_i}$ for every $a_i \in D$ and every $s_i \in S$. For an element $f = \sum a_i X^{t_i} \in K[X; G]$ with $a_i \neq 0$ and $t_i \neq t_j$ for every $i \neq j$, the D-module $\sum Da_i$ is denoted by $c_D(f)$ (or, $c(f)$) and $\cup(S + t_i)$ is denoted by $e_S(f)$ (or, $e(f)$).

A maximal member in the set $\{I \mid I$ is an ideal of S with $I \not\ni 0$ such that $I^\star \cap S = I\}$ is called a quasi-$\star$-maximal ideal of S. The set of quasi-$\star$-maximal ideals of S is denoted by QMax$^\star(S)$.

The g-monoid $\{s + kX \mid s \in S, k \in \mathbb{Z}_0\}$ is called the polynomial semigroup of X over S, and is denoted by $S[X]$.

In this paper, we will prove the following Propositions 2.1, 2.2, 2.4, Theorems 1.1′, 1.2′, 1.3′ and Corollaries 2.3, 2.5.

Proposition 2.1. *Let D be an integral domain, and let $\star$ be a stable and finite type semistar operation on D. Then the following assertions are equivalent:*

(1) *D is a* P$\star$MD$_1$.

(2) *D is a* P$\star$MD$_2$.

Theorem 1.1′. *Let D be an integral domain, let S be a g-monoid with quotient group G, let X be an indeterminate over G, and let $\star$ be a stable and finite type semistar operation on S. Then the following assertions are equivalent:*

(1) $e_S(fg)^\star = (e_S(f) + e_S(g))^\star$ *for every* $f, g \in D[X; G] - \{0\}$.

(2) S_P *is a valuation semigroup for every* $P \in \mathrm{QMax}^\star(S)$.

(3) S *is a* P⋆MS$_1$.

(4) S *is a* P⋆MS$_2$.

Proposition 2.2. *There is a canonical monomorphism from* $\mathrm{Cl}^v(D)$ *to* $\mathrm{Cl}^v(D[X; S])$.

Corollary 2.3. *There is a canonical monomorphism from* $\mathrm{Cl}^v(D)$ *to* $\mathrm{Cl}^v(D[X])$.

Proposition 2.4. *There is a canonical isomorphism from* $\mathrm{Cl}^v(S)$ *onto* $\mathrm{Cl}^v(S[X])$.

Corollary 2.5. *Let Γ be a subgroup of $\mathbb{R}$, let V be a one-dimensional valuation semigroup with value group Γ.*

(1) *If V is discrete, then* $\mathrm{Cl}^v(V[X]) = 0$.

(2) *If V is not discrete, then* $\mathrm{Cl}^v(V[X]) \cong \mathbb{R}/\Gamma$.

Theorem 1.2′. *Let Γ be a subgroup of $\mathbb{R}$ and V a one-dimensional valuation semigroup with value group Γ.*

(1) $\mathrm{Pic}(S) = 0$ *for every g-monoid S.*

(2) *If V is discrete, then* $\mathrm{Cl}^v(V) = 0$.

(3) *If V is not discrete, then* $\mathrm{Cl}^v(V) \cong \mathbb{R}/\Gamma$.

Theorem 1.3′. *Let V be a valuation semigroup with maximal ideal M and let $P \subsetneq M$ be a prime ideal of V. Then* $\mathrm{Cl}^v(V) \cong \mathrm{Cl}^v(V/P)$.

3 Proofs

Proof of Proposition 2.1. Assume that D is a P⋆MD$_2$. Let $f, g \in K[X] - \{0\}$. By the Dedekind–Mertens lemma (cf. [8, Theorem 28.1]), there is a positive integer m such that $c(f)^m c(fg) = c(f)^{m+1} c(g)$. There is $J \in \mathrm{f}(D)$ such that $(c(f)J)^\star = D^\star$. Since $(c(f)^m c(fg) J^m)^\star = (c(f)^{m+1} c(g) J^m)^\star$, it follows that $c(fg)^\star = (c(f)c(g))^\star$. By [2, Theorem 1.1], D is a P⋆MD$_1$. □

An ideal I of S is called a cancelation ideal if, for every ideals J_1 and J_2 of S, $I + J_1 = I + J_2$ implies $J_1 = J_2$. Clearly, every invertible ideal of S is a cancelation ideal. We will give an outline of the proof of lemma for the convenience of the reader.

Lemma 3.1 ([15, Theorem 8.2]). *Let I be a cancelation ideal of S. Then I is a principal ideal.*

Proof (outline). Let I be a cancelation ideal of S. Suppose that I is not principal. Then $I \neq S$, because $S = (0)$ is principal. Let M be the maximal ideal of S. If $I + S = I + M$, then $S = M$; a contradiction. Hence $I + M \subsetneqq I$. Choose an element $x \in I$ with $x \notin I + M$. Since I is not principal, we have $(x) \subsetneqq I$. Choose an element $y \in I$ with $y \notin (x)$, and put $a = x + y$. Then clearly, we have $a \notin (2x)$. There is a maximal member J in the set of ideals that do not contain a, and then $2x \in J$. Since $J \not\ni a$, and since I is a cancelation ideal, $I + J$ does not contain $I + a$. Hence there is $b \in I$ with $b + a \notin I + J$. The case where $b \in (x)$: Then $b + a \in (y + 2x) \subset I + J$; a contradiction. The case where $b \notin (x)$: If $a \in (b + y)$, then $a = b + y + s$ for some $s \in S$. Since $x \notin I + M$, we have $s \notin M$, and $b \in (x)$; a contradiction. Hence $a \notin (b + y)$ and $b + y \in J$. Then $b + a = x + (b + y) \in I + J$; a contradiction. □

Lemma 3.2. *We have* $\mathrm{Pic}(S) = 0$.

Proof. We have $\mathrm{Inv}^d(S) = \mathrm{Prin}(S)$ by Lemma 3.1. Hence $\mathrm{Pic}(S) = 0$. □

Let w be a valuation on G with value group Γ. For every subset $X \subset G$, we set $w(X) = \{w(x) \mid x \in X\} \subset \Gamma$.

Lemma 3.3 (A semigroup version of Theorem 1.2(1)). *Let V be a one-dimensional valuation semigroup. If V is discrete, then* $\mathrm{Cl}^v(V) = 0$.

Proof. Let w be a valuation on $G = \mathrm{q}(V)$ belonging to V, and let Γ be the value group of w. Since Γ is of rank 1 and discrete, we may assume that $\Gamma = \mathbb{Z}$. Let $I \in \mathrm{F}(V)$. Then there is $a \in I$ such that $\min w(I) = w(a)$, and we have $I = (a)$. Then $I^v = (a) = I^d$, hence $v = d$. By Lemma 3.2, we have $\mathrm{Cl}^v(V) = \mathrm{Cl}^d(V) = 0$. □

If X is a subset of $\mathbb{R}$, the infimum of X in $\mathbb{R}$ is denoted by $\inf_{\mathbb{R}} X$.

Lemma 3.4. *Let V be a one-dimensional valuation semigroup which is not discrete and with value group Γ, where Γ is a dense subgroup of $\mathbb{R}$. Let w be a valuation on $G = \mathrm{q}(V)$ belonging to V. For every $I \in \mathrm{F}(V)$, set $f(I) = \inf_{\mathbb{R}} w(I)$, and define a mapping f from $\mathrm{F}(V)$ to $\mathbb{R}$.*

(1) *Let $I \in \mathrm{F}(V)$ with $f(I) = \alpha$. Then $I^{-1} = \{x \in G \mid w(x) \geq -\alpha\}$ and $f(I^{-1}) = -\alpha$.*

(2) *Let $I \in \mathrm{F}(V)$ with $f(I) = \alpha$. Then $I^v = \{x \in G \mid w(x) \geq \alpha\}$ and $f(I^v) = \alpha$.*

(3) *$f(I + J) = f(I) + f(J)$ for every $I, J \in \mathrm{F}(V)$.*

Proof. (1) The first assertion follows from the definitions. And the second assertion follows from the first one.

(2) The assertion follows by applying (1) for I^{-1}.

(3) Set $f(I)=\alpha$ and $f(J)=\beta$. $f(I+J)\geq\alpha+\beta$ is obvious. Let $\mathbb{R}\ni\gamma>\alpha+\beta$. There are $\alpha_1>\alpha$ and $\beta_1>\beta$ such that $\alpha_1+\beta_1<\gamma$. Then there are $i\in I$ and $j\in J$ such that $w(i)<\alpha_1$ and $w(j)<\beta_1$. Then we have $w(i+j)<\alpha_1+\beta_1<\gamma$, hence $f(I+J)=\alpha+\beta$. □

Lemma 3.5 (A semigroup version of Theorem 1.2(2)). *Let V be a one-dimensional valuation semigroup which is not discrete and with value group Γ, where Γ is a subgroup of $\mathbb{R}$. Then $\mathrm{Cl}^v(V)\cong\mathbb{R}/\Gamma$.*

Proof. Let w be a valuation on $G=\mathrm{q}(V)$ with value group Γ. Since Γ is not discrete, Γ is a dense subgroup of $\mathbb{R}$. For every $I\in\mathrm{Inv}^v(V)$, set $\varphi(I)=\overline{f(I)}\in\mathbb{R}/\Gamma$, where f is as in Lemma 3.4. Throughout the proof, we will use Lemma 3.4.

- φ is a mapping onto $\mathbb{R}/\Gamma$.
  For, let $\alpha\in\mathbb{R}$. Let $I=\{x\in G\mid w(x)\geq\alpha\}$. Then we have $f(I)=\alpha$. Since $f(I^{-1})=-\alpha$, we have $f(I+I^{-1})=f(I)+f(I^{-1})=0$. Therefore $(I+I^{-1})^v=\{x\in G\mid w(x)\geq 0\}=V$, that is, I is v-invertible. On the other hand, we have $I^v=\{x\in G\mid w(x)\geq\alpha\}=I$. It follows that $I\in\mathrm{Inv}^v(V)$ and $\varphi(I)=\overline{f(I)}=\bar{\alpha}$.

- If $\varphi(I)=\bar{0}$, then $I\in\mathrm{Prin}(V)$.
  For, since $f(I)\in\Gamma$, there is $a\in G$ such that $f(I)=w(a)$. Then we have $I^v=\{x\in G\mid w(x)\geq w(a)\}=(a)$. Since $I\in\mathrm{Inv}^v(V)$, we have $I^v=I$, and hence $I=(a)\in\mathrm{Prin}(V)$.

- For every $I,J\in\mathrm{Inv}^v(V)$, we have $\varphi((I+J)^v)=\varphi(I)+\varphi(J)$.
  For, $f((I+J)^v)=f(I+J)=f(I)+f(J)$.

The proof is complete. □

Lemmas 3.2, 3.3 and 3.5 complete the proof of Theorem 1.2′.

An ideal I of a g-monoid is called v-ideal if $I^v=I$.

Lemma 3.6 (A semigroup version of [2, Lemma 3.3]). *Let V be a valuation semigroup with maximal ideal M, let P be a prime ideal of V with $M\supsetneqq P$, let I be an ideal of V with $I\supsetneqq P$. Then $I^v/P=(I/P)^v$. In particular, I/P is a v-ideal of V/P if and only if I is a v-ideal of V.*

Proof. Let w be a valuation on $G=\mathrm{q}(V)$ belonging to V. Set $\{a\in V\mid w(a)$ is a lower bound of $w(I)\}=\{a_\lambda\mid\lambda\in\Lambda\}$. Since I^v is the intersection of principal fractional ideals of V containing I, we have $\cap_\lambda(a_\lambda+V)=I^v$. Set $\{b\in V/P\mid w(b)$ is a lower bound of $w(I/P)\}=\{b_\sigma\mid\sigma\in\Sigma\}$. Similarly, we have $\cap_\sigma(b_\sigma+V/P)=(I/P)^v$. Easily we have $\{a_\lambda\mid\lambda\in\Lambda\}=\{s\in V-P\mid w(s)$ is a lower bound of $w(I)\}$ and $\{b_\sigma\mid\sigma\in\Sigma\}=\{s\in V-P\mid w(s)$ is a lower bound of $w(I)\}$, and hence $\{a_\lambda\mid\lambda\in\Lambda\}=\{b_\sigma\mid\sigma\in\Sigma\}$. It follows that $(I/P)^v=\cap_\sigma\{(b_\sigma+V)-P\}=\cap_\sigma(b_\sigma+V)-P=\cap_\lambda(a_\lambda+V)-P=I^v-P=I^v/P$. □

Let V be a valuation semigroup. We note that every two ideals of V are comparable.

Lemma 3.7 (A semigroup version of [2, Lemma 3.4]). *Let V be a valuation semigroup with maximal ideal M, let P be a prime ideal of V with $P \subsetneqq M$, and let I be an ideal of V with $I \supsetneqq P$. Then I/P is a v-invertible v-ideal of V/P if and only if I is a v-invertible v-ideal of V.*

Proof. We will use Lemma 3.6. Assume that I is a v-invertible v-ideal of V. We have $I + I^{-1} \supset M$. For, if $I + I^{-1} \not\supset M$, there is $a \in M$ such that $a \notin I + I^{-1}$. Then $I + I^{-1} \subset (a)$, hence $(I + I^{-1})^v \subset (a) \subsetneqq V$; a contradiction.

Choose $s \in I - P$, and set $J = s + I^{-1}$. J is an ideal of V. If $J \subset P$, then $s \in s + I^{-1} \subset P$; a contradiction. Hence $P \subsetneqq J$. Then we have $(I/P + J/P)^v = (\frac{I+J}{P})^v = \frac{s+V}{P} = s + V/P$. Hence I/P is v-invertible.

The proof of the necessity is similar. □

Proof of Theorem 1.3′. We will use Lemmas 3.6 and 3.7. Let I be a v-invertible v-ideal of V. Then $I + I^{-1} \supset M$ by a proof similar to that of Lemma 3.7. Since $P \subsetneqq I + I^{-1}$, there is $j \in I^{-1}$ such that $P \subsetneqq I + j$. Set $I + j = J$. Since J is a v-invertible v-ideal of V, J/P is a v-invertible v-ideal of V/P. For the element $[I] \in \mathrm{Cl}^v(V)$, we set $\varphi([I]) = [J/P] \in \mathrm{Cl}^v(V/P)$.

- Then φ is well-defined.

  For, let $I, J \in \mathrm{Inv}^v(V)$ such that $P \subsetneqq I \subset V$ and $P \subsetneqq J \subset V$. Assume that $[I] = [J]$. We may assume that $I = t + J$ for some $t \in V$. Then $t \notin P$, and then $I/P = \frac{t+J}{P} = t + J/P$.

- φ is injective.

  For, assume that $[I/P] = [J/P]$. We may assume there $I/P = t + J/P$ for some $t \in V/P$. Then $I = t + J$.

- φ is a mapping onto $\mathrm{Cl}^v(V/P)$.

  For, let $I/P \in \mathrm{Inv}^v(V/P)$ such that $P \subsetneqq I \subset V$. Then $I \in \mathrm{Inv}^v(V)$.

- φ is a homomorphism.

  For, let $I, I' \in \mathrm{Inv}^v(V)$ such that $P \subsetneqq I \subset V$ and $P \subsetneqq I' \subset V$. Then we have $\frac{(I+I')^v}{P} = (\frac{I+I'}{P})^v = (I/P + I'/P)^v$. □

Lemma 3.8. *Let $\star$ be semistar operation on a g-monoid S.*

(1) ([4], [14, §1, (2.6)(2)]) *The mapping $\tilde{\star} : E \longmapsto \cup\{(E : I) \mid I$ is a finitely generated ideal of S with $I^\star \ni 0\}$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation on S.*

(2) ([4, Corollary 3.9.(2)], [14, §1, (2.11)(2)]) $\star = \tilde{\star}$ *if and only if $\star$ is stable of finite type.*

(3) ([6, Corollary 2.7], [13, §2, (1.8)]) *If $\star$ is of finite type, then, for every $E \in \bar{\mathrm{F}}(S)$, $E^{\tilde{\star}} = \cap\{E + S_P \mid P \in \mathrm{QMax}^\star(S)\}$.*

Lemma 3.9 ([10, Proposition 6.2] The Dedekind–Mertens lemma for semigroups). *Let D be a domain with quotient field K, and let S be a g-monoid with quotient group G.*

For every $f, g \in K[X;G] - \{0\}$*, there is a positive integer* m *such that* $me_S(f) + e_S(fg) = (m+1)e_S(f) + e_S(g)$.

If every finitely generated ideal of an integral domain D is principal, then D is called a Bézout domain.

Lemma 3.10 ([16, Lemma 13]). *If every finitely generated ideal of a g-monoid* S *is principal, then* S *is a valuation semigroup.*

Lemma 3.11 ([8, Corollary 28.5, Theorem 28.6]). *Let* D *be an integral domain, and let* $G = q(S)$*. The following statements are equivalent:*

(1) S *is a valuation semigroup.*

(2) $e_S(fg) = e_S(f) + e_S(g)$ *for every* $f, g \in D[X;G] - \{0\}$.

Proof. (1) $\Longrightarrow$ (2). By Lemma 3.9.

(2) $\Longrightarrow$ (1). Let $s, t \in S$ with $s \neq t$. Set $f = X^s + X^t$ and $g = X^s - X^t$. Then $2(s,t) = (2s, 2t)$ by the assumption. Then $s + t \in (2s, 2t)$. It follows that $(s,t) = (s)$ or $(s,t) = (t)$. Lemma 3.10 completes the proof. □

Proof of Theorem 1.1′. For every $I \in \bar{\mathrm{F}}(S)$, $I^\star = \cap\{I + S_P \mid P \in \mathrm{QMax}^\star(S)\}$ by Lemma 3.8. Then easily we have $I^\star + S_P = I + S_P$ for every $P \in \mathrm{QMax}^\star(S)$.

(1) $\Longrightarrow$ (2). Let $f, g \in D[X;G] - \{0\}$, and let $P \in \mathrm{QMax}^\star(S)$. Then we have $e_{S_P}(fg) = e(fg) + S_P = (e(fg))^\star + S_P = (e(f) + e(g))^\star + S_P = (e(f) + e(g)) + S_P = e_{S_P}(f) + e_{S_P}(g)$.

By Lemma 3.11, we have that S_P is a valuation semigroup.

(2) $\Longrightarrow$ (3). Let $I \in \mathrm{f}(S)$. Then $(I + I^{-1})^\star = \cap\{(I + I^{-1}) + S_P \mid P \in \mathrm{QMax}^\star(S)\} = \cap\{I + S_P + I^{-1} + S_P \mid P \in \mathrm{QMax}^\star(S)\} = \cap\{I + S_P + (I + S_P)^{-1} \mid P \in \mathrm{QMax}^\star(S)\} = \cap\{S_P \mid P \in \mathrm{QMax}^\star(S)\} = S^\star$.

(3) $\Longrightarrow$ (1). Let $f, g \in D[X;G] - \{0\}$. By Lemma 3.9, there is a positive integer m such that $me(f) + e(fg) = (m+1)e(f) + e(g)$. Since $(e(f) + e(f)^{-1})^\star = S^\star$, we have $e(fg)^\star = (e(f) + e(g))^\star$.

(4) $\Longrightarrow$ (1). The proof follows from Lemma 3.9.

(3) $\Longrightarrow$ (4). Straightforward. □

Lemma 3.12. *The following hold:*

(1) *If* $I^v = I \in \mathrm{F}(D)$*, then* $(ID[X;S])^v = ID[X;S]$.

(2) *Let* $I^v = I \in \mathrm{F}(D)$*, and let* $b \in K - \{0\}$*. Then* $II^{-1} \subset (b)$ *if and only if* $I \subset Ib$.

(3) *Let* $I^v = I \in \mathrm{F}(D)$*, and let* $\varphi \in \mathrm{q}(D[X;S]) - \{0\}$*. Then*

$$ID[X;S](ID[X;S])^{-1} \subset \varphi D[X;S] \text{ if and only if } ID[X;S] \subset ID[X;S]\varphi.$$

(4) *Let* $I \in \mathrm{Inv}^v(D)$*, and set* $\alpha(I) = ID[X;S]$*. Then* $\alpha(I) \in \mathrm{Inv}^v(D[X;S])$.

(5) *Let* $I \in \mathrm{Inv}^v(D)$, $[I] \in \mathrm{Cl}^v(D)$, *and set* $\bar{\alpha}([I]) = [ID[X;S]] \in \mathrm{Cl}^v(D[X;S])$. *Then the mapping* $\bar{\alpha}$: $\mathrm{Cl}^v(D) \longrightarrow \mathrm{Cl}^v(D[X;S])$, $[I] \longmapsto [ID[X;S]]$ *is well-defined.*

(6) $\bar{\alpha}$ *is an injection.*

Proof. (1) There is a subset $\{a_\lambda \mid \lambda \in \Lambda\} \subset K$ such that $I = \cap_\lambda (a_\lambda)$. Then $ID[X;S] = \cap_\lambda a_\lambda D[X;S]$.

(2) If $II^{-1} \subset (b)$, then $(I\frac{1}{b})I^{-1} \subset D$. Hence $I\frac{1}{b} \subset I^v = I$, and hence $I \subset Ib$. The converse is similar.

(3) We have $(ID[X;S])^v = ID[X;S]$ by (1). Then (2) completes the proof.

(4) We have $(ID[X;S])^v = ID[X;S]$ by (1).

Let $ID[X;S](ID[X;S])^{-1} \subset \varphi D[X;S]$ with $\varphi \in \mathrm{q}(D[X;S])$. Set $\frac{1}{\varphi} = F$. Then $IF \subset ID[X;S]$ by (3). Hence $F \in K[X;S]$. Set $F = \sum b_i X^{s_i}$ with $b_i \in K - \{0\}$ and $s_i \neq s_j$ for $i \neq j$. Since $IF \subset ID[X;S]$, we have $Ib_i \subset I$ for every i. Hence $II^{-1} \subset (\frac{1}{b_i})$ for every i by (2). Since $(II^{-1})^v = D$, we have $(\frac{1}{b_i}) \ni 1$, and hence $b_i \in D$. Hence $\varphi D[X;S] = \frac{1}{F} D[X;S] \ni 1$, and hence $(ID[X;S](ID[X;S])^{-1})^v = D[X;S]$.

(5) Let $I, J \in \mathrm{Inv}^v(D)$ such that $[I] = [J]$. Easily we have $J = Ib$ for some $b \in K - \{0\}$. It follows that $[JD[X;S]] = [ID[X;S]b] = [ID[X;S]]$.

(6) Let $I, J \in \mathrm{Inv}^v(D)$ such that $[ID[X;S]] = [JD[X;S]]$. Then $JD[X;S] = ID[X;S]\varphi$ for some $\varphi \in \mathrm{q}(D[X;S])$. Since $I\varphi \subset JD[X;S]$, we have $\varphi \in K[X;S]$. Since $J \subset ID[X;S]\varphi$, we have $\varphi \in K - \{0\}$. Set $\varphi = b$. Then $JD[X;S] = ID[X;S]b$, hence $I = Jb$, and hence $[J] = [I]$. □

Lemma 3.13. *For every* $I \in \mathrm{F}(D)$, *we have* $(ID[X;S])^v = I^v D[X;S]$.

Proof. Since $ID[X;S] \subset I^v D[X;S]$, we have $(ID[X;S])^v \subset I^v D[X;S]$ by Lemma 3.12(1). Let $ID[X;S] \subset \varphi D[X;S]$ with $\varphi \in \mathrm{q}(D[X;S])$.

To prove $I^v D[X;S] \subset (ID[X;S])^v$, we need to show the inclusion $I^v D[X;S] \subset \varphi D[X;S]$. Set $\frac{1}{\varphi} = F$. Then we have $F \in K[X;S]$. Let $F = \sum b_i X^{s_i}$ with every $b_i \in K - \{0\}$ and $s_i \neq s_j$ for $i \neq j$. Since $IF \subset D[X;S]$, we have $Ib_i \subset D$, hence $b_i \in I^{-1}$. Since $I^v I^{-1} \subset D$, we have $I^v F \subset D[X;S]$, hence $I^v \subset \varphi D[X;S]$, and hence $I^v D[X;S] \subset \varphi D[X;S]$. □

Let $I, J \in \mathrm{Inv}^v(D)$. By Lemma 3.13, we have

$$(ID[X;S]JD[X;S])^v = (IJD[X;S])^v = (IJ)^v D[X;S].$$

Therefore $\bar{\alpha}$ is a homomorphism. The proof of Proposition 2.2 is complete.

Since $D[X;\mathbb{Z}_0] = D[X]$, Corollary 2.3 follows from Proposition 2.2.

Lemma 3.14. *The following hold:*

(1) *If* $I^v = I \in \mathrm{F}(S)$, *then* $(I + S[X])^v = I + S[X]$.

(2) *Let* $I^v = I \in \mathrm{F}(S)$, *and let* $b \in G$. *Then* $I + I^{-1} \subset (b)$ *if and only if* $I \subset I + b$.

(3) *Let* $I^v = I \in \mathrm{F}(S)$*, and let* $\varphi \in \mathrm{q}(S[X])$*. Then* $I + S[X] + (I + S[X])^{-1} \subset \varphi + S[X]$ *if and only if* $I + S[X] \subset (I + S[X]) + \varphi$*.*

(4) *Let* $I \in \mathrm{Inv}^v(S)$*, and set* $\alpha(I) = I + S[X]$*. Then* $\alpha(I) \in \mathrm{Inv}^v(S[X])$*.*

(5) *Let* $I \in \mathrm{Inv}^v(S)$*, let* $[I] \in \mathrm{Cl}^v(S)$*, and set* $\bar{\alpha}([I]) = [I + S[X]] \in \mathrm{Cl}^v(S[X])$*. Then the mapping* $\bar{\alpha}\colon \mathrm{Cl}^v(S) \longrightarrow \mathrm{Cl}^v(S[X])$*,* $[I] \longmapsto [I + S[X]]$ *is well-defined.*

(6) $\bar{\alpha}$ *is an injection.*

(7) $\bar{\alpha}$ *is a homomorphism.*

The proof is similar to those for Lemmas 3.12 and 3.13.

Lemma 3.15. $\bar{\alpha}$ *is a mapping onto* $\mathrm{Cl}^v(S[X])$.

Proof. Let $\mathrm{Inv}^v(S[X]) \ni J$ with $J \subset S[X]$, let $J = \{a_\alpha + k_\alpha X \mid \alpha \in A\}$ with $\min_\alpha k_\alpha = 0$, and let $I = \{a_\alpha \mid \alpha \in A\}$. Then I is an ideal of S. It is sufficient to show that $I \in \mathrm{Inv}^v(S)$ and $\alpha(I) = J$. Set $\{x \in G \mid I \subset S + x\} = \{x_\lambda \mid \lambda \in \Lambda\}$. Then we have $I^v = \cap_\lambda(S + x_\lambda)$. Set $L = \{x + lX \in \mathrm{q}(S[X]) \mid J \subset x + lX + S[X]\}$, and set $R = \{x_\lambda + lX \mid \lambda \in \Lambda, l \leq 0\}$.

We have $L = R$. For, let $x + lX \in L$. For every α, we have $a_\alpha + k_\alpha X \in J \subset x + lX + S[X]$, hence $a_\alpha \in x + S$. Hence $I \subset S + x$, and hence $x = x_\lambda$ for some λ. Therefore $L \subset R$. Conversely, let $x_\lambda + lX \in R$. Note that $l \leq 0$. Since $I \subset S + x_\lambda$, we have $J \subset I + S[X] \subset x_\lambda + S[X] \subset x_\lambda + lX + S[X]$. Hence $x_\lambda + lX \in L$. Therefore, $R \subset L$, and hence $L = R$.

- We have $J = \cap_\lambda(x_\lambda + S[X])$.
  For, $J = J^v = \cap\{x + lX + S[X] \mid x + lX \in L\} = \cap_\lambda(x_\lambda + S[X])$.

- $I^v = I$.
  For, let $x \in I^v$. Then $x \in S + x_\lambda$ for every λ. It follows that $x \in J$. Then $x = a_\alpha + k_\alpha X$ for some α. Hence $x = a_\alpha \in I$.

- $J = I + S[X]$.
  For, let $a \in I$. Since $a \in S + x_\lambda$ for every λ, we have $a \in \cap_\lambda(x_\lambda + S) \subset J$. Hence $I \subset J$.

- We have $(I + I^{-1})^v = S$.
  For, let $I + I^{-1} \subset b + S$ for $b \in G$. Then $I \subset I + b$ by Lemma 3.14(2), hence $I + S[X] \subset I + S[X] + b$, and hence $J + J^{-1} \subset b + S[X]$ by Lemma 3.14(3). Since $(J + J^{-1})^v = S[X]$, we have $0 \in b + S[X]$, hence $-b \in S$. Therefore $(I + I^{-1})^v = S$.

We have proved that $I \in \mathrm{Inv}^v(S)$ and $\alpha(I) = J$. □

The proof of Proposition 2.4 is complete.

The proof of Corollary 2.5 follows from Theorem 1.2′ and Proposition 2.4.

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Schubert varieties with inequidimensional singular locus

S. B. Mulay

Abstract. We present a class of determinantal ideals (of mixed type) whose singular loci fail to be equidimensional (with arbitrarily large dimension-gap). Since these ideals are the defining ideals of a class of Schubert varieties, we get a family of Schubert varieties (as subvarieties of a variety of full flags or of a Grassmannian) possessing inequidimensional singular locus.

Keywords. Schubert varieties, singularities.

AMS classification. 14M15, 14M18.

Examples of Schubert varieties whose singular locus consists of components of differing dimensions, have been of some interest (see [3]). The author of this article discovered a class of such examples, well over a dozen years ago, as a consequence of his investigations related to the intersections of Schubert subvarieties of the (classical) space of full flags. Since then, the existence of these examples has been orally communicated by the author in professional meetings etc (most recently in Abhyankar's 70th birthday conference, July 2000) but the details remained unpublished as yet. This article, it is hoped, will serve as a convenient reference.

In this article, k is assumed to be an integral domain and $X := [X_{ij}]$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, is assumed to be an $m \times n$ matrix whose entries are indeterminates over k. By $k[X]$ we mean the polynomial ring over k in the mn indeterminates X_{ij}. Let Y be the $m \times t$ submatrix of X consisting of the first t columns of X. Let p be a positive integer not exceeding the minimum of m, n and let $I(p, X)$ denote the ideal of $k[X]$ generated by all the $p \times p$ minors of X. It is well known that $I(p, X)$ is a prime ideal of $k[X]$ of height $(m - p + 1)(n - p + 1)$ (see [1] or [2]). More generally, the results illustrated in the introduction of [5] (which are consequences of Theorem 4 of that article) yield the following as a special case.

Theorem. *Assume q is a positive integer with $q \leq \min\{t, p\}$. Then $I(q, Y) + I(p, X)$ is a prime ideal of $k[X]$. Furthermore, if $(p-q) < (n-t)$, then the height of $I(q, Y) + I(p, X)$ is $(m - p + 1)(n - p + 1) + (p - q)(t - q + 1)$.*

In fact, the results of [5] also relate $I := I(q, Y) + I(p, X)$ to a Schubert variety by the results of [4]. This relationship is important for our considerations in this article. We proceed to provide a brief description of this relationship through a concrete example in which X has size 5×6, submatrix Y has size 5×3, $p = 4$ and $q = 2$. Temporarily assume k to be an algebraically closed field and let V denote the (affine) variety defined by $I := I(2, Y) + I(4, X)$ in the affine space over k whose coordinate ring is $k[X]$. Let $\mathrm{FL}(11, k)$ be the variety of full-flags on an 11-dimensional k-vector space and let σ denote the permutation of $\{1, 2, \dots, 11\}$ defined by

$$\sigma := (11, 6, 5, 10, 9, 4, 8, 7, 3, 2, 1)$$

where the vector on the right has $\sigma(i)$ as its i-th (from the left) component. Fix a Bruhat decomposition of $\mathrm{FL}(11, k)$. Then the Schubert variety C_σ corresponding to σ intersects the 'Opposite big cell' in an affine variety V_σ. The varieties V and V_σ are related by the equation

$$V_\sigma = V \times \mathbb{A}_k^{25}.$$

Thus, the singular locus of C_σ can be studied by investigating the singular locus of V. In passing we mention that expert readers will have no difficulty relating V to a Schubert subvariety of the Grassmannian of 6-dimensional subspaces of an 11-dimensional vector space over k.

Definitions. Let X and Y be as above. Assume

$$2 \leq p \leq \min\{m, n\} \quad \text{and} \quad 1 \leq t \leq n - p + 1.$$

(i) Given a subset S of the entries of X, let $k[S]$ denote the subring of $k[X]$ obtained by adjoining members of S to k. If A is a submatrix of X, then we also use A to denote (by abuse of notation) the set of entries of A. With this convention, given a sequence $A_1, \dots, A_r$ where each A_i is either a submatrix of X or a subset of the entries of X, by $k[A_1, \dots, A_r]$ we mean $k[A_1 \cup \cdots \cup A_r]$.

(ii) For each integer λ with $1 \leq \lambda < n$ define W_λ to be the $m \times (n - \lambda)$ submatrix of X consisting of the last $n - \lambda$ columns of X.

(iii) Define $Z := W_{n-p+1}$ and let D denote the $(p - 1) \times (p - 1)$ minor of Z determined by the first $(p - 1)$ rows of Z.

(iv) Let T_s denote the $(p - 1) \times t$ submatrix of X determined by the first $(p - 1)$ rows and the first s columns of X. Define $T := T_{n-p+1}$.

(v) For each (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$ let $N(i, j)$ be the $p \times p$ matrix determined by the row-sequence $\{1, \dots, p - 1, i\}$ and the column-sequence $\{j, (n - p + 2), \dots, n\}$ of X.

(vi) For $p \leq i \leq m$ let $H_1(i), \dots, H_{p-1}(i)$ be the (uniquely determined) homogeneous polynomials of degree $p - 1$ in $k[Z]$ such that given any j with $1 \leq j \leq n - p + 1$

$$\det N(i, j) = X_{ij} D - \left(\sum_{1 \leq r \leq p-1} X_{rj} H_r(i) \right)$$

where det stands for the determinant.

(vii) For each (i, j) with $p \leq i \leq m$ and $1 \leq j \leq s$, define

$$U_{ij} := X_{ij} - \left(\sum_{1 \leq r \leq p-1} X_{rj} \frac{H_r(i)}{D} \right).$$

Let U_s denote the set of these U_{ij} and $U := U_{n-p+1}$.

(viii) Define

$$h : k[X][1/D] \to k[X][1/D]$$

to be the k-homomorphism of rings such that if $p \le i \le m$ as well as $1 \le j \le n - p + 1$, then

$$h(X_{ij}) = \sum_{1 \le r \le p-1} X_{rj} \frac{H_r(i)}{D}$$

and for all other (i, j) we have $h(X_{ij}) = X_{ij}$.

The following Lemma 1 is basic in nature and crucial for our main results. Although most assertions of this lemma are known, their proofs are fairly short (some differing in details from those in the literature) and hence by providing them here, we wish to make our article more self-contained as well as easier to follow.

Lemma 1. *Along with the above notation, suppose q is a positive integer such that $q \le \min\{t, m\}$.*

(i) $k[X][1/D] = k[U, T, Z][1/D]$.

(ii) *There exists a $(m - p + 1) \times (p - 1)$ matrix $A := [A_{ij}]$ with $DA_{ij} \in k[Z]$ such that*

$$X \equiv \begin{bmatrix} I \\ A \end{bmatrix} T_n \mod Uk[X][1/D]$$

where (the block) I is the $(p - 1) \times (p - 1)$ identity matrix.

(iii) $I(q, Y)k[X][1/D] \subseteq U_t k[X][1/D] + I(q, T_t)k[X][1/D]$.

(iv) *Homomorphism h maps $k[X][1/D]$ onto $k[T, Z][1/D]$. Furthermore, the kernel of h is given by*

$$\operatorname{Ker} h = I(p, X)k[X][1/D] = Uk[X][1/D].$$

(v) *If $q \le p$ and $I := I(q, Y) + I(p, X)$, then*

$$h(I) = I(q, T_t)k[T, Z][1/D]$$

and $Ik[X][1/D]$ is a prime ideal of height

$$\operatorname{ht} I = (m - p + 1)(n - p + 1) + (p - q)(t - q + 1).$$

(vi) *There exists a $(p - 1) \times (n - p + 1)$ matrix $C := [C_{ij}]$ with $DC_{ij} \in k[T_n] \subset k[T, Z]$ and*

$$h(X) = Z[C \mid I]$$

where (the block) I is the $(p - 1) \times (p - 1)$ identity matrix.

(vii) *Let* $2 \leq e$ *be a positive integer. Then*

$$I(e, X) \subset I(e-1, x)^{(2)}.$$

Furthermore, the singular locus of $I(e, X)$ *is defined by* $I(e-1, X)$.

Proof. Assertion (i) is straightforward to verify. To prove (ii) observe that for each (i, j) with $p \leq i \leq m$ and $n-p+2 \leq j \leq n$ we have

$$0 = X_{ij} D - \left(\sum_{1 \leq r \leq p-1} X_{rj} H_r(i) \right)$$

since the sum on the right is the determinant of $N(i, j)$ and in this case $N(i, j)$ has two identical columns. Define

$$A_{ij} := \frac{H_i(p-1+j)}{D}$$

for $1 \leq i \leq m-p+1$ and $1 \leq j \leq p-1$. Clearly,

$$\det N(i, j) \equiv 0 \mod Uk[X][1/D]$$

for $p \leq i \leq m$ and $1 \leq j \leq n-p+1$. Hence (ii) follows.

To prove (iii) it suffices to show that given a $q \times q$ minor μ of Y the image μ is in the ideal generated by $I(q, T_t)$ modulo $Uk[X][1/D]$. From (ii) it follows that modulo $Uk[X][1/D]$ the rows of Y are linear combinations of the rows of T_t. Thus μ is easily seen to be in the ideal generated by the $I(q, T_t)$ modulo $Uk[X][1/D]$.

Let $h(X)$ denote the matrix whose ij-th entry is $h(X_{ij})$. Since U is contained in the kernel of h, assertion (ii) implies that $h(X)$ has rank at most $p-1$. Hence Ker(h) contains $I(p, X)$. Clearly, h is surjective onto $k[T, Z][1/D]$ and

$$Uk[X][1/D] \subseteq I(p, X)k[X][1/D].$$

Using (i) we get the asserted equality. In particular $h(X)$ has rank $p-1$. This establishes (iv).

Since $I(q, T_t)k[T, Z][1/D]$ is a prime ideal of height $(p-q)(t-q+1)$ assertion (v) follows from assertions (iii) and (iv).

For each j with $1 \leq j \leq n-p+1$ let $G_{n-p+2}(j), \ldots, G_n(j)$ be the (uniquely determined) homogeneous polynomials of degree $p-1$ in $k[T_n]$ such that given any i with $p \leq i \leq m$

$$\det N(i, j) = X_{ij} D - \left(\sum_{n-p+2 \leq r \leq n} X_{ir} G_r(j) \right).$$

Then, for each i with $1 \leq i \leq p-1$ the sum on the right is 0 since in this case $N(i, j)$ has two identical rows (and the sum is equal to the determinant of $N(i, j)$). Now define

$$C_{ij} := \frac{G_{n-p+i+1}(j)}{D}$$

for $1 \leq i \leq p-1$ and $1 \leq j \leq n-p+1$. From the very definition of h we have $\det h(N(i,j)) = 0$ for $p \leq i \leq m$ and $1 \leq j \leq n-p+1$. Thus (vi) follows.

Lastly, we prove (vii). If $e = 2$, then it is clear that $I(2, X) \subset I(1, X)^2$. Henceforth assume $3 \leq e$. Let M be an $e \times e$ submatrix of X. We claim that $\mu := \det M$ is in the symbolic square of the prime ideal $I(e-1, X)$. Since this claim is invariant under permutations of the rows and the columns of X, we may assume, without loss, M to be the submatrix determined by the first e rows and the last e columns of X. Letting $p = e-1$ in (iii) we deduce that $I(e-1, X)k[X][1/D] = Uk[X][1/D]$. For each (i, j) with $e-1 \leq i \leq e$ and $n-e+1 \leq j \leq n$ if the entry X_{ij} in M is replaced by

$$x_{ij} := X_{ij}D - \Bigg(\sum_{1 \leq r \leq e-2} X_{rj}H_r(i)\Bigg),$$

then the determinant of the resulting $e \times e$ matrix is $D^2\mu$. On the other hand, for each $n-e+3 \leq j \leq n$ we have $x_{ij} = 0$ since it is a determinant of $p-1 \times p-1$ matrix with a repeated column. Consequently,

$$D^2\mu = \pm D^3(U_{a\alpha}U_{b\beta} - U_{a\beta}U_{b\alpha})$$

where $(a, b) := (e-1, e)$ and $(\alpha, \beta) := (n-e+1, n-e+2)$. Hence μ is in $I(e-1, X)^{(2)}$ and thereby we have shown that $I(e-1, X)$ is in the singular locus of $I(e, X)$. Conversely, let $\operatorname{sing} I(e, X) \subset \operatorname{Spec} k[X]$ denote the singular locus of $I(e, X)$. Suppose $\operatorname{sing} I(e, X)$ has a prime not containing $I(e-1, X)$. Since $\operatorname{sing} I(e, X)$ remains invariant under permutations of rows and columns of X, it must also have a prime not containing E where E denotes the $e-1 \times e-1$ minor of X determined by the first $e-1$ rows and the last $e-1$ columns of X. But such a prime is necessarily in the singular locus of $I(e, X)k[X][1/E]$. By letting $p = e$ in assertions (ii) and (iii) we deduce that the singular locus of $I(e, X)k[X][1/E]$ is empty. In other words, each member of $\operatorname{sing} I(e, X)$ contains $I(e-1, X)$. □

Lemma 2. *We continue to use all the above notation. Again, $I := I(q, Y) + I(p, X)$ and Y is assumed to have at most $n-p+1$ columns, i.e. $t \leq n-p+1$.*

(i) *Assume $2 \leq q$ and e is a positive integer. Let Δ denote the $(q-1) \times (q-1)$ minor of Y determined by the first $q-1$ rows and the last $q-1$ columns of Y (i.e. columns $t-q+2, \ldots, t$). Then,*

$$I(e, X)k[X][1/\Delta] \subseteq (I(q, Y) + I(e, W_{t-q+1}))k[X][1/\Delta].$$

(ii) *Assume either $2 \leq q \leq p$ or $1 \leq q < p$. Then, the prime ideal*

$$Q := I(q, Y) + I(p-1, X)$$

is in the singular locus of I. Moreover, if $1 < q < p$ and J is a prime ideal (of $k[X]$) in the singular locus of I not containing Δ, then $Q \subseteq J$.

(iii) *Assume* $1 < q < p$. *Then, the prime ideal*

$$P := I(q-1, Y) + I(p, X)$$

is in the singular locus of I. *Moreover, the singular locus of* I *is defined by* $P \cap Q$.

Proof. Let S_l denote the submatrix of Y determined by the first $q-1$ rows and the first l columns and let $S := S_{t-q+1}$. If X, D, p are replaced by Y, Δ, q respectively, in assertions (iv) and (vi) of Lemma 1, then it follows that

$$R := \frac{k[X][1/\Delta]}{I(q,Y)k[X][1/\Delta]} = k[S, W_{t-q+1}][1/\Delta]$$

and the first $t-q+1$ columns of Y are $k[S_t]$-linear combinations of its last $q-1$ columns. Consequently, modulo $I(q,Y)k[X][1/\Delta]$, an $e \times e$ minor of X is in the ideal

$$I(e, W_{t-q+1})k[S, W_{t-q+1}][1/\Delta].$$

This proves (i).

By the theorem quoted above, Q is a prime ideal of $k[X]$. If $2 \leq q = p$, then $I = I(p, X)$ and $Q = I(p-1, X)$. In this case (vii) of Lemma 1 implies that Q is in the singular locus of I. If $1 = q < p$, then $I = I(1, Y) + I(p, W_t)$, $Q = I(p-1, W_t)$ and $k[X] = k[Y, W_t]$. In this case too, (vii) of Lemma 1 shows Q to be in the singular locus of I. Suppose $1 < q < p$. Now, Δ is obviously not in Q. Let R be the factor ring as in the proof of (i) and let $\overline{Q}, \overline{I}$ denote the canonical images of Q, I (respectively) in R. Applying (i) with $e = p-1, p$ we deduce the equalities

$$\overline{Q} = I(p-1, W_{t-q+1})k[S, W_{t-q+1}][1/\Delta],$$
$$\overline{I} = I(p, W_{t-q+1})k[S, W_{t-q+1}][1/\Delta].$$

Since $2 \leq p$, assertion (vii) of Lemma 1 implies that $\overline{Q}$ defines the singular locus of $\overline{I}$. Hence Q defines the singular locus of I away from Δ *i.e.* given a prime ideal $J \supset I$ (of $k[X]$) not containing Δ, the singular locus of I contains J if and only if $Q \subseteq J$. This establishes (ii).

Lastly we prove (iii). Again, by the theorem quoted above, P is a prime ideal of $k[X]$. Since $t \leq n-p+1$, the minor D is not an element of P. Thus it suffices to show that $Pk[X][1/D]$ is in the singular locus of $Ik[X][1/D]$. By assertion (v) of Lemma 1, $h(I) = I(q, T_t)k[T, Z][1/D]$ and replacing q by $q-1$ in that same assertion we get $h(P) = I(q-1, T_t)k[T, Z][1/D]$. It follows from (vii) of Lemma 1 that $h(P)$ is in the singular locus of $h(I)$. Hence P is in the singular locus of I. Next let J be a prime ideal contained in the singular locus of I such that J does not contain P. Clearly, J does not contain $I(q-1, Y)$. We claim that $Q \subseteq J$. Say ν is a $(q-1) \times (q-1)$ minor of Y which is not in J. Since $I(p-1, X), I(p, X), I(q-1, Y), I(q, Y)$ all are invariant under any permutation of the rows of X and also under any permutation of the columns of Y, we may assume $\nu = \Delta$ without any loss. But then our claim follows from (ii). □

Theorem 1. *Suppose* p, q *are positive integers such that*

(a) $1 < q \leq t \leq n - p + 1$ *and*

(b) $q < p \leq \min\{m, n\}$.

As above, let $I := I(q, Y) + I(p, X)$*,* $P := I(q-1, Y) + I(p, X)$ *and* $Q := I(q, Y) + I(p-1, X)$*. Then,*

(i) I, P, Q *are prime ideals of* $k[X]$ *having heights*

$$\begin{aligned}\operatorname{ht} I &= (m-p+1)(n-p+1) + (p-q)(t-q+1),\\ \operatorname{ht} P &= (m-p+1)(n-p+1) + (p-q+1)(t-q+2),\\ \operatorname{ht} Q &= (m-p+2)(n-p+2) + (p-q-1)(t-q+1)\end{aligned}$$

respectively.

(ii) *The (reduced) singular locus of* I *is defined by* $P \cap Q$*.*

(iii) *If* $(m+n) \neq [2t + 3(p-q)]$*, then the singular locus of* I *is not equidimensional. In fact, by choosing* m, n, t, p, q *appropriately the difference between the heights of* P *and* Q *can be made arbitrarily large.*

Proof. Our hypotheses (a), (b) allow us to apply Lemma 2 thereby deducing (i) as well as (ii). The first half of (iii) is straightforward to verify. Let d be a positive integer. Set $m = d + 3$, $n = 6$, $t = 3$, $p = 4$ and $q = 2$. Then the above hypotheses on m, n, t, p, q are satisfied. By (i), heights of P, Q are $3d + 9$, $4d + 6$ respectively. Hence, choosing d to be arbitrarily large, the difference between the heights of P, Q can be made as large as desired. □

Definition. Let r, m, N be positive integers such that

$$m \geq r + 3 \text{ and } N \geq m + r + 5.$$

(i) Let σ be a map from the set $[1, N] := \{1, \ldots, N\}$ into itself given by

$$\sigma(i) := \begin{cases} N + 1 - i & \text{if } 1 \leq i \leq r,\\ N + 1 + r - m - i & \text{if } r + 1 \leq i \leq r + 2,\\ N + 3 - i & \text{if } r + 3 \leq i \leq r + 4,\\ N - m - 2 & \text{if } i = r + 5,\\ N + 4 - i & \text{if } r + 6 \leq i \leq m + 3,\\ N + 1 - i & \text{if } m + 4 \leq i \leq N.\end{cases}$$

(ii) Define $q := r + 1$, $t := r + 2$, $p := r + 3$ and $n := r + 5$.

(iii) For an algebraically closed field k identify $k[X]$ as the affine coordinate ring of $\mathbb{A}_k^{mn}$ and let V be the subvariety of this affine space defined by the ideal

$$I := I(r + 1, Y) + I(r + 3, X).$$

(iv) For an algebraically closed field k let $\mathrm{FL}(N, k)$ denote the (smooth) variety of full flags on the k-vector-space k^N. Fix a Bruhat decomposition of $\mathrm{FL}(N, k)$ and for each permutation θ of $[1, N]$ let $X(\theta)$ denote the Schubert subvariety of $\mathrm{FL}(N, k)$ corresponding to θ.

Theorem 2. *For each positive integer r the above defined map σ is a permutation of $[1, N]$. The singular locus of $X(\sigma)$ has exactly two irreducible components whose dimensions differ by $m - r - 5$. In particular, the singular locus of $X(\sigma)$ is inequidimensional.*

Proof. It is easy to verify that for each positive integer r, the map σ is a permutation of $[1, N]$. Let W denote the 'Opposite big cell' (with respect to our fixed Bruhat decomposition) of $\mathrm{FL}(N, k)$. In this situation, Theorem 4 (or Lemma 10) of [5] asserts that the affine variety $X(\sigma) \cap W$ is isomorphic to the product of V with the $(N(N-1)/2 - mn)$-dimensional affine space over k. Since m, n, t, q, p are easily seen to satisfy requirements (a), (b) of Theorem 1, our assertion follows from Theorem 1. □

Remarks. (i) In the proof of Theorem 2 along with referring to Theorem 4 of [5], the reader may find it helpful to refer also to (2.9) of [4].

(ii) It is possible to replace $\mathrm{FL}(N, k)$ by the Grassmannian of n-dimensional subspaces of k^N and take the Schubert subvariety corresponding to σ in that Grassmannian.

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On Matlis domains and Prüfer sections of Noetherian domains

B. Olberding

Abstract. The class of Matlis domain, those integral domains whose quotient field has projective dimension 1, is surprisingly broad. However, whether every domain of Krull dimension 1 is a Matlis domain does not appear to have been resolved in the literature. In this note we construct a class of examples of one-dimensional domains (in fact, almost Dedekind domains) that are overrings of $K[X, Y]$ but are not Matlis domains. These examples fit into a larger context of what we term "Prüfer sections" of Noetherian domains, a notion we also consider briefly, and with emphasis on Prüfer sections of two-dimensional Noetherian domains.

Keywords. Matlis domain, Prüfer domain, almost Dedekind domain.

AMS classification. 13F05.

1 Introduction

In the monograph [1], Fuchs and Salce remark that, "In the category of R-modules, there exists one module which has a tremendous influence on the entire category once its projective dimension is declared to be ≤ 1: this is nothing else than Q, the quotient field of R." A domain whose quotient field has projective dimension 1 is said to be a *Matlis domain*, in honor of Eben Matlis, who was the first to recognize the powerful consequences of this assumption. Matlis domains have been well-studied; see [1] for an overview. As we recall in Section 3, a remarkable theorem of Lee shows that these domains can be characterized in an entirely non-homological way by the property that Q/R is the direct sum of countably generated submodules. Thus examples of Matlis domains include countable domains, one-dimensional Noetherian domains, and integral domains having a nonzero element contained in every nonzero prime ideal. However, there does not seem to appear in the literature an example of a domain of Krull dimension 1 that is not a Matlis domain. In this note we exhibit a class of such examples. Our examples are in fact *almost Dedekind domains*, meaning that each localization at a maximal ideal is a Dedekind domain (equivalently, a rank one discrete valuation ring, or *DVR*). Our examples are also overrings of $K[X, Y]$, where K can be chosen to be any uncountable non-algebraically closed field. The rings are instances of what we term "Prüfer sections" of Noetherian domains. More precisely, they are Prüfer domains formed by taking the intersection of valuation overrings of a Noetherian domain, each centered over a different maximal ideal of D. In Corollary 3.5 we give a somewhat organic construction of a Prüfer section of $K[X, Y]$ that is an almost Dedekind domain but is not a Matlis domain.

Notation. For a ring R, we denote by $\mathrm{Max}(R)$ the set of all maximal ideals of R. If I is an ideal of R, then $Z(I)$ is the set of all maximal ideals of R containing I. Since we sometimes consider several rings at once (e.g. $D \subseteq R$), we always reserve the notation $Z(I)$ for maximal ideals of the ring R (as opposed to, say, D).

2 Prüfer sections of Noetherian domains

An integral domain R is a *Prüfer domain* if for each maximal ideal M of R, R_M is a valuation domain. A domain R is a *QR-domain* if each overring of R is a quotient ring of R; i.e., a localization of R at a multiplicatively closed set. Thus a *QR*-domain is necessarily a Prüfer domain. Moreover, a Prüfer domain is a *QR*-domain if and only if the radical of each finitely generated ideal is the radical of a principal ideal [3, Theorem 27.5]. A special case of *QR*-domains are the *Bézout domains*, those domains for which every finitely generated ideal is a principal ideal.

Let D be a domain, and let Σ be a collection of valuation overrings of D. Then we say the ring $R = \bigcap_{V \in \Sigma} V$ is a *Prüfer section* of D if R is a Prüfer domain and each $V \in \Sigma$ is centered on a different maximal ideal of D. (The *center* of V in D is the prime ideal of D that is the intersection of the maximal ideal of V with D.) We say that each $V \in \Sigma$ is a *representative* of R, and that Σ is the *set of representatives* of R.[1]

We show in Theorem 2.3 that if R is a Prüfer section of D, then Σ is unique in the sense that it is irredundant (no member can be omitted), and it is the unique set of irredundant representatives of R. To motivate these notions, we collect in the next proposition some examples of how Prüfer sections arise. Versions of statement (1) can be found in [2], [10] and [15]. Statement (2) is a special case of (1): see the discussion on pp. 332–333 of [12]. Statement (3) is contained in Theorem 6.6 of [14], and statement (4) is proved in Lemma 4.2 of [13].

Proposition 2.1. *Let D be a domain, and let Σ be a collection of valuation overrings of D, each centered on a different maximal ideal of D.*

(1) *If there exists a nonconstant monic polynomial $f(X)$ in $D[X]$ that does not have a root in the residue field of any $V \in \Sigma$, then R is a QR-section of D.*

(2) *If D contains a non-algebraically closed field K, and the residue field of each $V \in \Sigma$ is contained in a purely transcendental extension of K, then R is a QR-section of D.*

(3) *If D contains a field K of cardinality greater than the cardinality of Σ, then R is a Bézout section of D.*

(4) *If Σ has finite character (meaning every nonzero element of D is a unit in all but finitely many members of Σ) and for each $V \in \Sigma$, each nonzero prime ideal of V contracts to a maximal ideal of D, then R is a Prüfer section of D.* □

[1]The terminology of sections is motivated by the fact that if R is a Prüfer section, then Σ is, as we will show, unique, and if $\mathrm{Zar}(D)$ is the space of valuation overrings of D and $f : \mathrm{Zar}(D) \to \mathrm{Spec}(D)$ denotes the continuous mapping that takes each valuation ring to its center, then the mapping $f^{-1}(f(\Sigma)) \to f(\Sigma)$ has a right inverse, i.e., a section.

Remark 2.2. Statements (1)–(3) of Proposition 2.1 remain true without the assumption that each valuation ring in Σ is centered on a distinct maximal ideal of D, in the sense that the resulting intersection is a *QR*-domain in the case of (1) and (2), and a Bézout domain in (3). Of course in this more general setting, these rings need no longer be sections of D.

In Theorem 2.3 we make some general observations regarding Prüfer sections of Noetherian domains, most of which are consequences of results in [4]. We use the following notation throughout the rest of this article. For each $V \in \Sigma$, let $\mathfrak{M}_V$ denote the maximal ideal of V, $M_V = \mathfrak{M}_V \cap R$ and $\mathfrak{m}_V = \mathfrak{M}_V \cap D$.

Theorem 2.3. *Let D be a Noetherian domain, let R be a Prüfer section of D and let Σ be a (by* (1) *below, "the") set of representatives of R. Then:*

(1) *$R = \bigcap_{V \in \Sigma} V$ is an irredundant intersection of valuation rings, and it the unique representation of R as an irredundant intersection of valuation overrings.*

(2) *For each $V \in \Sigma$, M_V is a maximal ideal of R and it is the radical of a finitely generated ideal of R.*

(3) *A maximal ideal M of R contains a finitely generated ideal that is contained in no other maximal ideal of R if and only if $M = M_V$ for some $V \in \Sigma$. Moreover, for each $V \in \Sigma$, this finitely generated ideal can be chosen to be $\mathfrak{m}_V R$.*

(4) *A maximal ideal M of R is the radical of a finitely generated ideal if and only if $M = M_V$ for some $V \in \Sigma$.*

(5) *A maximal ideal M of R is finitely generated if and only if $M = M_V$ for some valuation ring $V \in \Sigma$ having a principal maximal ideal.*

Proof. Since R is a Prüfer domain, for each $V \in \Sigma$, we have $V = R_{M_V}$ [3, Theorem 26.2]. Thus $R = \bigcap_{V \in \Sigma} R_{M_V}$. For each $V \in \Sigma$, there exists a finitely generated ideal of D, namely, $\mathfrak{m}_V$, that is contained in M_V but in no other M_W, where $W \in \Sigma$ with $W \neq V$. Thus by Proposition 1.4 of [4], the representation $R = \bigcap_{V \in \Sigma} R_{M_V}$ is irredundant. Also, Lemma 1.6 and Theorem 1.7 of [4] show that $\{M_V : V \in \Sigma\}$ is precisely the set of maximal ideals M of R such that M contains a finitely generated ideal that is contained in no other maximal ideal of R, and for this finitely generated ideal, one may choose $\mathfrak{m}_V R$, where V is such that $M = M_V$. This proves (3). Moreover, Theorem 1.7 of [4] implies that $R = \bigcap_{V \in \Sigma} V$ is the unique irredundant representation of R, and hence proves (1). Also, since a valuation overring of a Noetherian domain necessarily has finite Krull dimension [3, Theorem 25.8], the fact that for each $V \in \Sigma$, the finitely generated ideal $\mathfrak{m}_V$ is contained in M_V but no other maximal ideal implies easily that M_V is the radical of a finitely generated ideal. This proves (2) and (4). If also $V = R_{M_V}$ has a principal maximal ideal, say, $MR_{M_V} = xR_{M_V}$ for some $x \in M$, then since $\mathfrak{m}_V R$ is contained in M but no other maximal ideal of R, we have $M = xR + \mathfrak{m}_V R$, which proves (5). □

We restrict next to Krull dimension 2, and derive stronger results in this case. The following technical lemma needed for Theorem 2.6 will also be crucial in the next section.

Lemma 2.4. *Let D be an integrally closed Noetherian domain of Krull dimension 2, let R be a QR-section of D, and let Σ be the set of representatives of R.*

(1) *If M is a maximal ideal of R, then either $M = M_V$ for some $V \in \Sigma$ or $M = \mathfrak{p}D_\mathfrak{p} \cap R$ for some height 1 prime ideal $\mathfrak{p}$ of D such that $R_M = D_\mathfrak{p}$ and $\mathfrak{p}$ is contained in infinitely many members of $\{\mathfrak{m}_V : V \in \Sigma\}$.*

(2) *Each nonmaximal prime ideal of R is contracted from a nonmaximal prime ideal of a valuation ring in Σ.*

(3) *If $f \in D$ and $Z(f)$ is infinite, then f is an element of infinitely many members of $\{\mathfrak{m}_V : V \in \Sigma\}$.*

(4) *If $r = f/g \in R$ for some $f, g \in D$ such that $(f, g)D$ has height 2, then $Z(g)$ is finite, $Z(f) = Z(g) \cup Z(r)$ and $Z(f) \setminus Z(r)$ is finite.*

Proof. (1) Let M be a maximal ideal of R, and let $\mathfrak{p} = M \cap D$. Then since R is a *QR*-domain there exists $r \in R$ such that $\sqrt{\mathfrak{p}R} = \sqrt{rR}$. But then, since $R = \bigcap_{V \in \Sigma} R_{M_V}$, it follows that $r \in M_V$ for some $V \in \Sigma$, and hence $\mathfrak{p} \subseteq M_V \cap D = \mathfrak{m}_V$. If $\mathfrak{p}$ is a maximal ideal of D, then necessarily, $\mathfrak{m}_V = \mathfrak{p} \subseteq M$. But then Theorem 2.3(3) implies that M_V is the only maximal ideal of R containing $\mathfrak{p}$, and this forces $M = M_V$. Otherwise, if $\mathfrak{p}$ is not a maximal ideal of D (and hence $M \neq M_V$ for all $V \in \Sigma$), then since D is integrally closed and has Krull dimension 2, $D_\mathfrak{p}$ is a DVR with $D_\mathfrak{p} \subseteq R_M$. Consequently, $D_\mathfrak{p} = R_M$. Moreover, if $\mathfrak{p}$ is contained in only finitely many members of $\{\mathfrak{m}_V : V \in \Sigma\}$, then $\mathfrak{p}R$ is contained in only finitely many members of $\{M_V : V \in \Sigma\}$, say $M_{V_1}, M_{V_2}, \ldots, M_{V_n}$. In this case, choose $m \in M \setminus (M_{V_1} \cup \cdots \cup M_{V_n})$. Then the ideal $\mathfrak{p}R + mR \subseteq M$ is a proper finitely generated contained in no M_V, $V \in \Sigma$, which is impossible since every proper finitely generated ideal of R must be contained in at least one M_V, $V \in \Sigma$. (This follows from the fact that R is a *QR*-domain with $R = \bigcap_{V \in \Sigma} R_{M_V}$.) Therefore, $\mathfrak{p}$ is contained in infinitely many members of $\{\mathfrak{m}_V : V \in \Sigma\}$, and this proves (1).

(2) Let P be a nonmaximal prime ideal of R, and let M be a maximal ideal of R containing P. If $M = M_V$ for some $V \in \Sigma$, then PR_{M_V} is a prime ideal of the valuation ring $R_{M_V} = V \in \Sigma$, so P is contracted from a prime ideal of a valuation ring in Σ. Otherwise, if $M \neq M_V$ for all $V \in \Sigma$, then by (1), there exists a height 1 prime ideal $\mathfrak{p}$ of D such that $R_M = D_\mathfrak{p}$, which forces M to have height 1. Yet P is a nonmaximal prime ideal of R contained in M, so necessarily $P = 0$, and hence P is contracted from the 0 ideal of any valuation ring in Σ.

(3) The element f of D is contained in at most finitely many height 1 prime ideals of the Noetherian domain D, so by (1) all but finitely many members of $Z(f)$ are of the form M_V, $V \in \Sigma$. Thus since for each $V \in \Sigma$, $\mathfrak{m}_V = M_V \cap D$, and each M_V contracts to a unique maximal ideal of D, (3) follows.

(4) Since the ideal $(f, g)D$ of D has height 2, and the Krull dimension of D is 2, the ideal $(f, g)D$ is contained in finitely many members of $\{\mathfrak{m}_V : V \in \Sigma\}$. Thus if $Z(g)$ is infinite, then by (3), there must exist $V \in \Sigma$ such that $g \in \mathfrak{m}_V$ but $f \notin \mathfrak{m}_V$. But then f is a unit in V, so that $r^{-1} = g/f \in \mathfrak{M}_V$, contrary to the assumption that $r = f/g \in R \subseteq V$. Therefore, $Z(g)$ is finite, and since $f = gr$, we have $Z(f) = Z(g) \cup Z(r)$. In particular, $Z(f) \setminus Z(r)$ is finite. □

We also require a more general lemma, one that holds beyond the setting of *QR*-sections:

Lemma 2.5. *Let D be a Noetherian domain of Krull dimension 2, let Σ be a collection of valuation overrings of D, and let $R = \bigcap_{V \in \Sigma} V$. If $\mathfrak{p}$ is a height one prime ideal of D that is contained in infinitely many $\mathfrak{m}_V := \mathfrak{M}_V \cap D$, $V \in \Sigma$, then $R \subseteq D_{\mathfrak{p}}$ and $P := \mathfrak{p}D_{\mathfrak{p}} \cap R$ is a height 1 prime ideal of R with $R_P = D_{\mathfrak{p}}$.*

Proof. Let $r \in R$, and write $r = f/g$, where $f, g \in D$. Suppose by way of contradiction that $r \notin D_{\mathfrak{p}}$. Then since $D_{\mathfrak{p}}$ is a valuation ring, $g/f = r^{-1} \in \mathfrak{p}D_{\mathfrak{p}}$, so that we may assume $g \in \mathfrak{p}$ and $f \in D \setminus \mathfrak{p}$. Now since $D/\mathfrak{p}$ is a one-dimensional Noetherian domain, $\mathfrak{p}$ is the intersection of any set of infinitely many maximal ideals of D that contain $\mathfrak{p}$. In particular, $\mathfrak{p}$ is the intersection of the infinitely many $\mathfrak{m}_V$ that contain $\mathfrak{p}$. Thus since $f \notin \mathfrak{p}$, there exists $V \in \Sigma$ such that $\mathfrak{p} \subseteq \mathfrak{m}_V$ but $f \notin \mathfrak{m}_V$, and hence f is a unit in V. But then, since $g \in \mathfrak{p} \subseteq \mathfrak{M}_V$, we have $g/f \in \mathfrak{M}_V$, and at the same time, $f/g \in R \subseteq V$, a contradiction that implies $R \subseteq D_{\mathfrak{p}}$. Since $D_{\mathfrak{p}} \subseteq R_P$ and $D_{\mathfrak{p}}$ is a DVR, it follows that $D_{\mathfrak{p}} = R_P$. □

Theorem 2.6. *Let D be an integrally closed Noetherian domain of Krull dimension 2, let R be a QR-section of D, and let $\mathcal{P}$ be the set of height one prime ideals $\mathfrak{p}$ of D such that $\mathfrak{p}$ is contained in infinitely many $\mathfrak{m}_V := \mathfrak{M}_V \cap D$, $V \in \Sigma$.*

(1) $\mathrm{Spec}(R) = \{\mathfrak{P} \cap R : \mathfrak{P}$ *is a prime ideal of some* $V \in \Sigma\} \cup \{\mathfrak{p}D_{\mathfrak{p}} \cap R : \mathfrak{p} \in \mathcal{P}\}$.

(2) *If every representative of R has Krull dimension 1, then R has Krull dimension 1, every ideal of R can be represented uniquely as an irredundant intersection of irreducible ideals, and*

$$\mathrm{Max}(R) = \{M_V : V \in \Sigma\} \cup \{\mathfrak{p}D_{\mathfrak{p}} \cap R : \mathfrak{p} \in \mathcal{P}\}.$$

(3) *If every representative of R is a DVR, then R is an almost Dedekind domain such that every nonzero proper ideal of R can be represented uniquely as an irredundant intersection of powers of maximal ideals.*

Proof. (1) The inclusion $\subseteq$ is clear in light of Lemma 2.4(1) and (2). To see that the reverse inclusion holds it is enough to observe that by Lemma 2.5, for each $\mathfrak{p} \in \mathcal{P}$, $\mathfrak{p}D_{\mathfrak{p}} \cap R$ is a prime ideal of R.

(2) and (3). By Lemma 2.4(1), a localization of R at a maximal ideal is either a representative of R or a DVR of the form $D_{\mathfrak{p}}$, where $\mathfrak{p}$ is a height 1 prime ideal of D. Thus if every member of Σ has Krull dimension 1, then R has Krull dimension 1, and if every representative of R is a DVR, then R is an almost Dedekind domain. Since the calculation of $\mathrm{Max}(R)$ in (2) is a consequence of (1), it remains to prove the assertions about ideal decompositions in (2) and (3). To verify the decomposition in (2), since R is a Prüfer domain of Krull dimension 1, it suffices by Corollary 2.10 of [5] to show that for each nonzero proper ideal I of R, the ring R/I has at least one maximal ideal that is the radical of a finitely generated ideal. If this last property is satisfied and also R is an almost Dedekind domain, then Theorem 2.8 and Corollary 3.9 in [5] show that every nonzero proper ideal of R is an irredundant intersection of powers of maximal

ideals of R. Therefore, to complete the proof of both (2) and (3), we show that for every proper nonzero ideal I of R, R/I has a maximal ideal that is the radical of a finitely generated ideal.

Let I be a proper nonzero ideal of R. If I is contained in some M_V, $V \in \Sigma$, then by Theorem 2.3(4), the maximal ideal M_V/I of R/I is the radical of a finitely generated ideal. Otherwise, if I is not contained in any M_V, $V \in \Sigma$, then by Lemma 2.4(1), every maximal ideal of R containing I is contracted from the maximal ideal of a ring of the form $D_{\mathfrak{p}}$, where $\mathfrak{p}$ is a height 1 prime ideal of D. Since the collection of all such localizations $D_{\mathfrak{p}}$ of D has finite character, it follows in this case that I is contained in only finitely many maximal ideals of R. Let M be a maximal ideal of R containing I, and choose $m \in M$ such that m is not in any other maximal ideal of R containing I. Then the radical of $(mR + I)/I$ in R/I is M/I, and the theorem is proved. □

Remark 2.7. In Remarks 2.11 and 3.10 of [5], it is noted that a construction from [11] can be used to create examples of interesting (= non-Dedekind) almost Dedekind domains such that each nonzero proper ideal is an irredundant intersection of powers of maximal ideals. These examples have nonzero Jacobson radical. By contrast the order holomorphy rings we consider in Corollary 3.5 of the next section have trivial Jacobson radical, and satisfy Theorem 2.6(2). Thus the rings in the corollary provide a new class of examples of non-Dedekind domains for which every nonzero proper ideal can be represented uniquely as an irredundant intersection of powers of maximal ideals.

3 Matlis domains

In this section we first characterize among *QR*-domains of Krull dimension 1 those that are Matlis in terms of their maximal spectra. It is this characterization we use later to give examples of one-dimensional domains that are not Matlis domains. Our characterization in Lemma 3.3 is a consequence of the following theorem, due to Sang Bum Lee; it allows us to completely avoid homological arguments in what follows.

Theorem 3.1 (Lee [9]). *A domain R with quotient field Q is a Matlis domain if and only if Q/R is the direct sum of countably generated submodules.* □

To prove Lemma 3.3, we need the following routine observation about *QR*-domains.

Lemma 3.2. *Let R be a QR-domain with quotient field Q, and let $0 \neq q \in Q$. Then there exists $r \in R$ such that $R[q] = R[1/r]$.*

Proof. Since R is a Prüfer domain, the ideal $R \cap q^{-1}R$ is finitely generated; see for example [4, Proposition 1.2]. So since R is a *QR*-domain, there exists $r \in R$ such that $\sqrt{rR} = \sqrt{R \cap q^{-1}R}$. Also since R is a Prüfer domain, every overring of R is an intersection of localizations of R at prime ideals [3, Theorem 26.2]. Thus to prove that $R[q] = R[1/r]$, it is enough to check that for each prime ideal P of R, $R[q] \subseteq R_P$ if and only if $R[1/r] \subseteq R_P$. And since $\sqrt{R \cap q^{-1}R} = \sqrt{rR}$, this is indeed the case, for if P is a prime ideal of R, then $R[q] \subseteq R_P$ if and only if $R \cap q^{-1}R \not\subseteq P$; if and only if $r \notin P$; if and only if $R[1/r] \subseteq R_P$. □

Lemma 3.3. *Let R be a QR-domain.*

(1) *If R is a Matlis domain, then there exist a set $\mathcal{A}$ and nonzero elements $r_{\alpha,i} \in R$, where $\alpha \in \mathcal{A}$ and $i \in \mathbb{N}$, such that the sets $Z_\alpha = \bigcup_{i=1}^{\infty} Z(r_{\alpha,i})$ form a disjoint partition of* $\mathrm{Max}(R)$.

(2) *Conversely, if R has Krull dimension* 1 *and there exist a set $\mathcal{A}$ and nonzero elements $r_{\alpha,i} \in R$, with $\alpha \in \mathcal{A}$ and $i \in \mathbb{N}$, such that the sets $Z_\alpha = \bigcup_{i=1}^{\infty} Z(r_{\alpha,i})$ form a disjoint partition of* $\mathrm{Max}(R)$, *then R is a Matlis domain*

Proof. (1) Suppose that R is a Matlis domain. Then by Theorem 3.1, there exist countably generated R-submodules T_α of Q containing R such that $Q/R = \bigoplus_\alpha T_\alpha/R$. In such a decomposition, each T_α is necessarily a ring [1, Lemma IV.4.2]. Thus by Lemma 3.2, there exist for each α, elements $r_{\alpha,i}$, $i \in \mathbb{N}$, such that $T_\alpha = R[1/r_{\alpha,i} : i \in \mathbb{N}]$. We show that the sets $Z_\alpha := \bigcup_{i=1}^{\infty} Z(r_{\alpha,i})$ form a disjoint partition of $\mathrm{Max}(R)$. Let M be a maximal ideal of R. Then $\sum_\alpha T_\alpha = Q \not\subseteq R_M$, so there exist β and $j > 0$ such that $R[1/r_{\beta,j}] \not\subseteq R_M$, and hence $r_{\beta,j} \in M$. Consequently, $M \in Z(r_{\beta,j})$, and this shows that $\mathrm{Max}(R) = \bigcup_\alpha Z_\alpha$. To see that this is in fact a disjoint union, suppose by way of contradiction that $\alpha, \beta \in \mathcal{A}$ with $\alpha \neq \beta$, and there is a maximal ideal M with $M \in Z_\alpha \cap Z_\beta$. Then there exist $i, j > 0$ such that $M \in Z(r_{\alpha,i}) \cap Z(r_{\beta,j})$, and consequently, $T_\alpha \not\subseteq R_M$ and $T_\beta \not\subseteq R_M$. However, $R = T_\alpha \cap (\sum_{\gamma\neq\alpha} T_\gamma)$, and so $R_M = T_\alpha R_M \cap (\sum_{\gamma\neq\alpha} T_\gamma)R_M$. Since R_M is a valuation domain, it must be then that $T_\alpha \subseteq R_M$ or $\sum_{\gamma\neq\alpha} T_\gamma \subseteq R_M$. The former case we have already ruled out, and the latter is also impossible, since $T_\beta \subseteq \sum_{\gamma\neq\alpha} T_\gamma$, with $T_\beta \not\subseteq R_M$. Therefore, we conclude that the sets Z_α, $\alpha \in \mathcal{A}$, form a disjoint partition of $\mathrm{Max}(R)$.

(2) Conversely, suppose that R has Krull dimension 1 and there exist a set $\mathcal{A}$ and nonzero elements $r_{\alpha,i} \in R$, $\alpha \in \mathcal{A}$, $i \in \mathbb{N}$, such that the sets $Z_\alpha := \bigcup_{i=1}^{\infty} Z(r_{\alpha,i})$ form a disjoint partition of $\mathrm{Max}(R)$. For each α, define $T_\alpha = R[1/r_{\alpha,i} : i \in \mathbb{N}]$. We show that Q/R is a direct sum of the countably generated R-modules T_α/R. If $Q \neq \sum_\alpha T_\alpha$, then since R is a Prüfer domain, there exists a nonzero prime ideal P of R such that $\sum_\alpha T_\alpha \subseteq R_P$. But since R has Krull dimension 1, P is a maximal ideal of R, and hence by assumption there exist $\beta \in \mathcal{A}$ and $j > 0$ such that $r_{\beta,j} \in P$. But then $T_\beta \not\subseteq R_P$, a contradiction that forces us to conclude $Q = \sum_\alpha T_\alpha$. Next, fix $\beta \in \mathcal{A}$. We claim that $R = T_\beta \cap (\sum_{\alpha\neq\beta} T_\alpha)$. To this end, it suffices to show that for each maximal ideal M of R, $R_M = T_\beta R_M \cap (\sum_{\alpha\neq\beta} T_\alpha)R_M$. Let M be a maximal ideal of R, and suppose by way of contradiction that $R_M \neq T_\beta R_M \cap (\sum_{\alpha\neq\beta} T_\alpha)R_M$. Then necessarily, $T_\beta \not\subseteq R_M$ and $\sum_{\alpha\neq\beta} T_\alpha \not\subseteq R_M$. Hence there exist $\alpha \neq \beta$ and $i, j > 0$ such that $R[1/r_{\beta,i}] \not\subseteq R_M$ and $R[1/r_{\alpha,j}] \not\subseteq R_M$. But then $M \in Z(r_{\beta,i}) \cap Z(r_{\alpha,j}) \subseteq Z_\beta \cap Z_\alpha$, which is impossible since the latter intersection is empty. Therefore, for each $\beta \in \mathcal{A}$, $R = T_\beta \cap (\sum_{\alpha\neq\beta} T_\alpha)$, and we conclude that $Q/R = \bigoplus_\alpha T_\alpha/R$. Hence by Theorem 3.1, R is a Matlis domain. □

We consider now the special case $D = K[X, Y]$, where K is an uncountable field. If $E \subseteq K^2$, then we say that a Prüfer section with set of representatives Σ is *over E* when $\{\mathfrak{m}_V : V \in \Sigma\} = \{(X - a, Y - b)D : (a, b) \in E\}$.

Theorem 3.4. *Let K be an uncountable field, let A and B be uncountable subsets of K, and let E be a subset of K^2 containing $A \times B$. Then no QR-section of $K[X,Y]$ over E whose representatives have Krull dimension* 1 *is a Matlis domain.*

Proof. Let $D = K[X,Y]$, and suppose that R is a *QR*-section over E that is also a Matlis domain. Then by Lemma 3.3, there exists an index set $\mathcal{A}$ and a set of elements $\{r_{\alpha,i} : \alpha \in \mathcal{A}, i \in \mathbb{N}\}$ of R such that the sets $Z_\alpha := \bigcup_{i>0} Z(r_{\alpha,i})$ form a disjoint partition of $\text{Max}(R)$. We prove a series of claims to show that such a decomposition of $\text{Max}(R)$ is impossible. For each $\alpha \in \mathcal{A}, i \in \mathbb{N}$, write $r_{\alpha,i} = f_{\alpha,i}/g_{\alpha,i}$ for relatively prime elements $f_{\alpha,i}$ and $g_{\alpha,i}$ of D.

Claim 1. *For each α, there exist at most countably many $a \in A$ such that $Z(X - a) \setminus Z_\alpha$ is finite and countably many $b \in B$ such that $Z(Y - b) \setminus Z_\alpha$ is finite.*

Fix $\alpha \in \mathcal{A}$. To simplify notation for the proof of Claim 1, we let $Z = Z_\alpha$, $r_i = r_{\alpha,i}$, $f_i = f_{\alpha,i}$ and $g_i = g_{\alpha,i}$. First we verify that $\{a \in A : Z(X - a) \setminus Z \text{ is finite}\} \subseteq \{a \in A : Z(X - a) \cap Z(r_i) \text{ is uncountable for some } i > 0\}$. Suppose that $a \in A$ such that $Z(X - a) \setminus Z$ is finite. Now

$$Z(X - a) \cap Z = \bigcup_{i=1}^{\infty} Z(X - a) \cap Z(r_i),$$

and since $Z(X-a)$ is uncountable, it follows from the assumption that $Z(X-a)\setminus Z$ is finite that $Z(X-a)\cap Z$ is uncountable. Therefore, since $Z(X-a)\cap Z$ is represented by a countable union of the sets $Z(X - a) \cap Z(r_i)$, there necessarily exists $i > 0$ such that $Z(X - a) \cap Z(r_i)$ is uncountable.

Thus, in light of the inclusion, $\{a \in A : Z(X - a) \setminus Z \text{ is finite}\} \subseteq \{a \in A : Z(X-a)\cap Z(r_i) \text{ is uncountable for some } i > 0\}$, to prove the claim that there exist at most countably many $a \in A$ such that $Z(X - a) \setminus Z$ is finite, we need only verify that there are at most countably many $a \in A$ such that $Z(X-a)\cap Z(r_i)$ is uncountable for some i. And to show this is the case, it suffices to show for each i, there exist at most finitely many $a \in A$ such that $Z(X - a) \cap Z(r_i)$ is uncountable. To this end, suppose that $a \in A$ and there exists $i > 0$ such that $Z(X - a) \cap Z(r_i)$ is uncountable. Then since $Z(r_i) \subseteq Z(f_i)$, we have that $Z(X-a, f_i) = Z(X-a)\cap Z(f_i)$ is uncountable, so that $X - a$ and f_i are not relatively prime. Indeed, relative primeness would force $(X-a, f_i)D$ to be height 2, and hence contained in only finitely many maximal ideals of D; yet, as in the proof of Lemma 2.4(3), all but finitely many maximal ideals of R in $Z(X - a, f_i)$ contract to a different maximal ideal of D, and hence if $Z(X - a, f_i)$ is infinite, so is the set of maximal ideals of D containing $(X - a, f_i)D$. This contradiction implies that $X - a$ and f_i are not relatively prime. But then since $X - a$ is irreducible, it must be that $f_i \in (X - a)D$. Yet since D is a Noetherian domain, f_i is contained in only finitely many prime ideals of height 1, and so there are at most finitely many possible choices for a. This proves there exist at most countably many $a \in A$ such that $Z(X - a) \setminus Z$ is finite, and a similar argument shows that there are at most countably many $b \in B$ such that $Z(Y - b) \setminus Z$ is finite.

Claim 2. *For each $a \in A$, there exists a unique $\alpha \in \mathcal{A}$, which we denote by $\alpha(a)$, such that $Z(X - a) \setminus Z_\alpha$ is finite, and for each $b \in B$, there exists a unique $\beta = \beta(b) \in \mathcal{A}$ such that $Z(Y - b) \setminus Z_\beta$ is finite.*

Let $a \in A$. Then $X - a \in (X - a, X - b)D$ for all $b \in B$, so that since B is infinite, we have by Lemma 2.5 that $R \subseteq D_{(X-a)}$. Thus $P_{X-a} := (X-a)D_{(X-a)} \cap R$ is a prime ideal of R. By Theorem 2.6(2), $P_{X-a} \in \mathrm{Max}(R)$, so there exist α and j such that $f_{\alpha,j} = g_{\alpha,j} r_{\alpha,j} \in P_{X-a}$. Hence $f_{\alpha,j} \in P_{X-a} \cap D = (X - a)D$, and so $Z(X-a) \subseteq Z(f_{\alpha,j})$. By Lemma 2.4(4), $Z(f_{\alpha,j}) \setminus Z(r_{\alpha,j})$ is finite, so $Z(X-a) \setminus Z_\alpha$ is finite. In fact, the choice of α is unique, since otherwise there exists $\alpha' \neq \alpha$ such that $Z(X - a) \setminus Z_{\alpha'}$ is finite. But if this were the case, then since Z_α and $Z_{\alpha'}$ are disjoint, we would have $Z(X - a) \subseteq Z_\alpha^c \cup Z_{\alpha'}^c$ (where $(\ )^c$ denotes the complement in $\mathrm{Max}(R)$), so that

$$Z(X - a) = (Z(X - a) \setminus Z_\alpha) \cup (Z(X - a) \setminus Z_{\alpha'}),$$

a situation which forces $Z(X - a)$ to be finite, a contradiction. Therefore, for each $a \in A$, there exists a unique $\alpha = \alpha(a)$ such that $Z(X - a)/Z_\alpha$ is finite. A similar argument shows that for each $b \in B$, there exists a unique $\beta = \beta(b) \in \mathcal{A}$ such that $Z(Y - b)/Z_\beta$ is finite. This proves Claim 2.

Claim 3. *Let $S = \{\alpha(a) : a \in A\}$ and $T = \{\beta(b) : b \in B\}$. Then there exist uncountable sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $S_1 \cap T_1$ is empty.*

By Claim 1, given $\alpha \in \mathcal{A}$, there are at most countably many $a \in A$ such that $Z(X - a) \setminus Z_\alpha$ is finite, and hence for any $a \in A$, there exist at most countably many $a' \in A$ such that $\alpha(a) = \alpha(a')$. Therefore, since A is uncountable, it follows that S is uncountable. Similarly, since B is uncountable, T is uncountable. Now we may choose two uncountable sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $S_1 \cap S_2$ is empty. For suppose that $S \cap T$ is countable; then necessarily since S and T are uncountable, $S \setminus (S \cap T)$ and $T \setminus (S \cap T)$ are uncountable. In this case, set $S_1 = S \setminus (S \cap T)$ and $T_1 = T \setminus (S \cap T)$, and observe that S_1 and T_1 are disjoint. Otherwise, if $S \cap T$ is uncountable, then it is a standard fact of set theory that we may write $S \cap T$ as a disjoint union of two uncountable sets, S_1 and T_1 [7, Theorem 13, p. 41]. This proves Claim 3.

Claim 4. *For each $a \in A$ and $b \in B$, let $M_{(a,b)} = (X - a, Y - b)D$. Then there exist uncountable sets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that for all $a \in A_1$ and $b \in B_1$, the "lines" $\ell_a := \{(a, b) : b \in B_1$ and $M_{(a,b)} \in Z_{\alpha(a)}\}$ and $\ell^b := \{(a, b) : a \in A_1$ and $M_{(a,b)} \in Z_{\beta(b)}\}$ have empty intersection.*

For each $\gamma \in S_1$, we may choose $a_\gamma \in A$ such that $\gamma = \alpha(a_\gamma)$, and hence all but finitely many elements of $Z(X - a_\gamma)$ are in Z_γ. Similarly, for each $\delta \in T_1$, there exists $b_\delta \in B$ such that all but finitely many elements of $Z(Y - b_\delta)$ are in Z_δ. Let $A_1 = \{a_\gamma : \gamma \in S_1\}$ and $B_1 = \{b_\delta : \delta \in T_1\}$. Then A_1 has the same cardinality as S_1, since for any $\gamma, \gamma' \in S_1$, $a_\gamma = a_{\gamma'}$ implies $\gamma = \alpha(a_\gamma) = \alpha(a_{\gamma'}) = \gamma'$. Similarly, B_1 has the same cardinality as T_1.

We claim that for every $a \in A_1$ and $b \in B_1$, $\ell_a \cap \ell^b$ is empty. Indeed, the only element possibly in $\ell_a \cap \ell^b$ is (a, b). But if $(a, b) \in \ell_a \cap \ell^b$, then $M_{(a,b)} \in Z_{\alpha(a)} \cap Z_{\beta(b)}$. However, $\alpha(a)$ and $\beta(b)$ are distinct (they are in the disjoint sets S_1 and T_1, respectively), so $Z_{\alpha(a)}$ and $Z_{\beta(b)}$ are disjoint. Hence $\ell_a \cap \ell^b$ is empty, and Claim 4 is proved.

Claim 5. *R is not a Matlis domain.*

We define a matrix $[t_{a,b}]_{a\in A_1, b\in B_1}$ by $t_{a,b} = 1$ if $(a,b) \notin \ell_a$ and $t_{a,b} = 0$ if $(a,b) \in \ell_a$. Then for $a \in A_1$ and $b \in B_1$, we have $t_{a,b} = 1$ if and only if $(a,b) \notin \ell_a$; if and only if $M_{(a,b)} \in Z(X-a) \setminus Z_{\alpha(a)}$. But for $a \in A_1$, the set $Z(X-a) \setminus Z_{\alpha(a)}$ is finite, so there are at most finitely many $b \in B_1$ such that $M_{(a,b)} \notin Z_{\alpha(a)}$. Therefore, each row of the matrix $[t_{a,b}]$ has at most finitely many 1's.

We claim similarly that each column of the matrix $[t_{a,b}]$ has at most finitely many 0's. Indeed, let $b \in B_1$. Then for each $a \in A_1$, $t_{a,b} = 0$ if and only if $(a,b) \in \ell_a$. Thus since by Claim 4, $\ell_a \cap \ell^b$ is empty, if $t_{a,b} = 0$, then $(a,b) \notin \ell^b$, and hence $M_{(a,b)} \notin Z_{\beta(b)}$. By Claim 2, $Z(Y-b) \setminus Z_{\beta(b)}$ is finite, so there are at most finitely many $a \in A_1$ such that $M_{(a,b)} \notin Z_{\beta(b)}$. Therefore, for each $b \in B_1$, there are most finitely $a \in A_1$ such that $t_{a,b} = 0$, and this proves that each column of the matrix $[t_{a,b}]$ has at most finitely many 0's.

Now since each row of the matrix $[t_{a,b}]$ has finitely many 1's, and since A_1 is uncountable, it follows that there exists $n \geq 0$ such that infinitely many rows contain exactly n occurrences of 1. Moreover, $n > 0$, since otherwise there are infinitely many rows that consist entirely of 0's, which contradicts the fact that each column has at most finitely many 0's. In light of this, there exists a sequence of distinct elements $\{a_i\}_{i\in\mathbb{N}}$ of A_1 such that for each i, the i^{th} row $[t_{a_i,b}]_{b\in B_1}$ has exactly n occurrences of 1 in it. Similarly, there exists $m \geq 0$ such that infinitely many columns of $[t_{a,b}]$ contain exactly m occurrences of 0. Note also that $m > 0$, since otherwise there are infinitely many columns consisting of all 1's, which would force the existence of rows having infinitely many 1's, contrary to the fact that each row has finitely many 1's. Thus there exists a sequence of distinct elements $\{b_j\}_{j\in\mathbb{N}}$ of B_1 such that for each j, the j^{th} column $[t_{a,b_j}]_{a\in A_1}$ has exactly m occurrences of 0 in it.

Now we form a (finite) matrix $F = [t_{a_i,b_j}]$, where $i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n+1$. Let k denote the number of occurrences of 1 in the matrix F. We will calculate k first by counting the number of 1's in the rows, then recalculate it by counting the number of 1's in the columns, and we will see that these two calculations cannot be reconciled. Now each row of F has at most n occurrences of 1, so, since there are $2m$ rows of F, the matrix F contains at most $2mn$ occurrences of 1; i.e., $k \leq 2mn$. Next we count the number of 1's by using columns rather than rows. Each column has at most m occurrences of 0. In fact, since each entry of F is either 1 or 0, each column of F has at least $2m - m = m$ occurrences of 1. Since there are $2n+1$ columns of F, this means that the matrix F contains at least $m(2n+1)$ occurrences of 1. Thus $m(2n+1) \leq k$. But this shows that $m(2n+1) \leq k \leq 2mn$, a contradiction to $m \neq 0$. This proves that R is not a Matlis domain. □

It is not hard to use Theorem 3.4 to create examples of one-dimensional domains that are not Matlis domains. For example, one could choose a field K that has cardinality greater than the continuum, choose subsets A and B of K that have cardinality that of the continuum, and then choose for each point $(a,b) \in A \times B$, a valuation overring of $D = K[X,Y]$ of Krull dimension 1 that is centered on $(X-a, Y-b)D$.

By Proposition 2.1(3), the intersection R of all these valuation rings is a Bézout section over $A \times B$, and by Theorem 2.6, R has Krull dimension 1. Yet by Theorem 3.4, R is not a Matlis domain. Moreover, by choosing each representative of R to be a DVR, one obtains that R is an almost Dedekind domain (Theorem 2.6). In general, there are many ways to choose these one-dimensional valuation overrings; see for example [8] or [16].

However, to illustrate the theorem in more concrete terms, we give a direct construction of an almost Dedekind domain that is not a Matlis domain. We first recall the notion of an order valuation: Let D be a Noetherian domain, and let $\mathfrak{m}$ be a maximal ideal of D such that $D_{\mathfrak{m}}$ is a regular local ring. Define a mapping $\text{ord}_{\mathfrak{m}} : D_{\mathfrak{m}} \to \mathbb{Z} \cup \{\infty\}$ by $\text{ord}_{\mathfrak{m}}(0) = \infty$ and $\text{ord}_{\mathfrak{m}}(f) = \sup\{k : f \in \mathfrak{m}^k\}$ for all $f \in D_{\mathfrak{m}}$. Since $D_{\mathfrak{m}}$ is a regular local ring, the mapping $\text{ord}_{\mathfrak{m}}$ extends to a rank one discrete valuation on the quotient field of D, and this valuation (the *order valuation* with respect to $\mathfrak{m}$) has residue field purely transcendental over the residue field of $D_{\mathfrak{m}}$ of transcendence degree 1 less than the Krull dimension of $D_{\mathfrak{m}}$ [6, Theorem 6.7.9].

Now consider the setting of Theorem 3.4, so that K is an uncountable field and $D = K[X, Y]$. Suppose also that K is not algebraically closed. If E is a subset of K^2, then we say the *order holomorphy ring with respect to* E is the ring $R = \bigcap_{p \in E} V_p$, where for each $p = (a, b) \in E$, V_p is the order valuation ring of $D_{(X-a,Y-b)}$. It is clear that each member of $\Sigma := \{V_p : p \in E\}$ lies over a different maximal ideal of D. Therefore, by Proposition 2.1(2), since the residue field of each V_p is purely transcendental of transcendence degree 1 over the non-algebraically closed field K, we have that R is a *QR*-section of D. Moreover, by Theorem 2.6, R is an almost Dedekind domain. Making sure that E is large enough, we obtain by Theorem 3.4 that R is not a Matlis domain:

Corollary 3.5. *Let K be an uncountable non-algebraically closed field, let A and B be uncountable subsets of K, and let E be a subset of K^2 containing $A \times B$. Then the order holomorphy ring R of $K[X, Y]$ with respect to E is an almost Dedekind domain that is not a Matlis domain.* □

Remark 3.6. It is not enough in Theorem 3.4 or Corollary 3.5 to assume simply that E is uncountable. For example, let $D = \mathbb{R}[X, Y]$, let I be an infinite compact subset of $\mathbb{R}$, and let $E = \{(t, e^t) : t \in I\}$ (here e^x is the usual exponential function). Then the uncountable set of maximal ideals $\{(X - t, Y - e^t) : t \in I\}$ has the property that each nonzero element of D is contained in at most finitely many of these maximal ideals [13, Proposition 5.6]. Thus if we let R be the order holomorphy ring of D with respect to E, we have that R is an almost Dedekind domain for which every nonzero ideal is contained in at most finitely many maximal ideals, and therefore R is a Dedekind, hence Matlis, domain.

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Transfinite self-idealization and commutative rings of triangular matrices

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Abstract. The self-idealization of a commutative ring is iterated countably many times, producing an inverse-direct system of rings. We investigate the rings which are the inverse limit and the direct limit of this system.

Keywords. Idealization, upper triangular matrices, inverse-direct systems, groups of units.

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Introduction

There are many different ways to produce new commutative rings by means of finite or infinite matrices, starting with a commutative ring R with 1. For instance, looking at finite matrices, one can consider 2×2 matrices of the form

$$\begin{pmatrix} r & -s \\ s & r \end{pmatrix}$$

$(r, s \in R)$, that form a commutative ring containing the subring of the scalar matrices, which is isomorphic to R. This is the way the complex numbers are viewed as 2×2 matrices with real entries. Another example is furnished by circulant $n \times n$ matrices with entries in R, which form a commutative R-algebra isomorphic to $R[X]/(X^n - 1)$ (see [8, p. 70, Ex. 10]).

Passing to infinite matrices, one can consider $\omega \times \omega$ upper triangular matrices with entries in R, with equal elements along the diagonals parallel to the principal diagonal (from now on shortly called *diagonals*). These are a special kind of Toeplitz matrices, and form a commutative R-algebra isomorphic to $R[[X]]$, the ring of power series over R. The subring of the matrices with almost all zero diagonals is isomorphic to the ring $R[X]$ of polynomials over R. The finite $n \times n$ sections of these matrices form a ring isomorphic to $R[X]/(X^n)$.

Another well-known construction introduced by Nagata [15] in 1956 (see also [13]) is the idealization, which is a particular case of a more general construction, described by Shores in [16]. This construction, starting with the commutative ring R and an R-algebra B (neither commutative in general, nor with 1), produces a new R-algebra S, containing R as subring and B as two-sided ideal. Following Shores (but inverting the order), we denote this algebra by $R \circ B$. When the algebra B is commutative, the ring $R \circ B$ is also commutative; when B has the trivial multiplication, $R \circ B$ coincides with the Nagata idealization of B (called *trivial extension* in [10]), usually denoted by $R\,(+)\,B$. In case B is an ideal of the total ring of fractions of R, $R \circ B$ is called the *amalgamated duplication of R along B* by D'Anna and Fontana, and is investigated in [5], [7] and [6]. It is worthwhile to remark that the R-algebra $S = R \circ B$ constructed by Shores coincides, in the terminology used by Corner [4], with the *split extension* of R by its ideal B; this means that there are a ring embedding $\eta : R \to S$ and a ring projection $\pi : S \to R$ such that $\pi \cdot \eta = 1_R$ and $B = \operatorname{Ker} \pi$.

Shores considered in [16] a transfinite process of idealization, starting with a field F. While at the non-limit steps the process is just a self-idealization of the ring (i.e., idealization of the ring itself), the construction at limit steps is a bit artificial, since its goal is to produce a local perfect ring, which is not obtainable by a "canonical" construction.

The main goal of this note is to investigate the two rings which are obtained, starting from an arbitrary commutative ring R, by changing the ω-th step in Shores' construction, using the "canonical" processes of the inverse limit and of the direct limit, taking as ring homomorphisms the canonical projections of the self-idealization in the first case, and the canonical ring embeddings of the self-idealization in the latter case. Actually, these maps give rise to an inverse-direct system as defined by Eklof–Mekler in [9, Chapter XI], but of rings and not only of Abelian groups.

The ring obtained via the inverse limit is isomorphic to a ring denoted by $\hat{T}_\omega(R)$, and the ring obtained via the direct limit is isomorphic to a ring denoted by $T_\omega(R)$. The letter T reminds that we are dealing with upper triangular matrices. In fact, we will see in Section 2 that $\hat{T}_\omega(R)$ consists of the $\omega \times \omega$ upper triangular matrices identified by their first row and by a certain "diagonal rule", which describes how to insert in the k-th upper diagonal (the main diagonal has index 1) either 0 or a fixed element $r_k \in R$ (depending on the diagonal). The ring $\hat{T}_\omega(R)$ contains as subring the ring $T_\omega(R)$, that consists of the matrices in $\hat{T}_\omega(R)$ with almost all the diagonals zero. As expectable, the two rings are never Noetherian. We will see that $\hat{T}_\omega(R)$ is the completion of $T_\omega(R)$ with respect to a suitable topology.

Notice that the self-idealization of a ring R is isomorphic to the ring $R[X]/(X^2)$, hence the ring $\hat{T}_\omega(R)$ can be viewed also as the inverse limit of the factor rings $R[X_1, \dots, X_n]/(X_1^2, \dots, X_n^2)$ with respect to the canonical projections; similarly, $T_\omega(R)$ is isomorphic to the ring $R[X_n \mid n \in \omega]/(X_n^2 \mid n \in \omega)$. However we will prefer the matricial setting and notation, instead of those of polynomials.

We will show in Section 3 that $\hat{T}_\omega(R)$ is the idealization of its ideal of the so called *even matrices*, viewed as ideal of its subring of the so called *odd matrices* (which is isomorphic to the whole ring $\hat{T}_\omega(R)$). Odd (respectively, even) matrices are those matrices of $\hat{T}_\omega(R)$ such that the entries of the first row with even (respectively, odd) index are zero. From this decomposition of $\hat{T}_\omega(R)$ we obtain a strictly descending chain of subrings $\hat{T}_\omega(R) > \mathcal{O}_1 > \mathcal{O}_2 > \mathcal{O}_3 > \cdots$, and a strictly ascending chain of ideals $0 < \mathcal{E}_1 < \mathcal{E}_2 < \mathcal{E}_3 < \cdots$, such that, for each integer $n > 1$, $\hat{T}_\omega(R) = \mathcal{O}_n \circ \mathcal{E}_n$, the algebra obtained via the Shores construction.

Section 4 is devoted to determine the structure of the groups of the units of the two rings $T_\omega(R)$ and $\hat{T}_\omega(R)$. We will show that they are isomorphic, as additive Abelian groups, to the direct sum of the group of the units of R (written additively) and of a direct sum (respectively, a direct product) of countably many copies of the additive group of the ring R itself.

It is worthwhile to remark that a slight modification of the transfinite self-idealization given in Section 2, extended to the first uncountable ordinal, is performed in [11] to construct a commutative ring admitting modules whose behaviour reminds that of uncountably generated uniserial modules over valuation rings, and gives rise to non-isomorphic clones, as the non-standard uniserial modules (see [12, Chapter X]).

1 Self-idealization and Shores' constructions

We summarize the basic constructions quoted in the Introduction. Given a commutative ring R and a left R-algebra B, we can define the new R-algebra $R \circ B$, whose R-module structure is $R \oplus B$, endowed by the multiplication:

$$(r,b)\cdot(r',b') = (rr', rb' + r'b + bb').$$

Some properties of this R-algebra are mentioned in [16]. As recalled in the Introduction, when B is an R-submodule of the total ring of fractions of R, $R \circ B$ is called the *amalgamated duplication* of R along B (see [7] and papers quoted there).

If B is an R-module, we can endow B by the trivial multiplication. In this case the multiplication in $R \circ B$ is given by

$$(r,b)\cdot(r',b') = (rr', rb' + r'b)$$

so that we can identify the pair (r,b) with the 2x2 matrix

$$\begin{pmatrix} r & b \\ 0 & r \end{pmatrix}$$

and the multiplication in $R \circ B$ with the usual multiplication of matrices. The ring $R \circ B$ is denoted in this case by $R\ (+)\ B$ and is called the *idealization* of B. This construction, started by Nagata in [15], is described in [13] and has been investigated recently in [2], [3] and [1].

We are interested in the R-algebra $R\ (+)\ R$, called the *self-idealization* of R.

The following facts are well known (see [13], [16] and [3]).

(1) There is a canonical ring embedding $\eta : R \to R\ (+)\ R$ mapping r to $(r,0)$; in this way we can identify R and its action on $R\ (+)\ R$ with its image in $R\ (+)\ R$ and its action as subring.

(2) The subset $(0) \oplus R$ is an ideal of the ring $R\ (+)\ R$, and there is a canonical surjective ring homomorphism $\pi : R\ (+)\ R \to R$, mapping (r,s) to r, with kernel $(0) \oplus R$.

(3) The two maps η and π defined above satisfy $\pi \cdot \eta = 1_R$, that is, $R\,(+)\,R$ is a split extension of R by itself.

(4) If I and J are ideals of R such that $I \leq J$, then $I \oplus J$ is an ideal of $R\,(+)\,R$ and $(R\,(+)\,R)/(I \oplus J) \cong (R/I)\,(+)\,(R/J)$. Ideals of $R\,(+)\,R$ of this form are called *homogeneous*; if R is a domain,

every ideal of $R\,(+)\,R$ is homogeneous if and only if R is a field.

(5) $\mathrm{Spec}(R)$ corresponds bijectively to $\mathrm{Spec}(R\,(+)\,R)$, since every prime ideal of $R\,(+)\,R$ is homogeneous of the form $P \oplus R$ for some prime ideal P of R; hence the nilradical of $R\,(+)\,R$ coincides with $\mathrm{Nil}(R) \oplus R$.

(6) If R is local with maximal ideal P, then $R\,(+)\,R$ is local with maximal ideal $P \oplus R$ and socle $(R\,(+)\,R)[P \oplus R] = (0) \oplus R[P]$; furthermore, if $n \geq 1$, $(P \oplus R)^n = P^n \oplus P^{n-1}$ and the n-th socle of $R\,(+)\,R$ is $R[P^{n-1}] \oplus R[P^n]$.

Shores proves [16, Theorem 7.5] that, given any non-limit ordinal α, there exists a commutative local semiartinian ring R_α whose Loewy length is α. His construction is by transfinite induction, starting with a field F. We repeat part of this construction, but starting with an arbitrary commutative ring R.

The finite steps are iterated self-idealizations; let $R_1 = R$ and, for each $n > 1$, set

$$R_{n+1} = R_n\,(+)\,R_n.$$

Each R_n is a commutative ring and its elements can be viewed as $2^{n-1} \times 2^{n-1}$ upper triangular matrices with entries in R; for $n > 1$ they are in block form:

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$$

where $A, B \in R_{n-1}$ are $2^{n-2} \times 2^{n-2}$ matrices. R_n embeds into R_{n+1} by means of the canonical embedding $\eta_n : R_n \to R_{n+1}$ defined by:

$$\eta_n(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \qquad (A \in R_n).$$

We have also the canonical ring projections $\pi_n : R_{n+1} \to R_n$ defined by:

$$\pi_n \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = A \qquad (A, B \in R_n).$$

For every $n \geq 1$, the ideal J_n of R_n consisting of the matrices with entries 0 on the main diagonal satisfies $J_n^n = 0$, as is easily checked by induction on n. The maximal ideals of R_n are the ideals of the form $P \oplus J_n$, where P is a maximal ideal of R. Obviously, J_n is a prime (respectively, maximal) ideal if and only if R is a domain (respectively, field).

The following result is quite obvious by fact (6) above.

Proposition 1.1. *If R is a local perfect ring of Loewy length α, then the ring R_{n+1} obtained by iterating n times the self-idealization is a local perfect ring of Loewy length $\alpha + n$.* □

For the sake of completeness, we remind Shores' construction at ω for $R = F$. In the above notation, let $J_\omega = \bigoplus_n J_n$, which is an F-vector space of dimension ω and an F-algebra with the pointwise multiplication. Let $R'_\omega = F \times J_\omega$. Its unique maximal ideal is $J'_\omega = 0 \oplus J_\omega$, and its socle is isomorphic to the direct sum of the socles of the rings R_n (this requires some effort).

Consider now the submodule Δ of R'_ω generated by the differences $x_n - x_m$ ($m < n$), where x_n is a generator of the socle of R_n. Then set $R_\omega = R'_\omega/\Delta$. It can be shown that all the dimensions of the factors of the Loewy series of R'_ω and R_ω coincide, except the first one, since R_ω has simple socle.

2 Inverse and direct limits of self-idealizations

We introduce now a set $\hat{T}_\omega(R)$ of upper triangular $\omega \times \omega$ matrices with entries in a fixed commutative ring R, which turns out to be a commutative subring of the ring of all upper triangular $\omega \times \omega$ matrices with entries in R. The elements of the set $\hat{T}_\omega(R)$ are those $\omega \times \omega$ triangular matrices which have the k-th diagonal containing either 0 or a fixed element r_k of R depending on the diagonal itself, according to the following rule (we emphasize that the main diagonal has index 1).

The Diagonal Rule

- The 1-st diagonal is constantly equal to r_1: $[r_1, r_1, r_1, r_1, \dots]$.
- The k-th diagonal ($k > 1$) is defined inductively on $n \geq 0$, for $2^n < k \leq 2^{n+1}$:

  Case $n = 0$: $k = 2$; the 2-nd diagonal is $[r_2, 0, r_2, 0, r_2, 0, \dots]$,

  Case $n > 0$: assume $2^n < k \leq 2^{n+1}$ and the h-th diagonal defined for $h \leq 2^n$;
- the k-th diagonal is periodical of period 2^{n+1};
- the first 2^n entries are equal to the first 2^n entries of the $(k - 2^n)$-th diagonal, with r_k replacing r_{k-2^n};
- the entries from the $(2^n + 1)$-th to the $(2^{n+1} - 1)$-th are 0.

Thus such an $\omega \times \omega$ upper triangular matrix is uniquely determined by its first row $[r_1, r_2, r_3, \dots]$ and by the Diagonal Rule. We represent it as $T(r_1, r_2, r_3, \dots)$, or, shortly, as $T(r_n)$, to remind that it is an upper triangular matrix determined by the first row. We say that the matrix $T(r_n)$ is *generated* by the sequence $(r_n)_{n \geq 1}$; in particular, if almost all the elements r_n are zero, we say that $T(r_n)$ is *finitely generated*. $T(r_n)$ has the following shape (where $-$ means 0):

$$\begin{pmatrix}
r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & r_9 & r_{10} & r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} & \cdots \\
- & r_1 & - & r_3 & - & r_5 & - & r_7 & - & r_9 & - & r_{11} & - & r_{13} & - & r_{15} & \cdots \\
- & - & r_1 & r_2 & - & - & r_5 & r_6 & - & - & r_9 & r_{10} & - & - & r_{13} & r_{14} & \cdots \\
- & - & - & r_1 & - & - & - & r_5 & - & - & - & r_9 & - & - & - & r_{13} & \cdots \\
- & - & - & - & r_1 & r_2 & r_3 & r_4 & - & - & - & - & r_9 & r_{10} & r_{11} & r_{12} & \cdots \\
- & - & - & - & - & r_1 & - & r_3 & - & - & - & - & - & r_9 & - & r_{11} & \cdots \\
- & - & - & - & - & - & r_1 & r_2 & - & - & - & - & - & - & r_9 & r_{10} & \cdots \\
- & - & - & - & - & - & - & r_1 & - & - & - & - & - & - & - & r_9 & \cdots \\
- & - & - & - & - & - & - & - & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & \cdots \\
- & - & - & - & - & - & - & - & - & r_1 & - & r_3 & - & r_5 & - & r_7 & \cdots \\
- & - & - & - & - & - & - & - & - & - & r_1 & r_2 & - & - & r_5 & r_6 & \cdots \\
- & - & - & - & - & - & - & - & - & - & - & r_1 & - & - & - & r_5 & \cdots \\
- & - & - & - & - & - & - & - & - & - & - & - & r_1 & r_2 & r_3 & r_4 & \cdots \\
- & - & - & - & - & - & - & - & - & - & - & - & - & r_1 & - & r_3 & \cdots \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & r_1 & r_2 & \cdots \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & r_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{pmatrix}$$

This shape shows that the diagonal blocks of size 2^n, for a fixed positive integer n, are all equal and correspond to a matrix in R_{n+1}, the ring obtained by iterating n times the self-idealization described in the preceding section, starting with the ring $R_1 = R$, as is easily seen by induction on n. The next two results describe the structure of $\hat{T}_\omega(R)$.

Proposition 2.1. *$\hat{T}_\omega(R)$ is a commutative subring of the ring of the $\omega \times \omega$ upper triangular matrices with entries in R.*

Proof. $\hat{T}_\omega(R)$ is obviously closed under pointwise addition and it contains the unit matrix $I = T(1, 0, 0, 0, \dots)$. Let $T(r_n)$ and $T(s_n)$ be two matrices in $\hat{T}_\omega(R)$. We must prove that the upper triangular matrix $T(r_n) \cdot T(s_n)$ obtained by the "rows by columns" product still satisfies the Diagonal Rule. Fix an index $k \geq 1$ and look at the entries in the k-th diagonal of $T(r_n) \cdot T(s_n)$. Choose n such that $k \leq 2^n$. All the $2^n \times 2^n$ diagonal blocks of $T(r_n) \cdot T(s_n)$ are the products of the two corresponding $2^n \times 2^n$ diagonal blocks of $T(r_n)$ and $T(s_n)$, so they are all the same. The entries of the k-th diagonal of $T(r_n) \cdot T(s_n)$ outside of these blocks are all 0, as one can see by decomposing $T(r_n)$ and $T(s_n)$ into $2^n \times 2^n$ upper triangular blocks. This shows that the k-th diagonal is periodical of period 2^n and that the other two conditions of the Diagonal Rule are satisfied. Since the product of the diagonal blocks commute, the product in $\hat{T}_\omega(R)$ is commutative. $\square$

The iteration of the self-idealization exposed in Section 1, starting from the ring R, produces the inverse system of commutative rings $\{R_n \mid \pi_n : R_{n+1} \to R_n\}$. The inverse limit of this system is isomorphic to the ring $\hat{T}_\omega(R)$.

Proposition 2.2. *The ring $\hat{T}_\omega(R)$ is isomorphic to the inverse limit $\varprojlim R_n$, where the connecting maps are the canonical projections $\pi_n : R_{n+1} \to R_n$.*

Proof. Define for each $n \geq 1$ the epimorphism $\rho_n : \hat{T}_\omega(R) \to R_n$ by sending the matrix $T(r_k)$ to its first $2^{n-1} \times 2^{n-1}$ diagonal block. Note that $\operatorname{Ker}(\rho_n) = \{T(r_k) \mid r_k = 0 \text{ for all } k \leq 2^{n-1}\}$. Then we have the commutative diagrams

$$\begin{array}{ccc} \hat{T}_\omega(R) & = & \hat{T}_\omega(R) \\ {\scriptstyle \rho_{n+1}}\downarrow & & \downarrow{\scriptstyle \rho_n} \\ R_{n+1} & \xrightarrow{\pi_n} & R_n \end{array}$$

If, for a commutative ring S, we have commutative diagrams

$$\begin{array}{ccc} S & = & S \\ {\scriptstyle \phi_{n+1}}\downarrow & & \downarrow{\scriptstyle \phi_n} \\ R_{n+1} & \xrightarrow{\pi_n} & R_n \end{array}$$

then there is a ring homomorphism $\phi : S \to \hat{T}_\omega(R)$ making all the diagrams

$$\begin{array}{ccc} S & = & S \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \phi_n}\downarrow \\ \hat{T}_\omega(R) & \xrightarrow{\rho_n} & R_n \end{array}$$

commutative. The map ϕ is defined in the following way: for $s \in S$, $\phi(s) = T(r_n)$, where, for each $n \geq 1$, $[r_1, r_2, \ldots, r_{2^n}]$ is the first row of the matrix $\phi_{n+1}(s)$. □

As an R-module, $\hat{T}_\omega(R)$ is isomorphic to $\prod_\omega R$, the direct product of ω copies of R. Let us set $K_n = \operatorname{Ker}(\rho_n)$; then $\hat{T}_\omega(R)$, as an inverse limit, is complete in the topology having the chain of ideals

$$K_1 > K_2 > K_3 > K_4 > \cdots$$

as basis of neighborhoods of 0.

The ring $\hat{T}_\omega(R)$ contains the subset $T_\omega(R)$ consisting of the finitely generated matrices, which is an R-module isomorphic to $\bigoplus_\omega R$, the direct sum of ω copies of R. Clearly

$$T_\omega(R) = \bigcup_{n \geq 1} T_n(R)$$

where $T_n(R)$ is the subset of $T_\omega(R)$ consisting of the matrices generated by the sequences $(r_k)_{k\geq 1}$ such that $r_k = 0$ for $k > 2^{n-1}$. These matrices are of the form $\mathrm{Diag}(A, A, A, \dots)$, with $A \in R_n$. Hence $T_n(R)$ is a ring, isomorphic to the ring R_n via the map which sends $\mathrm{Diag}(A, A, A, \dots)$ into A. Consequently, $T_\omega(R)$, as union of a chain of subrings, is a ring. Thus we have the following

Proposition 2.3. *The ring $T_\omega(R)$ is isomorphic to the direct limit $\varinjlim R_n$, where the connecting maps are the canonical embeddings $\eta_n : R_n \to R_{n+1}$.*

Proof. It is enough to note that the commutative diagrams

$$\begin{array}{ccc} T_\omega(R) & = & T_\omega(R) \\ {\scriptstyle\sigma_n}\uparrow & & \uparrow{\scriptstyle\sigma_{n+1}} \\ R_n & \xrightarrow{\eta_n} & R_{n+1} \end{array}$$

are obtained defining $\sigma_n(A) = \mathrm{Diag}(A, A, A, \dots)$ for each $A \in R_n$. □

Proposition 2.4. *The two rings $\hat{T}_\omega(R)$ and $T_\omega(R)$ are both split extensions of the rings R_n, for every $n \geq 1$.*

Proof. Let $\sigma'_n : R_n \to \hat{T}_\omega(R)$ be the composition of the map σ_n followed by the inclusion map of $T_\omega(R)$ into $\hat{T}_\omega(R)$, and let $\rho'_n : T_\omega(R) \to R_n$ be the restriction of ρ_n to $T_\omega(R)$. Then $\rho_n \cdot \sigma'_n = 1_{R_n} = \rho'_n \cdot \sigma_n$ gives the claim. □

Let us set, for each $n \geq 1$, $H_n = K_n \cap T_\omega(R) = \mathrm{Ker}(\rho'_n)$. The H_n form a descending chain of ideals of $T_\omega(R)$ and we have

$$T_\omega(R)/H_n \cong (K_n + T_\omega(R))/K_n = \hat{T}_\omega(R)/K_n$$

since $K_n + T_\omega(R) = \hat{T}_\omega(R)$, as is easily seen. So $T_\omega(R)$ is dense in $\hat{T}_\omega(R)$ with respect to the topology which has the K_n's as basis of neighborhoods of 0, and $\hat{T}_\omega(R)$ is the completion of $T_\omega(R)$ in the topology of the ideals H_n.

3 Even and odd matrices

We continue by assuming that R is an arbitrary commutative ring. Given two matrices $T(r_n)$ and $T(s_n)$ in $\hat{T}_\omega(R)$, their product is a certain matrix $T(t_n) \in \hat{T}_\omega(R)$, where the elements $t_n \in R$ ($n \geq 1$) are sums of products $r_i s_{n-i}$, for suitable $0 \leq i \leq n$.

The particular shape of the matrices in $\hat{T}_\omega(R)$ ensures that the following conditions are satisfied:

(i) t_{2i+1} has no summands of the form $r_{2m+1}s_{2n}$ or $r_{2m}s_{2n+1}$.

(ii) t_{2i} has no summands of the form $r_{2m}s_{2n}$ or $r_{2m+1}s_{2n+1}$.

(iii) t_{2i+1} has no summands of the form $r_{2m}s_{2n}$.

Conditions (i) and (ii) depend on the fact that in every column of a matrix in $\hat{T}_\omega(R)$ one has alternately elements with odd and even indices; in order to check (iii), it is enough to notice that, in the column of the matrix $T(s_n)$ whose first element is s_{2i+1}, the second, fourth, sixth, ... entries are all zero.

We introduce now a basic distinction among the matrices in $\hat{T}_\omega(R)$.

Definition. A matrix $T(r_n) \in \hat{T}_\omega(R)$ is called *even* (respectively, *odd*) if $r_{2n+1} = 0$ (respectively, $r_{2n} = 0$) for all n.

We shall need in this section a special family of matrices, which will play a distinguished role for the ideal theoretic structure of $\hat{T}_\omega(R)$. For all $n \geq 1$ let $I(n) = T(0, \ldots, 0, 1, 0, \ldots)$ be the matrix whose first row has a unique non-zero entry equal to 1 at the n-th place. The matrices we are interested in are the matrices $I(2^n + 1)$ for $n \geq 0$. It is easy to see that $I(2^n + 1)$ is the $\omega \times \omega$ block diagonal matrix $\mathrm{Diag}(J_n, J_n, J_n, \ldots)$, where J_n is the $2^{n+1} \times 2^{n+1}$ matrix

$$J_n = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}.$$

Theorem 3.1. (a) *The set $\mathcal{O}$ of the odd matrices is a subring of $\hat{T}_\omega(R)$ isomorphic to $\hat{T}_\omega(R)$;*

(b) *the set $\mathcal{E}$ of the even matrices is a principal ideal of $\hat{T}_\omega(R)$, generated by the matrix $I(2)$ and satisfying $\mathcal{E}^2 = 0$;*

(c) *$\hat{T}_\omega(R) = \mathcal{O} \,(+)\, \mathcal{E}$, the idealization of the $\mathcal{O}$-module $\mathcal{E}$.*

Proof. (a) The set $\mathcal{O}$ is obviously closed under differences; it is also closed under products by property (ii) above. Since the unit matrix is an odd matrix, $\mathcal{O}$ is a subring of $\hat{T}_\omega(R)$. Consider now the map $\delta : \hat{T}_\omega(R) \to \mathcal{O}$ defined by setting

$$\delta(T(r_1, r_2, r_3, r_4, \ldots)) = T(r_1, 0, r_2, 0, r_3, 0, r_4, \ldots).$$

The map δ is obviously a bijection. It is additive and it sends the unit matrix I to itself; in order to see that it is also multiplicative, it is better to look at δ^{-1}, which sends the odd matrix $O = T(r_1, 0, r_3, 0, r_5, 0, \ldots)$ to the matrix $\delta^{-1}(O) = T(r_1, r_3, r_5, r_7, \ldots)$. The map δ^{-1} sends each 2×2 block

$$\begin{pmatrix} r_{2n+1} & 0 \\ 0 & r_{2n+1} \end{pmatrix}$$

to the single element r_{2n+1} of $\delta^{-1}(O)$. Since the multiplication in O can be made on the 2×2 blocks, δ^{-1} is a ring homomorphism.

(b) It is easy to check that $I(2) \cdot T(r_1, r_2, r_3, \ldots) = T(0, r_1, 0, r_3, 0, r_5, 0, \ldots)$, hence $I(2)\hat{T}_\omega(R) = \mathcal{E}$. The square of $\mathcal{E}$ vanishes, since $I(2)^2 = 0$.

(c) It is obvious that every matrix $T \in \hat{T}_\omega(R)$ can be uniquely written as $T = O + E$, with $O \in \mathcal{O}$ and $E \in \mathcal{E}$; for the multiplication we have that $(O + E)\cdot(O' + E') = OO' + OE' + O'E$, since $EE' = 0$, which is the multiplication in the idealization ring. □

We call the maps δ the "diluting" isomorphism and δ^{-1} the "squeezing" isomorphism. Note that

$$\delta(I(2)) = I(3),\ \delta(I(3)) = I(5),\ \ldots,\ \delta(I(2^n + 1) = I(2^{n+1} + 1),\ \ldots.$$

Consider now the ring embedding $\phi : \hat{T}_\omega(R) \to \hat{T}_\omega(R)$ acting as the diluting isomorphism δ. Setting $\mathcal{O}_n = \mathrm{Im}(\phi^n)$, we have an infinite descending chain of subrings of $\hat{T}_\omega(R)$

$$\hat{T}_\omega(R) > \mathcal{O} = \mathcal{O}_1 > \mathcal{O}_2 > \mathcal{O}_3 > \cdots$$

such that $\mathcal{O}_n \cong \hat{T}_\omega(R)$ for all n, and $\bigcap_n \mathcal{O}_n = T_1(R) \cong R$.

It is immediate to check that, for each $n \geq 1$, the matrix $T(r_i)$ belongs to $\mathcal{O}_n$ if and only if the only possibly non-zero elements r_i are those indexed by $2^n \cdot k + 1$ for $k \geq 0$.

On the other hand, the ideal $\mathcal{E}$ of the even matrices coincides with $\mathrm{Ker}(\psi)$, where

$$\psi : \hat{T}_\omega(R) \xrightarrow{\pi} \mathcal{O} \xrightarrow{\delta^{-1}} \hat{T}_\omega(R)$$

is the surjective ring homomorphism obtained by composing the canonical projection $\pi : \hat{T}_\omega(R) \to \mathcal{O}$ defined by $\pi(T(r_n)) = T(r_1, 0, r_3, 0, r_5, \ldots)$, with the squeezing isomorphism $\delta^{-1} : \mathcal{O} \to \hat{T}_\omega(R)$; in fact, we have

$$\psi(T(r_1, r_2, r_3, r_4, r_5, \ldots)) = T(r_1, r_3, r_5, \ldots),$$

thus $\psi(T(r_i)) = 0$ if and only if $T(r_i) \in \mathcal{E}$.

For every $n \geq 1$ set $\mathcal{E}_n = \mathrm{Ker}(\psi^n)$. Since for each $n \geq 1$ one has

$$\psi^n(T(r_i)) = T(r_1, r_{2^n+1}, r_{2^n 2+1}, r_{2^n 3+1}, r_{2^n 4+1}, \ldots)$$

we derive that

$$\mathcal{E}_n = \{T(r_i) \mid r_{2^n k+1} = 0 \text{ for all } k\}.$$

Therefore we have the infinite ascending chain of ideals of $\hat{T}_\omega(R)$:

$$0 < \mathcal{E} = \mathcal{E}_1 < \mathcal{E}_2 < \mathcal{E}_3 < \cdots$$

which shows that $\hat{T}_\omega(R)$ is not a Noetherian ring. For each n we have a ring isomorphism $\hat{T}_\omega(R)/\mathcal{E}_n \cong \hat{T}_\omega(R)$.

Also the chain of ideals of $T_\omega(R)$:

$$0 < \mathcal{E}_1 \cap T_\omega(R) < \mathcal{E}_2 \cap T_\omega(R) < \mathcal{E}_3 \cap T_\omega(R) < \cdots$$

is strictly increasing, hence also the ring $T_\omega(R)$ is not Noetherian. Since the restriction of ψ to $T_\omega(R)$ induces a surjective ring homomorphism onto $T_\omega(R)$, we have analogously that $T_\omega(R)/(\mathcal{E}_n \cap T_\omega(R)) \cong T_\omega(R)$ for all n.

The following technical result will be needed.

Lemma 3.2. (a) *A matrix $T(r_i)$ belongs to the principal ideal $I(2^n+1)\cdot\hat{T}_\omega(R)$ for $n \geq 0$ if and only if $r_i = 0$ for all indices i satisfying.*

$$2^n 2k < i \leq 2^n(2k+1) \quad (k = 0, 1, 2, \ldots).$$

(b) *If $r < s$, then $I(2^r+1)\cdot I(2^s+1) = I(2^r+2^s+1)$.*

Proof. (a) It is enough to observe that, given a matrix $T(r_i) \in \hat{T}_\omega(R)$, $I(2^n+1)\cdot T(r_i)$ coincides with the matrix $T(s_i)$ whose first row coincides with the (2^n+1)-th row of $T(r_i)$; this row starts with 2^n zeros, then has 2^n entries $r_{2^n+1}, r_{2^n+2}, \ldots, r_{2^{n+1}}$, then there are again 2^n zeros, and then the 2^n entries $r_{2^{n+1}+1}, r_{2^{n+1}+2}, \ldots, r_{2^{n+2}}$, and so on.

(b) Recall that $I(2^r+1)\cdot I(2^s+1)$ coincides with the matrix $T(r_i)$, where $[r_i]$ is the (2^r+1)-th row of $I(2^s+1)$. An inspection to this row completes the proof. □

The following result collects some properties of the subrings $\mathcal{O}_n$ and of the ideals $\mathcal{E}_n$.

Theorem 3.3. (a) *For every $n \geq 1$, $\hat{T}_\omega(R) = \mathcal{O}_n \circ \mathcal{E}_n$;*

(b) *the ideal $\mathcal{E}_n$ is generated by the matrices $I(2), I(3), \ldots, I(2^{n-1}+1)$;*

(c) *for every $n \geq 1$, $\mathcal{E}_n$ is a nilpotent ideal of index $n+1$.*

Proof. (a) The above discussion shows that every matrix $T \in \hat{T}_\omega(R)$ can be written in a unique way as $T = O_n + E_n$, with $O_n \in \mathcal{O}_n$ and $E_n \in \mathcal{E}_n$; note that $\mathcal{E}_n^2 \neq 0$ for $n > 1$. For the multiplication we have that $(O_n + E_n)\cdot(O_n' + E_n') = O_n O_n' + O_n E_n' + O_n' E_n + E_n E_n'$, which is the multiplication in the Shores' construction.

(b) Recall that $\mathcal{E}_n = \{T(r_i) \mid r_{2^n k+1} = 0 \text{ for all } k \geq 0\}$. The only indices such that the corresponding entries of the first row in each of $I(2), I(3), \ldots, I(2^{n-1}+1)$ are zero are exactly those of the form $2^n k + 1$ for all $k \geq 0$, by Lemma 3.2 (a); hence the claim follows.

(c) The elements of $\mathcal{E}_n^{n+1}$ are linear combination with coefficients in $\hat{T}_\omega(R)$ of products of $n+1$ matrices taken in the set $I(2), I(3), \ldots, I(2^{n-1}+1)$; hence in each of these products it appears one of these matrices with exponent 2, hence the product vanishes. Moreover, Lemma 3.2 (b) shows that the product $I(2)\cdot I(3)\cdot\ldots\cdot I(2^{n-1}+1) \neq 0$. □

By Theorem 3.1 (c), $\hat{T}_\omega(R) = \mathcal{O}_1 \,(+)\, \mathcal{E}_1$, and, by Theorem 3.1 (a), we have the ring isomorphism $\delta : \hat{T}_\omega(R) \to \mathcal{O}_1$; we deduce from Theorem 3.1 (b) that

$$\mathcal{O}_1 = \delta(\mathcal{O}_1) \,(+)\, \delta(\mathcal{E}_1) = \mathcal{O}_2 \,(+)\, \delta(I(2))\mathcal{O}_1 = \mathcal{O}_2 \,(+)\, I(3)\mathcal{O}_1.$$

From these facts we derive that

$$\hat{T}_\omega(R) = [\mathcal{O}_2 \,(+)\, I(3)\mathcal{O}_1] \,(+)\, I(2)\mathcal{O}_0$$

where we have set $\hat{T}_\omega(R) = \mathcal{O}_0$.

Iterating this process, and recalling that $\delta(I(2^n+1)) = I(2^{n+1}+1)$, we have the following

Corollary 3.4. *For every $n \geq 1$, $\hat{T}_\omega(R)$ decomposes in the following way:*

$$[[\cdots[\mathcal{O}_n\,(+)\,I(2^{n-1}+1)\mathcal{O}_{n-1}]\,(+)\,I(2^{n-2}+1)\mathcal{O}_{n-2}]\,(+)\ldots]\,(+)\,I(3)\mathcal{O}_1]\,(+)\,I(2)\mathcal{O}_0.$$

□

The next result gives information on the ideal $\mathcal{N}$ which is the union of the ideals $\mathcal{E}_n$.

Corollary 3.5. (a) *The ideal $\mathcal{N} = \bigcup_{n\geq 1}\mathcal{E}_n$ of $\hat{T}_\omega(R)$ is contained in the nilradical of $\hat{T}_\omega(R)$ and it is not nilpotent;*

(b) $\mathcal{N} = \bigoplus_{n\geq 1} I_{2^{n-1}+1}\mathcal{O}_{n-1}$.

Proof. (a) From Theorem 3.3 (c) it follows that every element of $\mathcal{N}$ is nilpotent. Lemma 3.2 (b) shows that, fixed $1 \leq r_1 < r_2 < \cdots < r_k$, the product $I(2^{r_1}+1)\cdot \ldots \cdot I(2^{r_k}+1)$ is not zero, hence $\mathcal{N}$ is not nilpotent.

(b) From Corollary 3.4 we derive that, for every $n \geq 1$, we have the direct decomposition of R-modules:

$$\mathcal{E}_n = I(2^{n-1}+1)\mathcal{O}_{n-1} \oplus I(2^{n-2}+1)\mathcal{O}_{n-2} \oplus \cdots \oplus I(2)\mathcal{O}_0$$

from which the conclusion immediately follows. □

4 Units

Our goal in this section is to describe the groups of the units of the two rings $T_\omega(R)$ and $\hat{T}_\omega(R)$. Furthermore, in view of Theorem 3.1 (c) and Theorem 3.3 (a), we are interested in describing also the groups of the units of rings which are idealizations of a commutative ring R by an R-module B, and, more generally, of rings of the form $R \circ B$, where B is an R-algebra (possibly without 1).

We start with the following very simple result.

Lemma 4.1. *Let $S = R\,(+)\,B$ be the idealization of the commutative ring R by the R-module B, and look at R as a subring of S via the canonical embedding. Then*

(a) *for the group of the units of S we have that $U(S) = U(R) \times V$, where V is the subgroup of $U(S)$: $V = \{[1\ b] \mid b \in B\}$;*

(b) *the multiplicative group V is isomorphic to the additive group of B.*

Proof. (a) The vector $[r\ b] \in S$ $(r \in R, b \in B)$ is a unit of S if and only if $r \in U(R)$ and $r^{-1}b = -rb'$ for some $b' \in B$; hence b is an arbitrary element of B. In this case, $[r\ b]$ can be written in a unique way as a product

$$[r\ 0]\cdot[1\ r^{-1}b]$$

hence $U(S)$ is the direct product of its subgroups $U(R)$ and V.

(b) It is enough to note that $[1\ b]\cdot[1\ b'] = [1\ b+b']$. □

From Lemma 4.1 we can easily derive the structure of the group of the units of the ring $T_\omega(R)$.

Proposition 4.2. *Given a commutative ring R with 1, the group of the units of the ring $T_\omega(R)$ is isomorphic to the additive Abelian group $U \oplus F$, where U is the group of the units of R written additively, and F is the additive group of the free R-module $\bigoplus_{\aleph_0} R$.*

Proof. As in Section 1, denote by R_1 the ring R itself, and by R_n the ring which is the $(n-1)$-th step in the process of self-idealization, starting with R_1. Applying repeatedly Lemma 4.1, we see that, for $n > 1$, $U(R_n) = U(R) \times V_1 \times \cdots \times V_n$, where V_n is a multiplicative subgroup of $U(R_n)$ isomorphic to the additive group of R_{n-1}. But R_{n-1} is isomorphic as R-module to R^{n-1}, and $T_\omega(R)$ is the union of the subrings $T_n(R) \cong R_n$, hence $U(T_\omega(R))$ is isomorphic to the direct limit, with respect to the canonical embeddings, of the Abelian groups $U(R_n) = U(R) \times V_1 \times \cdots \times V_n$. This direct limit is clearly isomorphic to $U \oplus F$, where F is the additive group of the free R-module $\bigoplus_{\aleph_0} R$. □

The structure of the group of the units of the ring $\hat{T}_\omega(R)$ is similar to that of $U(T_\omega(R))$, but putting the direct product in place of the direct sum.

Proposition 4.3. *Given a commutative ring R with 1, the group of the units of the ring $\hat{T}_\omega(R)$ is isomorphic to the additive Abelian group $U \oplus F'$, where U is the group of the units of R written additively, and F' is the additive group of the R-module $\prod_{\aleph_0} R$.*

Proof. We have the isomorphisms

$$U(\hat{T}_\omega(R)) \cong U(\varprojlim R_n) \cong \varprojlim U(R_n).$$

In the proof of Proposition 4.2 we have seen that $U(R_n) = U(R) \times V_1 \times \cdots \times V_n$, and the maps in the inverse system considered above are the canonical projections, therefore the inverse limit is isomorphic to the additive Abelian group $U \oplus F'$, where F' is the additive group of the direct product $\prod_{\aleph_0} R$. □

From Lemma 4.1 and Theorem 3.1, we derive another characterization of the group of the units of $\hat{T}_\omega(R)$, which, however, does not say anything new on its structure, since obviously $\prod_{\aleph_0} R$ is isomorphic to its square.

Corollary 4.4. $U(\hat{T}_\omega(R)) \cong U(\hat{T}_\omega(R)) \times W$, *where W is a multiplicative subgroup of $U(\hat{T}_\omega(R))$ isomorphic to the additive group of $\prod_{\aleph_0} R$.*

Proof. Apply Lemma 4.1 to the idealization $\hat{T}_\omega(R) = \mathcal{O}(+)\mathcal{E}$ given in Theorem 3.1 (c), and remind that the ring $\mathcal{O}$ is isomorphic to $\hat{T}_\omega(R)$. Moreover, the R-module $\mathcal{E}$ is isomorphic to $\prod_{\aleph_0} R$, so the claim follows. □

Looking at Theorem 3.3 (a), we are now interested in describing the group $U(S)$, where $S = R \circ B$, for R an arbitrary commutative ring and B an R-algebra.

We need to introduce a new notion. Given a commutative algebra (possibly without 1) B, consider the new binary operation $[+]$ defined on B in the following way:

$$b \,[+]\, b' = b + b' + bb'.$$

It is easy to check that this operation is associative and commutative, and that the neutral element of $(B, [+])$ is 0, the neutral element with respect to the original sum $+$. The operation $[+]$ is similar to the *circle operation* defined in [14, p. 55] for a ring (not necessarily unitary), where the product bb' is replaced by its opposite $-bb'$.

An element $b \in B$ has an opposite with respect to $[+]$ exactly if there exists a $b' \in B$ such that $b + b' + bb' = 0$. If such a b' does exist, it is unique, but in general no such b' exists, as one can see by considering easy examples. As a matter of fact, if B has a unit 1, then -1 has never an opposite with respect to $[+]$. Note that, if B is a zero-ring, then the new sum $[+]$ coincides with the original sum $+$.

Let us denote by $B^{[+]}$ the subset of the elements of B which have an opposite with respect to $[+]$. Obviously $(B^{[+]}, [+])$ is an Abelian group.

We give some examples of groups $B^{[+]}$, for some simple R-algebras B. We leave to the reader the easy check.

Examples. (1) If $R = \mathbb{Z} = B$, then $B^{[+]} = \{0, -2\}$.

(2) If $R = \mathbb{Z}$ and $B = p\mathbb{Z}$, where p is an odd prime number, then $B^{[+]} = \{0\}$.

(3) If $R = K = B$ is a field, then $B^{[+]} = K \setminus \{-1\}$.

(4) If the algebra B has a unit 1, then $B^{[+]} = \{b \in B \mid b \in (1 + b)B\}$.

We can now prove the result we are interested in.

Proposition 4.5. *Given the commutative ring $S = R \circ B$, where R is a commutative ring and B a commutative R-algebra, and looking at R as a subring of S, then $U(S) = U(R) \times W$, where $W = \{[1\ b] \mid b \in B^{[+]}\}$ is a multiplicative group isomorphic to the additive group $B^{[+]}$.*

Proof. The vector $[r\ b]$ $(r \in R, b \in B)$ is a unit of S if and only if $r \in U(R)$ and there exists an element $b' \in B$ such that $r^{-1}b + rb' + bb' = 0$, or, equivalently, $r^{-1}b + rb' + (r^{-1}b)(rb') = 0$. In this case, $[r\ b]$ can be written in a unique way as a product

$$[r\ 0] \cdot [1\ r^{-1}b],$$

where, by what we have seen above, $r^{-1}b \in B^{[+]}$. Hence $U(S)$ is the direct product of its subgroups $U(R)$ and W. To conclude, note that W is isomorphic to the additive group of $B^{[+]}$, via the map which sends $[1\ b]$ to b; in fact we have

$$[1\ b] \cdot [1\ b'] = [1\ b + b' + bb'] = [1\ \ b\,[+]\,b']. \qquad \square$$

As an immediate application of Proposition 4.5 and Theorem 3.3 (a) we get another characterization of the group of the units of the ring $\hat{T}_\omega(R)$.

Corollary 4.6. *For every commutative ring R and $n > 1$ we have*

$$U(\hat{T}_\omega(R)) = U(\mathcal{O}_n) \times W_n$$

where W_n is a multiplicative subgroup of $U(\hat{T}_\omega(R))$ isomorphic to the additive group $\mathcal{E}_n^{[+]}$. □

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Bass numbers and semidualizing complexes

Sean Sather-Wagstaff

Abstract. Let R be a commutative local Noetherian ring. We prove that the existence of a chain of semidualizing R-complexes of length $(d + 1)$ yields a degree-d polynomial lower bound for the Bass numbers of R. We also show how information about certain Bass numbers of R provides restrictions on the lengths of chains of semidualizing R-complexes. To make this article somewhat self-contained, we also include a survey of some of the basic properties of semidualizing modules, semidualizing complexes and derived categories.

Keywords. Bass number, semidualizing complex, semidualizing module, totally reflexive.

AMS classification. 13D02, 13D05, 13D07, 13D25.

Introduction

Throughout this paper $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring.

A classical maxim from module theory states that the existence of certain types of R-modules forces ring-theoretic conditions on R. For instance, if R has a dualizing module, then R is Cohen–Macaulay and a homomorphic image of a Gorenstein ring.

This paper is concerned with the consequences of the existence of nontrivial *semidualizing* R-modules and, more generally, semidualizing R-complexes. In this introduction, we restrict our attention to the modules. Essentially, a semidualizing module differs from a dualizing module in that the semidualizing module is not required to have finite injective dimension. (See Section 1 for definitions and background information.) The set of isomorphism classes of semidualizing R-modules has a rich structure. For instance, it comes equipped with an ordering based on the notion of total reflexivity.

It is not clear that the existence of nontrivial semidualizing R-complexes should have any deep implications for R. For instance, every ring has at least one semidualizing R-module, namely, the free R-module of rank 1. However, Gerko [21] has shown that, when R is artinian, the existence of certain collections of semidualizing R-modules implies the existence of a lower bound for the Loewy length of R; moreover, if this lower bound is achieved, then the Poincaré series of k has a very specific form.

The first point of this paper is to show how the existence of nontrivial semidualizing modules gives some insight into the following questions of Huneke about the *Bass numbers* $\mu^i_R(R) = \operatorname{rank}_k(\operatorname{Ext}^i_R(k, R))$.

Question A. Let R be a local Cohen–Macaulay ring.

(a) If the sequence $\{\mu^i_R(R)\}$ is bounded, must it be eventually 0, that is, must R be Gorenstein?

(b) If the sequence $\{\mu^i_R(R)\}$ is bounded above by a polynomial in i, must R be Gorenstein?

(c) If R is not Gorenstein, must the sequence $\{\mu^i_R(R)\}$ grow exponentially?

Some progress on these questions has been made by Borna Lorestani, Sather-Wagstaff and Yassemi [8], Christensen, Striuli and Veliche [14], and Jorgensen and Leuschke [26]. However, each of these questions is still open in general. The following result gives the connection with semidualizing modules. It is contained in Theorem 3.5 and Corollary 3.6. Note that this result does not assume that R is Cohen–Macaulay.

Theorem B. *Let R be a local ring. If R has a semidualizing module that is neither dualizing nor free, then the sequence of Bass numbers $\{\mu^i_R(R)\}$ is bounded below by a linear polynomial in i and hence is not eventually constant. Moreover, if R has a chain of semidualizing modules of length $d+1$, then the sequence of Bass numbers $\{\mu^i_R(R)\}$ is bounded below by a polynomial in i of degree d.*

For readers who are familiar with semidualizing modules, the proof of this result is relatively straightforward when R is Cohen–Macaulay. We outline the proof here. Pass to the completion of R in order to assume that R is complete, and hence has a dualizing module D. The Bass series $I^R_R(t)$ of R then agrees with the Poincaré series $P^R_D(t)$ of D, up to a shift. Because of a result of Gerko [21, (3.3)] the given chain of semidualizing modules yields a factorization $P^R_D(t) = P_1(t)\cdots P_{d+1}(t)$ where each $P_i(t)$ is a power series with positive integer coefficients. The result now follows from straightforward numerics. The proof in the general case is essentially the same: after passing to the completion, use semidualizing *complexes* and the Poincaré series of a dualizing complex for R.

The second point of this paper is to show how information about certain Bass numbers of R force restrictions on the set of isomorphism classes of semidualizing R-modules. By way of motivation, we recall one of the main open questions about this set: must it be finite? Christensen and Sather-Wagstaff [13] have made some progress on this question, but the general question is still open. While the current paper does not address this question directly, we do show that this set cannot contain chains of arbitrary length under the reflexivity ordering. This is contained in the next result which summarizes Theorems 4.1 and 4.2. Note that the integer $\mu^g_R(R)$ in part (b) is the Cohen–Macaulay type of R.

Theorem C. *Let R be a local ring of depth g.*

(a) *If R has a chain of semidualizing modules of length d, then $d \leqslant \mu^{g+1}_R(R)$. Thus, the ring R does not have arbitrarily long chains of semidualizing modules.*

(b) *Assume that R is Cohen–Macaulay. Let h denote the number of prime factors of the integer $\mu^g_R(R)$, counted with multiplicity. If R has a chain of semidualizing modules of length d, then $d \leqslant h \leqslant \mu^g_R(R)$. In particular, if $\mu^g_R(R)$ is prime, then every semidualizing R-module is either free or dualizing for R.*

As an introductory application of these ideas, we have the following:

Example D. Let k be a field and set $R = k[\![X,Y]\!]/(X^2, XY)$. For each semidualizing R-module C, one has $C \cong R$. Indeed, the semidualizing property implies that $\beta_0^R(C)\mu_R^0(C) = \mu_R^0(R) = 1$ where $\beta_0^R(C)$ is the minimal number of generators of C. It follows that C is cyclic, so $C \cong R/\operatorname{Ann}_R(C) \cong R$. See Facts 1.4 and 1.20.

We prove more facts about the semidualizing objects for this ring in Example 4.4.

We now summarize the contents of and philosophy behind this paper. Section 1 contains the basic properties of semidualizing modules needed for the proofs of Theorems B and C. Section 2 outlines the necessary background on semidualizing complexes needed for the more general versions of Theorems B and C, which are the subjects of Sections 3 and 4. Because the natural habitat for semidualizing complexes is the derived category $\mathsf{D}(R)$, we include a brief introduction to this category in Appendix A for readers who desire some background.

Sections 1 and 2 are arguably longer than necessary for the proofs of the results of Sections 3 and 4. Moreover, Section 1 is essentially a special case of Section 2. This is justified by the third point of this paper: We hope that, after seeing our applications to Question A, some readers will be motivated to learn more about semidualizing objects. To further encourage this, Section 1 is a brief survey of the theory for modules. We hope this will be helpful for readers who are familiar with dualizing modules, but possibly not familiar with dualizing complexes.

Section 2 is a parallel survey of the more general semidualizing complexes. It is written for readers who are familiar with dualizing complexes and the category of chain complexes and who have at least some knowledge about the derived category.

For readers who find their background on the derived category lacking, Appendix A contains background material on this subject. Our hope is to impart enough information about this category so that most readers get a feeling for the ideas behind our proofs. As such, we stress the connections between this category and the category of R-modules.

1 Semidualizing modules

This section contains an introduction to our main players when they are modules. These are the semidualizing modules, which were introduced independently (with different terminology) by Foxby [17], Golod [22], Vasconcelos [30] and Wakamatsu [31]. They generalize Grothendieck's notion of a dualizing module [24] and encompasses duality theories with respect to dualizing modules and with respect to the ring R.

Definition 1.1. Let C be an R-module. The *homothety homomorphism* associated to C is the R-module homomorphism $\chi_C^R\colon R \to \operatorname{Hom}_R(C,C)$ given by $\chi_C^R(r)(c) = rc$. The R-module C is *semidualizing* if it satisfies the following conditions:

(1) the R-module C is finitely generated;

(2) the homothety map $\chi_C^R\colon R \to \operatorname{Hom}_R(C,C)$ is an isomorphism; and

(3) for all $i \geqslant 1$, we have $\operatorname{Ext}_R^i(C,C) = 0$.

An R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$, and the isomorphism class of a given semidualizing R-module C is denoted $[C]$.

Example 1.2. The R-module R is semidualizing, so R has a semidualizing module.

Remark 1.3. For this article, we have assumed that the ring R is local. While this assumption is not necessary for the definitions and basic properties of semidualizing modules, it does make the theory somewhat simpler.

Specifically, let S be a commutative Noetherian ring, not necessarily local, and let C be an S-module. Define the homothety homomorphism $\overline{\chi}^S_C\colon S \to \operatorname{Hom}_S(C,C)$, the semidualizing property, and the set $\mathfrak{S}_0(S)$ as in 1.1. It is straightforward to show that the semidualizing property is local, that is, that C is a semidualizing S-module if and only if $C_{\mathfrak{n}}$ is a semidualizing $S_{\mathfrak{n}}$-module for each maximal ideal $\mathfrak{n} \subset S$. For instance, every finitely generated projective S-module of rank 1 is semidualizing. In other words, the Picard group $\operatorname{Pic}(S)$ is a subset of $\mathfrak{S}_0(S)$. Also, the group $\operatorname{Pic}(S)$ acts on $\mathfrak{S}_0(S)$ in a natural way: for each semidualizing S-module C and each finitely generated projective S-module L of rank 1, the S-module $L \otimes_S C$ is semidualizing. This action is trivial when S is local as the Picard group of a local ring contains only the free module of rank 1.

While this gives the nonlocal theory more structure to investigate, one can view this additional structure as problematic, for the following reason. Fix a semidualizing S-module C and a finitely generated projective S-module L of rank 1. Define the terms "totally C-reflexive" and "totally $L \otimes_S C$-reflexive" as in 1.10. It is straightforward to show that an S-module G is totally C-reflexive if and only if it is totally $L \otimes_S C$-reflexive. In particular, when $\operatorname{Pic}(S)$ is nontrivial, the reflexivity ordering on $\mathfrak{S}_0(S)$, defined as in 1.17, is not antisymmetric. Indeed, one has $[C] \trianglelefteq [L \otimes_S C] \trianglelefteq [C]$, even though $[C] = [L \otimes_S C]$ if and only if $L \cong S$.

One can overcome the lack of antisymmetry by considering the set $\overline{\mathfrak{S}_0}(S)$ of orbits in $\mathfrak{S}_0(S)$ under the Picard group action. (Indeed, investigations of $\overline{\mathfrak{S}_0}(S)$ can be found in the work of Avramov, Iyengar, and Lipman [7] and Frankild, Sather-Wagstaff and Taylor [19].) However, we choose to avoid this level of generality in the current paper, not only for the sake of simplicity, but also because our applications in Section 3 and 4 are explicitly for local rings.

For the record, we note that another level of complexity arises when the ring S is of the form $S_1 \times S_2$ where S_1 and S_2 are (nonzero) commutative Noetherian rings. In this setting, the semidualizing S-modules are all of the form $C_1 \oplus C_2$ where each C_i is a semidualizing S_i-module. In other words, each connected component of $\operatorname{Spec}(S)$ contributes a degree of freedom to the elements of $\mathfrak{S}_0(S)$, and to $\overline{\mathfrak{S}_0}(S)$. For further discussion, see [18, 19].

The next three facts contain fundamental properties of semidualizing modules.

Fact 1.4. Let C be a semidualizing R-module. The isomorphism $R \cong \operatorname{Hom}_R(C,C)$ implies that $\operatorname{Ann}_R(C) = 0$. It follows that $\operatorname{Supp}_R(C) = \operatorname{Spec}(R)$ and so $\dim(C) = \dim(R)$. Furthermore C is cyclic if and only if $C \cong R$: for the nontrivial implication, if C is cyclic, then $C \cong R/\operatorname{Ann}_R(C) \cong R$. In particular, if $C \not\cong R$, then $\beta_0^R(C) \geqslant 2$. Here $\beta_0^R(C)$ is the 0th Betti number of C, i.e., the minimal number of generators of C.

Furthermore, the isomorphism $R \cong \operatorname{Hom}_R(C, C)$ also implies that $\operatorname{Ass}_R(C) = \operatorname{Ass}(R)$. It follows that an element $x \in \mathfrak{m}$ is C-regular if and only if it is R-regular. When x is R-regular, one can show that the R/xR-module C/xC is semidualizing; see [18, (4.5)]. Hence, by induction, we have $\operatorname{depth}_R(C) = \operatorname{depth}(R)$.

Fact 1.5. If R is Gorenstein, then every semidualizing R-module is isomorphic to R; see [11, (8.6)] or Theorem 4.1. (Note that the assumption that R is local is crucial here because of Remark 1.3.) The converse of this statement holds when R has a dualizing module by [11, (8.6)]; the converse can fail when R does not have a dualizing module by [12, (5.5)]. Compare this with Fact 2.6.

Fact 1.6. A result of Foxby [17, (4.1)], Reiten [28, (3)] and Sharp [29, (3.1)] says that R has a dualizing module if and only if R is Cohen–Macaulay and a homomorphic image of a Gorenstein ring. Hence, if R is complete and Cohen–Macaulay, then Cohen's structure theorem implies that R has a dualizing module. Compare this with Fact 2.7.

We next give the first link between semidualizing modules and Bass numbers.

Fact 1.7. Assume that R is Cohen–Macaulay of depth g. If R has a dualizing module D, then for every integer $i \geqslant 0$ we have $\mu_R^{i+g}(R) = \beta_i^R(D)$. Moreover, if D' is a dualizing module for $\widehat{R}$, then for each $i \geqslant 0$ we have $\mu_R^{i+g}(R) = \mu_{\widehat{R}}^{i+g}(\widehat{R}) = \beta_i^{\widehat{R}}(D')$; see e.g. [4, (1.5.3),(2.6)] and [23, (V.3.4)]. Compare this with Fact 2.8.

Here is one of the main open questions in this subject. An affirmative answer for the case when R is Cohen–Macaulay and equicharacteristic is given in [13, (1)]. Note that it is crucial that R be local; see Remark 1.3. Also note that, while Theorem 4.2 shows that chains in $\mathfrak{S}_0(R)$ cannot have arbitrarily large length, the methods of this paper do not answer this question.

Question 1.8. Is the set $\mathfrak{S}_0(R)$ finite?

The next fact documents some fundamental properties.

Fact 1.9. When C is a finitely generated R-module, it is semidualizing for R if and only if the completion $\widehat{C}$ is semidualizing for $\widehat{R}$. See [11, (5.6)]. The essential point of the proof is that there are isomorphisms

$$\operatorname{Ext}^i_{\widehat{R}}(\widehat{C}, \widehat{C}) \cong \widehat{R} \otimes_R \operatorname{Ext}^i_R(C, C).$$

(The analogous result holds for the dualizing property by, e.g., [9, (3.3.14)].) Thus, the assignment $C \mapsto \widehat{C}$ induces a well-defined function $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(\widehat{R})$; this function is injective since, for finitely generated R-modules B and C, we have $B \cong C$ if and only if $\widehat{B} \cong \widehat{C}$. From [12, (5.5)] we know that this map can fail to be surjective. Compare this with Fact 2.13.

Next we summarize the aspects of duality with respect to semidualizing modules that are relevant for our results.

Definition 1.10. Let C and G be R-modules. The *biduality homomorphism* associated to C and G is the map $\overline{\delta}^C_G \colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G,C),C)$ given by $\overline{\delta}^C_G(x)(\phi) = \phi(x)$.

Assume that C is a semidualizing R-module. The R-module G is *totally C-reflexive* when it satisfies the following conditions:

(1) The R-module G is finitely generated;

(2) The biduality map $\overline{\delta}^C_G \colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G,C),C)$ is an isomorphism; and

(3) For all $i \geqslant 1$, we have $\operatorname{Ext}^i_R(G,C) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(G,C),C)$.

Fact 1.11. Let C be a semidualizing R-module. It is straightforward to show that every finitely generated free R-module is totally C-reflexive. The essential point of the proof is that there are isomorphisms

$$\operatorname{Ext}^i_R(R^n,C) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ C^n & \text{if } i = 0, \end{cases}$$

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(R^n,C),C) \cong \operatorname{Ext}^i_R(C^n,C) \cong \operatorname{Ext}^i_R(C,C)^n \cong \begin{cases} 0 & \text{if } i \neq 0, \\ R^n & \text{if } i = 0. \end{cases}$$

It follows that every finitely generated R-module M has a resolution by totally C-reflexive R-modules $\cdots \to G_1 \to G_0 \to M \to 0$. It is similarly straightforward to show that C is totally C-reflexive because

$$\operatorname{Ext}^i_R(C,C) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ R & \text{if } i = 0, \end{cases}$$

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,C),C) \cong \operatorname{Ext}^i_R(R,C) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ C & \text{if } i = 0. \end{cases}$$

Compare this with Facts 2.19 and 2.20.

The next definition was introduced by Golod [22].

Definition 1.12. Let C be a semidualizing R-module, and let M be a finitely generated R-module. If M has a bounded resolution by totally C-reflexive R-modules, then it has *finite G_C-dimension* and its *G_C-dimension*, denoted $\mathrm{G}_C\text{-}\dim_R(M)$ is the length of the shortest such resolution.

The next fact contains the ever-useful "AB-formula" for G_C-dimension and is followed by some of its consequences.

Fact 1.13. Let C be a semidualizing R-module. If B is a finitely generated R-module of finite G_C-dimension, then $\mathrm{G}_C\text{-}\dim_R(B) = \operatorname{depth}(R) - \operatorname{depth}_R(B)$; see [11, (3.14)] or [22]. When B is semidualizing, Facts 1.4 and 1.13 combine to show that B has finite G_C-dimension if and only if B is totally C-reflexive.

Fact 1.14. Let C be a semidualizing R-module. If $\operatorname{pd}_R(C) < \infty$, then $C \cong R$ Indeed, using Fact 1.4, the Auslander–Buchsbaum formula shows that C must be free, and the isomorphism $\operatorname{Hom}_R(C, C) \cong R$ implies that C is free of rank 1. (Note that this depends on the assumption that R is local; see Remark 1.3.) It follows that, if C is a non-free semidualizing R-module, then the Betti number $\beta_i^R(C)$ is positive for each integer $i \geqslant 0$. Compare this with Fact 2.14 and Lemma 3.2. Questions about the Betti numbers of semidualizing modules akin to those in Question A are contained in 4.5.

The next facts contain some fundamental properties of this notion of reflexivity.

Fact 1.15. Let C be a semidualizing R-module. A finitely generated R-module G is totally C-reflexive if and only if the completion $\widehat{G}$ is totally $\widehat{C}$-reflexive. The essential point of the proof is that there are isomorphisms

$$\begin{aligned}
\operatorname{Ext}^i_{\widehat{R}}(\widehat{G}, \widehat{C}) &\cong \widehat{R} \otimes_R \operatorname{Ext}^i_R(G, C), \\
\operatorname{Ext}^i_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{G}, \widehat{C}), \widehat{C}) &\cong \operatorname{Ext}^i_{\widehat{R}}(\widehat{R} \otimes_R \operatorname{Hom}_R(G, C), \widehat{R} \otimes_R C) \\
&\cong \widehat{R} \otimes_R \operatorname{Ext}^i_R(\operatorname{Hom}_R(G, C), C).
\end{aligned}$$

Furthermore, a finitely generated R-module M has finite G_C-dimension if and only if $\widehat{M}$ has finite $\mathrm{G}_{\widehat{C}}$-dimension. See [11, (5.10)] or [22]. Compare this with Fact 2.22.

Fact 1.16. Let C be a semidualizing R-module. If M is a finitely generated R-module of finite projective dimension, then M has finite G_C-dimension by Fact 1.11.

Let D be a dualizing R-module. If M is a maximal Cohen–Macaulay R-module, then M is totally D-reflexive by [9, (3.3.10)]. The converse holds because of the AB-formula 1.13. It follows that every finitely generated R-module N has finite G_D-dimension, as the fact that R is Cohen–Macaulay (cf. Fact 1.6) implies that some syzygy of N is maximal Cohen–Macaulay. Compare this with Fact 2.20.

Here is the ordering on $\mathfrak{S}_0(R)$ that gives the chains discussed in the introduction.

Definition 1.17. Given two classes $[B], [C] \in \mathfrak{S}_0(R)$, we write $[B] \trianglelefteq [C]$ when C is totally B-reflexive, that is, when C has finite G_B-dimension; see Fact 1.13. We write $[B] \triangleleft [C]$ when $[B] \trianglelefteq [C]$ and $[B] \neq [C]$.

The next facts contain some fundamental properties of this ordering.

Fact 1.18. Let C be a semidualizing R-module. Fact 1.16 implies that $[C] \trianglelefteq [R]$ and, if D is a dualizing R-module, then $[D] \trianglelefteq [C]$.

Fact 1.15 says that $[B] \trianglelefteq [C]$ in $\mathfrak{S}_0(R)$ if and only if $[\widehat{B}] \trianglelefteq [\widehat{C}]$ in $\mathfrak{S}_0(\widehat{R})$; also $[B] \triangleleft [C]$ in $\mathfrak{S}_0(R)$ if and only if $[\widehat{B}] \triangleleft [\widehat{C}]$ in $\mathfrak{S}_0(\widehat{R})$ by Fact 1.9. In other words, the injection $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(\widehat{R})$ perfectly respects the orderings on these two sets. Compare this with Fact 2.25.

Fact 1.19. Let B and C be semidualizing R-modules such that C is totally B-reflexive, that is, such that $[B] \trianglelefteq [C]$. By definition, this implies that $\operatorname{Ext}^i_R(C, B) = 0$ for all $i \geqslant 1$. In addition, the R-module $\operatorname{Hom}_R(C, B)$ is semidualizing and totally B-reflexive; see [11, (2.11)]. Compare this with Fact 2.26.

Here is the key to the proofs of our main results when R is Cohen–Macaulay.

Fact 1.20. Consider a chain $[C^0] \trianglelefteq [C^1] \trianglelefteq \cdots \trianglelefteq [C^d]$ in $\mathfrak{S}_0(R)$. Gerko [21, (3.3)] shows that there is an isomorphism

$$C^0 \cong \operatorname{Hom}_R(C^1, C^0) \otimes_R \cdots \otimes_R \operatorname{Hom}_R(C^d, C^{d-1}) \otimes_R C^d.$$

(Note that Fact 1.19 implies that each factor in the tensor product is a semidualizing R-module.) The proof is by induction on d, with the case $d = 1$ being the most important: The natural evaluation homomorphism $\xi\colon \operatorname{Hom}_R(C^1, C^0) \otimes_R C^1 \to C^0$ given by $\phi \otimes x \mapsto \phi(x)$ fits into the following commutative diagram:

$$\begin{array}{ccc}
R & \xrightarrow[\cong]{\overline{\chi}^R_{\operatorname{Hom}_R(C^1,C^0)}} & \operatorname{Hom}_R(\operatorname{Hom}_R(C^1, C^0), \operatorname{Hom}_R(C^1, C^0)) \\
{\scriptstyle \overline{\chi}^R_{C^0}}\Big\downarrow{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle \cong} \\
\operatorname{Hom}_R(C^0, C^0) & \xrightarrow{\operatorname{Hom}_R(\xi, C^0)} & \operatorname{Hom}_R(\operatorname{Hom}_R(C^1, C^0) \otimes_R C^1, C^0).
\end{array}$$

The unspecified isomorphism is Hom-tensor adjointness. The homomorphisms $\overline{\chi}^R_{C^0}$ and $\overline{\chi}^R_{\operatorname{Hom}_R(C^1,C^0)}$ are isomorphisms because C^0 and $\operatorname{Hom}_R(C^1, C^0)$ are semidualizing; see Fact 1.19. It follows that the homomorphism $\operatorname{Hom}_R(\xi, C^0)$ is an isomorphism. Since C^0 is semidualizing, it follows that ξ is also an isomorphism; see [10, (A.8.11)].

Moreover, if F^i is a free resolution of $\operatorname{Hom}_R(C^i, C^{i-1})$ for $i = 1, \ldots, d$ and F^{d+1} is a projective resolution of C^d, then the tensor product from Definition A.16

$$F^1 \otimes_R \cdots \otimes_R F^d \otimes_R F^{d+1}$$

is a free resolution of C^0. Compare this with Fact 2.27.

The final fact of this section demonstrates the utility of 1.20. It compares to 2.28.

Fact 1.21. The ordering on $\mathfrak{S}_0(R)$ is reflexive by Fact 1.11. Also, it is antisymmetric by [2, (5.3)]. The essential point in the proof of antisymmetry comes from Fact 1.20. Indeed, if $[B] \trianglelefteq [C] \trianglelefteq [B]$, then

$$B \cong \operatorname{Hom}_R(C, B) \otimes_R \operatorname{Hom}_R(B, C) \otimes_R B.$$

It follows that there is an equality of Betti numbers

$$\beta_0^R(B) = \beta_0^R(\mathrm{Hom}_R(C,B))\beta_0^R(\mathrm{Hom}_R(B,C))\beta_0^R(B)$$

and so $\mathrm{Hom}_R(C,B)$ and $\mathrm{Hom}_R(B,C)$ are cyclic. Fact 1.19 implies that $\mathrm{Hom}_R(C,B)$ is semidualizing, so we have $\mathrm{Hom}_R(C,B) \cong R$ by Fact 1.4. This yields the second isomorphism in the next sequence:

$$C \cong \mathrm{Hom}_R(\mathrm{Hom}_R(C,B),B) \cong \mathrm{Hom}_R(R,B) \cong B.$$

The first isomorphism follows from the fact that C is totally B-reflexive, and the third isomorphism is standard. We conclude that $[C] = [B]$.

The question of transitivity for this relation is another open question in this area. It is open, even for artinian rings containing a field. Compare this to Question 2.29.

Question 1.22. Let A, B and C be semidualizing R-modules. If B is totally A-reflexive and C is totally B-reflexive, must C be totally A-reflexive?

2 Semidualizing complexes

This section contains definitions and background material on semidualizing complexes. In a sense, these are derived-category versions of the semidualizing modules from the previous section. (For notation and background information on the derived category $\mathsf{D}(R)$, consult Appendix A.) Motivation also comes from Grothendieck's notion of a dualizing complex [23] and Avramov and Foxby's notion of a relative dualizing complex [4]. The general definition is due to Christensen [11].

Definition 2.1. Let C be an R-complex. The *homothety morphism* associated to C in the category of R-complexes $\mathsf{C}(R)$ is the morphism $\overline{\chi}_C^R\colon R \to \mathrm{Hom}_R(C,C)$ given by $\overline{\chi}_C^R(r)(c) = rc$. This induces a well-defined *homothety morphism* associated to C in $\mathsf{D}(R)$ which is denoted $\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C,C)$.

The R-complex C is *semidualizing* if it is homologically finite, and the homothety morphism $\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C,C)$ is an isomorphism in $\mathsf{D}(R)$. An R-complex D is *dualizing* if it is semidualizing and has finite injective dimension.

The first fact of this section describes this definition in terms of resolutions.

Fact 2.2. Let C be an R-complex. The morphism $\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C,C)$ in $\mathsf{D}(R)$ can be described using a free resolution F of C, in which case it is represented by the morphism $\overline{\chi}_F^R\colon R \to \mathrm{Hom}_R(F,F)$ in $\mathsf{C}(R)$. It can also be described using an injective resolution I of C, in which case it is represented by $\overline{\chi}_I^R\colon R \to \mathrm{Hom}_R(I,I)$. Compare this with [10, (2.1.2)]. As this suggests, the semidualizing property can be detected by any free (or injective) resolution of C; and, when C is semidualizing, the semidualizing property is embodied by every free resolution and every injective resolution. Here is the essence of the argument of one aspect of this statement; the others are similar. The resolutions F and I are connected by a quasiisomorphism $\alpha\colon F \xrightarrow{\simeq} I$ which yields the next commutative diagram in $\mathsf{C}(R)$:

$$\begin{array}{ccc} R & \xrightarrow{\overline{\chi}^R_F} & \operatorname{Hom}_R(F,F) \\ {\scriptstyle \overline{\chi}^R_I}\downarrow & & \simeq \downarrow {\scriptstyle \operatorname{Hom}_R(F,\alpha)} \\ \operatorname{Hom}_R(I,I) & \underset{\simeq}{\xrightarrow{\operatorname{Hom}_R(\alpha,I)}} & \operatorname{Hom}_R(F,I). \end{array}$$

Hence, $\overline{\chi}^R_F$ is a quasiisomorphism if and only if $\overline{\chi}^R_I$ is a quasiisomorphism.

The next fact compares Definitions 1.1 and 2.1.

Fact 2.3. An R-module C is semidualizing as an R-module if and only if it is semidualizing as an R-complex. To see this, let F be a free resolution of M. The condition $\operatorname{Ext}^i_R(C,C) = 0$ is equivalent to the condition $\operatorname{H}_{-i}(\operatorname{Hom}_R(F,F)) = 0$ because of the following isomorphisms:

$$\operatorname{Ext}^i_R(C,C) \cong \operatorname{H}_{-i}(\mathbf{R}\operatorname{Hom}_R(C,C)) \cong \operatorname{H}_{-i}(\operatorname{Hom}_R(F,F)).$$

(See, e.g., Fact A.21.) Thus, we assume that $\operatorname{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$. In particular, since $\operatorname{H}_i(R) = 0$ for all $i \neq 0$, the map $\operatorname{H}_i(\chi^R_C)\colon \operatorname{H}_i(R) \to \operatorname{H}_i(\mathbf{R}\operatorname{Hom}_R(C,C))$ is an isomorphism for all $i \neq 0$. Next, there is a commutative diagram of R-module homomorphisms where the unspecified isomorphisms are from Facts A.3 and A.23:

$$\begin{array}{ccc} R & \xrightarrow{\overline{\chi}^R_C} & \operatorname{Hom}_R(C,C) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{H}_0(R) & \xrightarrow{\operatorname{H}_0(\chi^R_C)} & \operatorname{H}_0(\mathbf{R}\operatorname{Hom}_R(C,C)). \end{array}$$

It follows that $\overline{\chi}^R_C$ is an isomorphism if and only if $\operatorname{H}_0(\chi^R_C)$ is an isomorphism, that is, if and only if χ^R_C is an isomorphism in $\mathsf{D}(R)$.

The next fact documents the interplay between the semidualizing property and the suspension operator.

Fact 2.4. It is straightforward to show that an R-complex C is semidualizing if and only if some (equivalently, every) shift $\Sigma^i C$ is semidualizing; see [11, (2.4)]. The essential point of the proof is that Fact A.22 yields natural isomorphisms

$$\mathbf{R}\operatorname{Hom}_R(\Sigma^i C, \Sigma^i C) \simeq \Sigma^{i-i}\mathbf{R}\operatorname{Hom}_R(C,C) \simeq \mathbf{R}\operatorname{Hom}_R(C,C)$$

that are compatible with the homothety morphisms χ^R_C and $\chi^R_{\Sigma^i C}$. The analogous statement for dualizing complexes follows from this because of the equality $\operatorname{id}_R(\Sigma^i C) = \operatorname{id}_R(C) - i$ from Fact A.15.

Remark 2.5. As in Remark 1.3, we pause to explain some of the issues that arise when investigating semidualizing complexes in the non-local setting. Let S be a commutative Noetherian ring, not necessarily local, and let C be an S-complex. Define the homothety homomorphism $\overline{\chi}^S_C\colon S \to \operatorname{Hom}_S(C, C)$, the semidualizing property, and the set $\mathfrak{S}(S)$ as in 2.1.

When $\operatorname{Spec}(S)$ is connected, the set $\mathfrak{S}(S)$ behaves similarly to $\mathfrak{S}_0(S)$: a nontrivial Picard group makes the ordering on $\mathfrak{S}(S)$ non-antisymmetric, and one can overcome this by looking at an appropriate set of orbits.

However, when $\operatorname{Spec}(S)$ is disconnected (that is, when S is of the form $S_1 \times S_2$ for (nonzero) commutative Noetherian rings S_1 and S_2) things are even more complicated than in the module-setting. Indeed, the semidualizing S-complexes are all of the form $\Sigma^i C_1 \oplus \Sigma^j C_2$ where each C_i is a semidualizing S_i-complex. In other words, each connected component of $\operatorname{Spec}(S)$ contributes essentially two degrees of freedom to the elements of $\mathfrak{S}(S)$. For further discussion, see [7, 18, 19].

The next two facts are versions of 1.5 and 1.6 for semidualizing complexes.

Fact 2.6. If R is Gorenstein, then every semidualizing R-complex C is isomorphic in $\mathsf{D}(R)$ to $\Sigma^i R$ for some integer i by [11, (8.6)]; see also Theorem 4.1. (Note that the assumption that R is local is crucial here because of Remark 2.5.) If R is Cohen–Macaulay, then every semidualizing R-complex C is isomorphic in $\mathsf{D}(R)$ to $\Sigma^i B$ for some integer i and some semidualizing R-module B by [11, (3.4)]. (In each case, we have $i = \inf(C)$. In the second case, we have $B \cong \operatorname{H}_i(C)$; see Facts A.4 and A.8. Again, this hinges on the assumption that R is local.) The converses of these statements hold when R has a dualizing complex by [11, (8.6)] and Fact 1.6; the converses can fail when R does not have a dualizing complex; see [12, (5.5)].

Fact 2.7. Grothendieck and Hartshorne [23, (V.10)] and Kawasaki [27, (1.4)] show that R has a dualizing complex if and only if R is a homomorphic image of a Gorenstein ring. In particular, if R is complete, then Cohen's structure theorem implies that R has a dualizing complex.

The next fact generalizes 1.7.

Fact 2.8. Assume for this paragraph that R has a dualizing complex D. Then there is a coefficientwise equality $I^R_R(t) = t^s P^R_D(t)$ where $s = \dim(R) - \sup(D)$; that is, for all $i \in \mathbb{Z}$ we have $\mu^i_R(R) = \beta^R_{i-s}(D)$; see e.g. [4, (1.5.3), (2.6)] and [23, (V.3.4)]. Also, we have $\sup(D) - \inf(D) = \dim(R) - \operatorname{depth}(R)$, that is, the range of nonvanishing homology of D is the same as the Cohen–Macaulay defect of R; see [11, (3.5)].

More generally, let D' be a dualizing complex for $\widehat{R}$. Then we have

$$I^R_R(t) = I^{\widehat{R}}_{\widehat{R}}(t) = t^s P^{\widehat{R}}_{D'}(t)$$

where $s = \dim(\widehat{R}) - \sup(D')$; in other words, for all $i \in \mathbb{Z}$ we have $\mu^i_R(R) =$

$\beta_{i-s}^{\widehat{R}}(D')$. Furthermore, we have

$$\sup(D') - \inf(D') = \dim(\widehat{R}) - \operatorname{depth}(\widehat{R}) = \dim(R) - \operatorname{depth}(R).$$

Compare this with Fact 1.7.

Fact 1.4 implies that a cyclic semidualizing R-module must be isomorphic to the ring R. Using the previous fact, we show next that a version of this statement for semidualizing complexes fails in general. Specifically, there exists a ring R that has a semidualizing R-complex C that is not shift-isomorphic to R even though its first nonzero Betti number is 1. See Example 4.4 for more on this ring.

Example 2.9. Let k be a field and set $R = k[\![X, Y]\!]/(X^2, XY)$. Then R is a complete local ring of dimension 1 and depth 0. Hence R has a dualizing complex D. Apply a shift if necessary to assume that $\inf(D) = 0$. Then Fact 2.8 provides the first equality in the next sequence:

$$P_D^R(t) = I_R^R(t) = 1 + 2t + 2t^2 + \cdots$$

while the second equality is from, e.g., [14, Ex. 1]. In particular, we have $\beta_0^R(D) = 1$ and $\beta_i^R(D) = 0$ for all $i < 0$ even though $D \not\simeq \Sigma^j R$ for all $j \in \mathbb{Z}$.

We shall use the next definition to equate semidualizing complexes that are essentially the same. This compares with the identification of isomorphic modules in Definition 2.1; see Fact 2.11.

Definition 2.10. Given two R-complexes B and C, if there is an integer i such that $C \simeq \Sigma^i B$, then B and C are *shift-isomorphic*.[1] The set of "shift-isomorphism classes" of semidualizing R-complexes is denoted $\mathfrak{S}(R)$, and the shift-isomorphism class of a semidualizing R-complex C is denoted $[C]$.

The next fact compares Definitions 1.1 and 2.10.

Fact 2.11. It is straightforward to show that the natural embedding of $\mathsf{M}(R)$ inside $\mathsf{D}(R)$ induces a natural injection $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$; see Facts 2.3 and A.3. This injection is surjective when R is Cohen–Macaulay by Fact 2.6. (Note that the assumption that R is local is essential here because of Remark 2.5.)

Here is the version of Question 1.8 for semidualizing complexes. Again, Remark 2.5 shows that the assumption that R is local is crucial. Fact 2.11 shows that an affirmative answer to Question 2.12 would yield an affirmative answer to Question 1.8. Also note that the methods of this paper do not answer this question, even though Theorem 4.2 shows that $\mathfrak{S}(R)$ cannot have arbitrarily long chains.

Question 2.12. Is the set $\mathfrak{S}(R)$ finite?

[1] This yields an equivalence relation on the class of all semidualizing R-complexes: (1) One has $C \simeq \Sigma^0 C$; (2) If $C \simeq \Sigma^i B$, then $B \simeq \Sigma^{-i} C$; (3) If $C \simeq \Sigma^i B$ and $B \simeq \Sigma^j A$, then $C \simeq \Sigma^{i+j} A$.

The next properties compare to those in Fact 1.9.

Fact 2.13. When C is a homologically finite R-complex, it is semidualizing for R if and only if the base-changed complex $\widehat{R} \otimes^{\mathbf{L}}_R C$ is semidualizing for $\widehat{R}$. The essential point of the proof is that Fact A.22 provides the following isomorphism in $\mathsf{D}(\widehat{R})$

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R C, \widehat{R} \otimes^{\mathbf{L}}_R C) \simeq \widehat{R} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, C)$$

which is compatible with the corresponding homothety morphisms. The parallel statement for dualizing objects also holds; see [11, (5.6)] and [23, (V.3.5)].

Given two homologically finite R-complexes B and C, we have $C \simeq \Sigma^i B$ if and only if $\widehat{R} \otimes^{\mathbf{L}}_R C \simeq \widehat{R} \otimes^{\mathbf{L}}_R \Sigma^i B$ by [18, (1.11)]. Combining this with the previous paragraph, we see that the assignment $C \mapsto \widehat{R} \otimes^{\mathbf{L}}_R C$ induces a well-defined injective function $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(\widehat{R})$. The restriction to $\mathfrak{S}_0(R)$ is precisely the induced map from Fact 1.9, and thus there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_0(R) & \hookrightarrow & \mathfrak{S}_0(\widehat{R}) \\ \downarrow & & \downarrow \\ \mathfrak{S}(R) & \hookrightarrow & \mathfrak{S}(\widehat{R}). \end{array}$$

The following fact compares to 1.14; see also Lemma 3.2 and Question 4.5.

Fact 2.14. If C is a semidualizing R-complex and $\mathrm{pd}_R(C) < \infty$, then $C \simeq \Sigma^i R$ where $i = \inf(C)$ by [11, (8.1)]. (As in Fact 1.14, this relies on the local assumption on R.)

Here is a version of Definition 1.10 for semidualizing complexes. It originates with the special cases of "reflexive complexes" from [23, 32]. The definition in this generality is from [11].

Definition 2.15. Let C and X be R-complexes. The *biduality morphism* associated to C and X in $\mathsf{C}(R)$ is the morphism $\overline{\delta}^C_X \colon X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, C), C)$ given by $((\overline{\delta}^C_X)_p(x))_q(\{\phi_j\}_{j\in\mathbb{Z}}) = (-1)^{pq}\phi_p(x)$. This yields a well-defined *biduality morphism* $\delta^C_X \colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ associated to C and X in $\mathsf{D}(R)$.

Assume that C is a semidualizing R-complex. The R-complex X is *C-reflexive* when it satisfies the following properties:

(1) the complex X is homologically finite;

(2) the biduality morphism $\delta^C_X \colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ in $\mathsf{D}(R)$ is an isomorphism; and

(3) the complex $\mathbf{R}\mathrm{Hom}_R(X, C)$ is homologically bounded, i.e., finite.

Remark 2.16. When C is a semidualizing R-complex, every homologically finite R-complex X has a well-defined G_C-dimension which is finite precisely when X is nonzero and C-reflexive. (Note that this invariant is not described in terms of resolutions.) We shall not need this invariant here; the interested reader should consult [11].

Remark 2.17. Avramov and Iyengar [6, (1.5)] have shown that condition (3) of Definition 2.15 is redundant when $C = R$. The same proof shows that this condition is redundant in general. However, the proof of this fact is outside the scope of the present article, so we continue to state this condition explicitly.

The next fact shows that, as with the semidualizing property, the reflexivity property is independent of the choice of resolutions.

Fact 2.18. Let C and X be R-complexes and assume that C is semidualizing. The biduality morphism $\delta_X^C\colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,C),C)$ in $\mathsf{D}(R)$ can be described using an injective resolution I of C, in which case it is represented by the morphism $\overline{\delta}_X^I\colon X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X,I),I)$. Compare this with [10, (2.1.4)]. As with the semidualizing property, reflexivity can be detected by any injective resolution of C; and, when X is C-reflexive, the reflexivity is embodied by every injective resolution. Here is the essence of the argument. Let I and J be injective resolutions of C. Fact A.21 implies that

$$\mathrm{Hom}_R(X,I) \simeq \mathbf{R}\mathrm{Hom}_R(X,C) \simeq \mathrm{Hom}_R(X,J)$$

and so $\mathrm{Hom}_R(X,I)$ is homologically bounded if and only if $\mathrm{Hom}_R(X,J)$ is homologically bounded. Furthermore, there is a quasiisomorphism $\alpha\colon I \xrightarrow{\simeq} J$, and this yields the next commutative diagram in $\mathsf{C}(R)$:

$$\begin{array}{ccc}
X & \xrightarrow{\quad\overline{\delta}_X^I\quad} & \mathrm{Hom}_R(\mathrm{Hom}_R(X,I),I) \\
{\scriptstyle \overline{\delta}_X^J}\downarrow & & \simeq\downarrow{\scriptstyle \mathrm{Hom}_R(\mathrm{Hom}_R(X,I),\alpha)} \\
\mathrm{Hom}_R(\mathrm{Hom}_R(X,J),J) & \xrightarrow[\simeq]{\mathrm{Hom}_R(\mathrm{Hom}_R(X,\alpha),J)} & \mathrm{Hom}_R(\mathrm{Hom}_R(X,I),J).
\end{array}$$

Hence $\overline{\delta}_X^I$ is a quasiisomorphism if and only if $\overline{\delta}_X^J$ is a quasiisomorphism.

We next compare Definition 2.15 with the corresponding notions from Section 1.

Fact 2.19. Let C be a semidualizing R-module, and let G be a finitely generated R-module. If G is totally C-reflexive, then it is C-reflexive as a complex. Indeed, the following isomorphisms imply that $\mathbf{R}\mathrm{Hom}_R(G,C)$ is homologically bounded.

$$\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(G,C)) \cong \mathrm{Ext}^{-i}(G,C) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ \mathrm{Hom}_R(G,C) & \text{if } i = 0. \end{cases}$$

(See Fact A.21.) Fact A.4 explains the first isomorphism in the next sequence

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(G,C) \simeq \mathrm{H}_0(\mathbf{R}\mathrm{Hom}_R(G,C)) &\cong \mathrm{Ext}^0_R(G,C) \cong \mathrm{Hom}_R(G,C) \\
\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(G,C),C)) &\cong \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(\mathrm{Hom}_R(G,C),C)) \\
&\cong \mathrm{Ext}^{-i}_R(\mathrm{Hom}_R(G,C),C) \\
&\cong \begin{cases} 0 & \text{if } i \neq 0, \\ G & \text{if } i = 0 \end{cases}
\end{aligned}$$

and the others follow from the previous display and Fact A.21. Thus, for all $i \neq 0$, the function $\mathrm{H}_i(\delta^C_G)\colon \mathrm{H}_i(G) \to \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(G,C),C))$ maps from 0 to 0 and thus is an isomorphism. To show that G is C-reflexive, it remains to show that the map $\mathrm{H}_0(\delta^C_G)$ is an isomorphism. Check that there is a commutative diagram

$$\begin{array}{ccc}
G & \xrightarrow[\cong]{\overline{\delta}^C_G} & \mathrm{Hom}_R(\mathrm{Hom}_R(G,C),C) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{H}_0(G) & \xrightarrow{\mathrm{H}_0(\delta^C_G)} & \mathrm{H}_0(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(G,C),C))
\end{array}$$

where the unspecified isomorphisms are essentially from Fact A.3. Thus $\mathrm{H}_0(\delta^C_G)$ is an isomorphism as desired.

More generally, a finitely generated R-module has finite G_C-dimension if and only if it is C-reflexive as an R-complex. (Thus, the converse of the second sentence of the previous paragraph fails in general.) Furthermore, a homologically finite R-complex X is C-reflexive if and only if there is an isomorphism $X \simeq H$ in $\mathsf{D}(R)$ where H is a bounded complex of totally C-reflexive R-modules. See [25, (3.1)].

The next fact includes versions of 1.11 and 1.16 for semidualizing complexes.

Fact 2.20. Let C be a semidualizing R-complex. Every finitely generated free R-module is C-reflexive, as is C itself. The essential point of the proof is that the following isomorphisms are compatible with the corresponding biduality morphisms:

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(R,C),C) \simeq \mathbf{R}\mathrm{Hom}_R(C,C) \simeq R,$$
$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,C),C) \simeq \mathbf{R}\mathrm{Hom}_R(R,C) \simeq C.$$

See [11, (2.8)] and Fact A.22.

If X is a homologically finite R-complex of finite projective dimension, then X is C-reflexive by [11, (2.9)]. If D is a dualizing R-complex, then every homologically finite R-complex is D-reflexive. Conversely, if the residue field k is C-reflexive, then C is dualizing. See [11, (8.4)] or [23, (V.2.1)].

As with the semidualizing property, reflexivity is independent of shift.

Fact 2.21. Let C be a semidualizing R-complex. It is straightforward to show that an R-complex X is C-reflexive if and only if some (equivalently, every) shift $\Sigma^i X$ is $\Sigma^j C$-reflexive for some (equivalently, every) integer j. The point is that Fact A.22 yields natural isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\Sigma^i X, \Sigma^j C) &\simeq \Sigma^{j-i}\mathbf{R}\mathrm{Hom}_R(X, C),\\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(\Sigma^i X, \Sigma^j C), \Sigma^j C) &\simeq \mathbf{R}\mathrm{Hom}_R(\Sigma^{j-i}\mathbf{R}\mathrm{Hom}_R(X, C), \Sigma^j C)\\ &\simeq \Sigma^{j-(j-i)}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)\\ &\simeq \Sigma^{i}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)\end{aligned}$$

that are compatible with δ_X^C and $\delta_{\Sigma^i X}^{\Sigma^j C}$.

The next fact is a version of 1.15 for semidualizing complexes.

Fact 2.22. If C is a semidualizing R-complex, then a given homologically finite R-complex X is C-reflexive if and only if the base-changed complex $\widehat{R}\otimes^{\mathbf{L}}_R X$ is $\widehat{R}\otimes^{\mathbf{L}}_R C$-reflexive; see [11, (5.10)]. The main point of the proof is that Fact A.22 provides the following isomorphisms in $\mathsf{D}(\widehat{R})$

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X, \widehat{R}\otimes^{\mathbf{L}}_R C) \simeq \widehat{R}\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C),\\ &\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X, \widehat{R}\otimes^{\mathbf{L}}_R C), \widehat{R}\otimes^{\mathbf{L}}_R C)\\ &\qquad\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, C), \widehat{R}\otimes^{\mathbf{L}}_R C)\\ &\qquad\simeq \widehat{R}\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)\end{aligned}$$

and that these isomorphisms are compatible with δ_X^C and $\delta_{\widehat{R}\otimes^{\mathbf{L}}_R X}^{\widehat{R}\otimes^{\mathbf{L}}_R C}$.

Here is the ordering on $\mathfrak{S}(R)$ used in our main results.

Definition 2.23. Given two classes $[B], [C] \in \mathfrak{S}(R)$, we write $[B] \trianglelefteq [C]$ when C is B-reflexive; we write $[B] \triangleleft [C]$ when $[B] \trianglelefteq [C]$ and $[B] \neq [C]$.

The following fact compares this relation with the one from Definition 1.17.

Fact 2.24. Combining Fact 1.13 and the last paragraph of Fact 2.19, we see that, if B and C are semidualizing R-modules, then $[B] \trianglelefteq [C]$ in $\mathfrak{S}(R)$ if and only if $[B] \trianglelefteq [C]$ in $\mathfrak{S}_0(R)$, and $[B] \triangleleft [C]$ in $\mathfrak{S}(R)$ if and only if $[B] \triangleleft [C]$ in $\mathfrak{S}_0(R)$ That is, the map $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$ perfectly respects the orderings on these two sets.

The next facts compare with 1.18 and 1.19.

Fact 2.25. Let C be a semidualizing R-complex. Fact 2.20 implies that $[C] \trianglelefteq [R]$, and if D is a dualizing R-complex, then $[D] \trianglelefteq [C]$.

Fact 2.22 says that $[B] \trianglelefteq [C]$ in $\mathfrak{S}(R)$ if and only if $[\widehat{R} \otimes^{\mathbf{L}}_R B] \trianglelefteq [\widehat{R} \otimes^{\mathbf{L}}_R C]$ in $\mathfrak{S}(\widehat{R})$; also $[B] \triangleleft [C]$ in $\mathfrak{S}(R)$ if and only if $[\widehat{R} \otimes^{\mathbf{L}}_R B] \triangleleft [\widehat{R} \otimes^{\mathbf{L}}_R C]$ in $\mathfrak{S}(\widehat{R})$ by Fact 2.13. That is, the injection $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(\widehat{R})$ perfectly respects the orderings on these two sets.

Fact 2.26. Let B and C be semidualizing R-complexes such that C is B-reflexive, that is, such that $[B] \trianglelefteq [C]$. This implies that the complex $\mathbf{R}\mathrm{Hom}_R(C, B)$ is homologically finite, by definition. Moreover [11, (2.11)] shows that $\mathbf{R}\mathrm{Hom}_R(C, B)$ is semidualizing and B-reflexive. The main point of the proof is that there is a sequence of isomorphisms

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, B), \mathbf{R}\mathrm{Hom}_R(C, B)) \\
&\qquad\qquad \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, B) \otimes^{\mathbf{L}}_R C, B) \\
&\qquad\qquad \simeq \mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, B), B)) \\
&\qquad\qquad \simeq \mathbf{R}\mathrm{Hom}_R(C, C) \\
&\qquad\qquad \simeq R.
\end{aligned}$$

The first two isomorphisms are Hom-tensor adjointness A.22. The third isomorphism is from the assumption that C is B-reflexive, and the fourth isomorphism is from the fact that C is semidualizing.

The next fact compares to 1.20. It is the key tool for our main results.

Fact 2.27. Consider a chain $[C^0] \trianglelefteq [C^1] \trianglelefteq \cdots \trianglelefteq [C^d]$ in $\mathfrak{S}(R)$. Gerko [21, (3.3)] shows that there is an isomorphism

$$C^0 \simeq \mathbf{R}\mathrm{Hom}_R(C^1, C^0) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C^d, C^{d-1}) \otimes^{\mathbf{L}}_R C^d.$$

(Note that each factor in the tensor product is a semidualizing R-complex by Fact 2.26.) The proof is by induction on d, with the case $d = 1$ being the most important. Consider the natural evaluation morphism

$$\xi \colon \mathbf{R}\mathrm{Hom}_R(C^1, C^0) \otimes^{\mathbf{L}}_R C^1 \to C^0$$

which fits into the following commutative diagram:

$$\begin{array}{ccc}
R & \xrightarrow[\simeq]{\chi^R_{\mathbf{R}\mathrm{Hom}_R(C^1,C^0)}} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C^1, C^0), \mathbf{R}\mathrm{Hom}_R(C^1, C^0)) \\
{\scriptstyle \chi^R_{C^0}}\downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} \\
\mathbf{R}\mathrm{Hom}_R(C^0, C^0) & \xrightarrow{\mathbf{R}\mathrm{Hom}_R(\xi, C^0)} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C^1, C^0) \otimes^{\mathbf{L}}_R C^1, C^0).
\end{array}$$

The unspecified isomorphism is adjointness A.22. The morphisms $\chi^R_{\mathbf{R}\mathrm{Hom}_R(C^1,C^0)}$ and $\chi^R_{C^0}$ are isomorphisms in $\mathsf{D}(R)$ since C^0 and $\mathbf{R}\mathrm{Hom}_R(C^1, C^0)$ are semidualizing; see Fact 2.26. Hence, the morphism $\mathbf{R}\mathrm{Hom}_R(\xi, C^0)$ is an isomorphism in $\mathsf{D}(R)$. Since C^0 is semidualizing, it follows that ξ is also an isomorphism; see [10, (A.8.11)].

The final fact in this section compares to 1.21.

Fact 2.28. The ordering on $\mathfrak{S}(R)$ is reflexive by Fact 2.19. Also, it is antisymmetric by [2, (5.3)]. The essential point in the proof of antisymmetry comes from Fact 2.27. Indeed, if $[B] \trianglelefteq [C] \trianglelefteq [B]$, then

$$B \simeq \mathbf{R}\mathrm{Hom}_R(C, B) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B, C) \otimes^{\mathbf{L}}_R B.$$

It follows that there is an equality of Poincaré series

$$P^R_B(t) = P^R_{\mathbf{R}\mathrm{Hom}_R(C,B)}(t) P^R_{\mathbf{R}\mathrm{Hom}_R(B,C)}(t) P^R_B(t).$$

Since each Poincaré series has nonnegative integer coefficients, this display implies that $P^R_{\mathbf{R}\mathrm{Hom}_R(C,B)}(t) = t^r$ and $P^R_{\mathbf{R}\mathrm{Hom}_R(B,C)}(t) = t^{-r}$ for some integer r. So, we have $\mathbf{R}\mathrm{Hom}_R(C, B) \simeq \Sigma^r R$. This yields the second isomorphism in the next sequence

$$C \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, B), B) \simeq \mathbf{R}\mathrm{Hom}_R(\Sigma^r R, B) \simeq \Sigma^r B.$$

The first isomorphism follows from the fact that C is B-reflexive, and the third isomorphism is cancellation A.22. We conclude that $[C] = [B]$.

As in the module-setting, the question of the transitivity of this order remains open. An affirmative answer to Question 2.29 would yield an affirmative answer to Question 1.22 as the map $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$ is order-preserving by Fact 2.24. Questions 1.22 and 2.29 are equivalent when R is Cohen–Macaulay since, in this case, the map $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$ is surjective by Fact 2.11. (Again, this hinges on the local assumption for R by Remark 2.5.)

Question 2.29. Let A, B and C be semidualizing R-complexes. If B is A-reflexive and C is B-reflexive, must C be A-reflexive?

3 Bounding Bass numbers

We begin with three lemmas, the first of which essentially says that semidualizing complexes over local rings are indecomposable. Note that Remark 2.5 shows that the local hypothesis is essential.

Lemma 3.1. *Let R be a local ring and let C be a semidualizing R-complex. If X and Y are R-complexes such that $C \simeq X \oplus Y$, then either $X \simeq 0$ or $Y \simeq 0$.*

Proof. The condition $C \simeq X \oplus Y$ implies that $\mathrm{H}_i(C) \cong \mathrm{H}_i(X) \oplus \mathrm{H}_i(Y)$ for each index i. Hence, the fact that C is homologically finite implies that X and Y are both homologically finite as well.

We assume that $X \not\simeq 0$ and show that $Y \simeq 0$. Fact A.27 yields the following equality of formal Laurent series:

$$I_R^{\mathbf{R}\mathrm{Hom}_R(X,X)}(t) = P^R_X(t) I^X_R(t).$$

The condition $X \not\simeq 0$ implies $P_X^R(t) \neq 0$ and $I_R^X(t) \neq 0$ by Fact A.27. The display implies that $I_R^{\mathbf{R}\mathrm{Hom}_R(X,X)}(t) \neq 0$, and thus $\mathbf{R}\mathrm{Hom}_R(X,X) \not\simeq 0$. The fact that C is a semidualizing R-complex yields the first isomorphism in the next sequence:

$$\begin{aligned} R &\simeq \mathbf{R}\mathrm{Hom}_R(C,C) \simeq \mathbf{R}\mathrm{Hom}_R(X \oplus Y, X \oplus Y) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X,X) \oplus \mathbf{R}\mathrm{Hom}_R(X,Y) \oplus \mathbf{R}\mathrm{Hom}_R(Y,X) \oplus \mathbf{R}\mathrm{Hom}_R(Y,Y). \end{aligned}$$

The third isomorphism is additivity A.22. Because R is local, it is indecomposible as an R-module. By taking homology, we conclude that three of the summands in the second line of the previous sequence are homologically trivial, that is $\simeq 0$. Since $\mathbf{R}\mathrm{Hom}_R(X,X) \not\simeq 0$, it follows that $\mathbf{R}\mathrm{Hom}_R(Y,Y) \simeq 0$. Another application of Fact A.27 implies that

$$0 = I_R^{\mathbf{R}\mathrm{Hom}_R(Y,Y)}(t) = P_Y^R(t) I_R^Y(t).$$

Hence, either $P_Y^R(t) = 0$ or $I_R^Y(t) = 0$. In either case, we conclude that $Y \simeq 0$. □

The next lemma generalizes Fact 2.14. See also Fact 1.14 and Question 4.5. It is essentially a corollary of Lemma 3.1.

Lemma 3.2. *Let R be a local ring and let C be a semidualizing R-complex. Set $i = \inf(C)$. If there is an integer $j \geqslant i$ such that $\beta_j^R(C) = 0$, then $C \simeq \Sigma^i R$.*

Proof. By Fact 2.14, it suffices to show that $\mathrm{pd}_R(C) < \infty$. Let F be a minimal free resolution of C. The assumption $\beta_j^R(C) = 0$ implies that $F_j = 0$ by Fact A.27. Note that $F_i \neq 0$ since $H_i(C) \neq 0$, so we have $j > i$. Thus F has the following form:

$$F = \cdots \xrightarrow{\partial_{j+2}^F} F_{j+1} \to 0 \to F_{j-1} \xrightarrow{\partial_{j-1}^F} \cdots \xrightarrow{\partial_{i+1}^F} F_i \to 0.$$

Hence, we have $C \simeq F \cong F^1 \oplus F^2$ where

$$\begin{aligned} F^1 &= \cdots \longrightarrow 0 \longrightarrow 0 \to F_{j-1} \xrightarrow{\partial_{j-1}^F} \cdots \xrightarrow{\partial_{i+1}^F} F_i \to 0, \\ F^2 &= \cdots \xrightarrow{\partial_{j+2}^F} F_{j+1} \to 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \to 0. \end{aligned}$$

The condition $F_i \neq 0$ implies $F^1 \not\simeq 0$ as F^1 is minimal; see Fact A.14. Lemma 3.1 yields $F^2 \simeq 0$, so $C \simeq F^1 \oplus F^2 \simeq F^1$, which has finite projective dimension. □

When R is Cohen–Macaulay, the gist of the proof of the next lemma is found in Fact 1.20: the minimal free resolution of D factors as a tensor product of $d + 1$ minimal free resolutions of modules of infinite projective dimension. Note that the Cohen–Macaulay hypothesis in the final sentence of the statement is essential because of Example 2.9.

Lemma 3.3. *Let R be a local ring of depth g such that $\mathfrak{S}(R)$ contains a chain of length $d+1$. Then there exist power series $P_0(t), \ldots, P_d(t)$ with positive integer coefficients such that*

$$I_R^R(t) = t^g P_0(t) \cdots P_d(t).$$

If, in addition, R is Cohen–Macaulay, and p is the smallest prime factor of $\mu_R^g(R)$, then the constant term of each $P_i(t)$ is at least p.

Proof. Assume that $\mathfrak{S}(R)$ contains a chain $[C^0] \triangleleft [C^1] \triangleleft \cdots \triangleleft [C^d] \triangleleft [C^{d+1}]$.

We begin by proving the result in the case where R has a dualizing complex D. Applying a suspension if necessary, we assume that $\sup(D) = \dim(R)$; see Fact 2.4. It follows that $\inf(D) = g$ by Fact 2.8. From Fact 2.8 we conclude that there is a formal equality of power series $I_R^R(t) = P_D^R(t)$. Fact 2.25 implies that $[D] \trianglelefteq [C^0]$. Hence, we may extend the given chain by adding the link $[D] \trianglelefteq [C^0]$ if necessary in order to assume that $C^0 = D$. Similarly, we assume that $C^{d+1} = R$.

Fact 2.26 implies that, for $i = 0, \ldots, d$ the R-complex $\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i)$ is semidualizing and C^i-reflexive. We observe that $[\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i)] \neq [R]$. Indeed, if not, then $\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i) \simeq \Sigma^j R$ for some j, and this explains the second isomorphism in the following sequence:

$$C^{i+1} \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i), C^i) \simeq \mathbf{R}\mathrm{Hom}_R(\Sigma^j R, C^i) \simeq \Sigma^j C^i.$$

The first isomorphism is by Definition 2.15(2), and the third one is cancellation A.22. These isomorphisms imply $[C^{i+1}] = [C^i]$, contradicting our assumption $[C^{i+1}] \triangleleft [C^i]$.

Set $m_i = \inf(\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i))$. Lemma 3.2 implies that

$$\beta_j^R(\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i)) \geqslant 1$$

for each $j \geqslant m_i$. It follows that the series

$$P_i(t) = \sum_{n=0}^{\infty} \beta_{n+m_i}^R(\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i)) t^n$$

is a power series with positive integer coefficients such that

$$P^R_{\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i)}(t) = t^{m_i} P_i(t). \tag{3.3.1}$$

Fact 2.27 yields the first isomorphism in the following sequence:

$$\begin{aligned} D = C^0 &\simeq \mathbf{R}\mathrm{Hom}_R(C^1, C^0) \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(C^{d+1}, C^d) \otimes_R^{\mathbf{L}} C^{d+1} \\ &\simeq \mathbf{R}\mathrm{Hom}_R(C^1, C^0) \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(C^{d+1}, C^d). \end{aligned} \tag{3.3.2}$$

The equality and the second isomorphism are from the assumptions $C^0 = D$ and $C^{d+1} = R$. It follows from Fact A.23 that

$$g = \inf(D) = \sum_{i=0}^{d} m_i. \tag{3.3.3}$$

The second equality in the next sequence follows from (3.3.2) using Fact A.27:

$$\begin{aligned} I_R^R(t) &= P_D^R(t) \\ &= P^R_{\mathbf{R}\mathrm{Hom}_R(C^1,C^0)}(t) \cdots P^R_{\mathbf{R}\mathrm{Hom}_R(C^{d+1},C^d)}(t) \\ &= (t^{m_0} P_0(t)) \cdots (t^{m_d} P_d(t)) \\ &= t^g P_0(t) \cdots P_d(t). \end{aligned}$$

The first equality is by the choice of D; the third equality is from (3.3.1); and the fourth equality is from (3.3.3).

Assume for this paragraph that R is Cohen–Macaulay. Fact 2.6 yields an isomorphism $\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i) \simeq \Sigma^{s_i} B^i$ where $s_i = \inf(\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i))$ and B^i is the semidualizing R-module $\mathrm{H}_{s_i}(\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i))$. Since $\mathbf{R}\mathrm{Hom}_R(C^{i+1}, C^i)$ is non-free, Fact 1.4 implies that $\beta_0^R(B^i) \geqslant 2$; this is the constant term of $P_i(t)$. The formula $I_R^R(t) = t^g P_0(t) \cdots P_d(t)$ implies that $\mu_R^g(R)$ is the product of the constant terms of the $P_i(t)$; since each constant term is at least 2, it must be at least p. This completes the proof in the case where R has a dualizing complex.

Finally, we prove the result in general. The completion $\widehat{R}$ has a dualizing complex by Fact 2.7. Also, the given chain gives rise to the following chain in $\mathfrak{S}(\widehat{R})$

$$[\widehat{R} \otimes^{\mathbf{L}}_R C^0] \triangleleft [\widehat{R} \otimes^{\mathbf{L}}_R C^1] \triangleleft \cdots \triangleleft [\widehat{R} \otimes^{\mathbf{L}}_R C^d] \triangleleft [\widehat{R} \otimes^{\mathbf{L}}_R C^{d+1}]$$

by Fact 2.25. The previous case yields power series $P_0(t), \ldots, P_d(t)$ with positive integer coefficients such that $I_{\widehat{R}}^{\widehat{R}}(t) = t^{\mathrm{depth}(\widehat{R})} P_0(t) \cdots P_d(t)$. Hence, the desired conclusion follows from the equalities $g = \mathrm{depth}(R) = \mathrm{depth}(\widehat{R})$ and $I_R^R(t) = I_{\widehat{R}}^{\widehat{R}}(t)$, and the fact that R is Cohen–Macaulay if and only if $\widehat{R}$ is Cohen–Macaulay. □

Remark 3.4. It is straightforward to use Fact 2.25 to give a slight strengthening of Lemma 3.3. Indeed, the condition "$\mathfrak{S}(R)$ contains a chain of length $d + 1$" is stronger than necessary; the proof shows that one can derive the same conclusions only assuming that $\mathfrak{S}(\widehat{R})$ contains a chain of length $d + 1$. Similar comments hold true for the remaining results in this section and for the results of Section 4.

The next two results contain Theorem B from the introduction and follow almost directly from Lemma 3.3.

Theorem 3.5. *Let R be a local ring. If $\mathfrak{S}(R)$ contains a chain of length $d + 1$, then the sequence of Bass numbers $\{\mu_R^i(R)\}$ is bounded below by a polynomial in i of degree d.*

Proof. Assume that $\mathfrak{S}(R)$ contains a chain of length $d + 1$. Lemma 3.3 implies that there exist power series $P_0(t), \ldots, P_d(t)$ with positive integer coefficients satisfying the equality in the following sequence:

$$I_R^R(t) = t^{\operatorname{depth}(R)} P_0(t) \cdots P_d(t) \succeq t^{\operatorname{depth}(R)} \left(\sum_{n=0}^{\infty} t^n \right)^{d+1}.$$

The inequality follows from the fact that each coefficient of $P_j(t)$ is a positive integer.

It is well known that the degree-i coefficient of the series $\left(\sum_{n=0}^{\infty} t^n\right)^{d+1}$ is given by a polynomial in i of degree d. It follows that the same is true of the coefficients of the series $t^{\operatorname{depth}(R)} \left(\sum_{n=0}^{\infty} t^n\right)^{d+1}$. Hence, the degree-$i$ coefficient of the Bass series $I_R^R(t)$, i.e., the ith Bass number $\mu_R^i(R)$, is bounded below by such a polynomial. □

Corollary 3.6. *Let R be a local ring. If R has a semidualizing complex that is neither dualizing nor free, then the sequence of Bass numbers $\{\mu_R^i(R)\}$ is bounded below by a linear polynomial in i and hence is not eventually constant.*

Proof. The assumption on R yields a chain in $\mathfrak{S}(\widehat{R})$ of the form $[D'] \lhd [\widehat{C}] \lhd [\widehat{R}]$, so the result follows from Theorem 3.5 using the equality $\mu_R^i(R) = \mu_{\widehat{R}}^i(\widehat{R})$. □

4 Bounding lengths of chains of semidualizing complexes

In this section we use Lemma 3.3 to show how the Bass numbers of R in low degree can be used to bound the lengths of chains in $\mathfrak{S}(R)$. The first two results contain Theorem C from the introduction and focus on the first two nonzero Bass numbers. The results of this section are not exhaustive. Instead, they are meant to give a sampling of applications of Lemma 3.3. For instance, the same technique can be used to give similar bounds in terms of higher-degree Bass numbers.

Theorem 4.1. *Let R be a local Cohen–Macaulay ring of depth g, and let h denote the number of prime factors of the integer $\mu_R^g(R)$, counted with multiplicity. If R has a chain of semidualizing modules of length d, then $d \leqslant h \leqslant \mu_R^g(R)$. In particular, if $\mu_R^g(R)$ is prime, then every semidualizing R-module is either free or dualizing for R.*

Proof. By Lemma 3.3, the existence of a chain in $\mathfrak{S}_0(R)$ of length d yields a factorization $I_R^R(t) = t^g P_1(t) \cdots P_d(t)$ where each $P_i(t)$ is a power series with positive integer coefficients and constant term $a_i \geqslant 2$. We then have

$$\mu_R^g(R) = a_1 \cdots a_d$$

so the inequalities $d \leqslant h \leqslant \mu_R^g(R)$ follow from the basic properties of factorizations of integers.

Assume now that $\mu_R^g(R)$ is prime and let C be a semidualizing R-module. The ring $\widehat{R}$ has a dualizing module D' by Fact 1.6, and Fact 1.18 shows that there is a chain $[D'] \trianglelefteq [\widehat{C}] \trianglelefteq [\widehat{R}]$ in $\mathfrak{S}(\widehat{R})$. This chain must have length at most 1 since the Bass number $\mu_{\widehat{R}}^g(\widehat{R}) = \mu_R^g(R)$ is prime. Hence, either $\widehat{C} \cong \widehat{R}$ or $\widehat{C} \cong D'$. From Fact 1.9, it follows that the R-module C is either free or dualizing for R. □

Theorem 4.2. *Let R be a local ring of depth g. If R has a chain of semidualizing complexes of length d, then $d \leqslant \mu_R^{g+1}(R)$. In particular, the set $\mathfrak{S}(R)$ does not contain arbitrarily long chains.*

Proof. Assume that $\mathfrak{S}(R)$ contains a chain of length d. Lemma 3.3 yields power series $P_1(t), \dots, P_d(t)$ with positive integer coefficients such that

$$I_R^R(t) = t^g P_1(t) \cdots P_d(t). \tag{4.2.1}$$

For each index i, write $P_i(t) = \sum_{j=0}^{\infty} a_{i,j} t^j$. By calculating the degree $g+1$ coefficient in (4.2.1), we obtain the first equality in the following sequence:

$$\mu_R^{g+1}(R) = \sum_{i=1}^{d} \frac{a_{1,0} \cdots a_{d,0}}{a_{i,0}} a_{i,1} \geqslant \sum_{i=1}^{d} a_{i,1} \geqslant \sum_{i=1}^{d} 1 = d.$$

The inequalities are from the conditions $a_{j,0}, a_{i,1} \geqslant 1$. □

The next result gives an indication how other Bass numbers can also give information about the chains in $\mathfrak{S}(R)$.

Proposition 4.3. *Let R be a local ring of depth g.*

(a) *If $\mu_R^i(R) \leqslant i - g$ for some index $i \geqslant g$, then every semidualizing R-complex is either free or dualizing for R.*

(b) *Assume that R is Cohen–Macaulay and let p be the smallest prime divisor of $\mu_R^g(R)$. If $\mu_R^i(R) < 2p + i - g - 1$ for some index $i > g$, then every semidualizing R-module is either free or dualizing for R.*

Proof. We prove the contrapositive of each statement. Assume that R has a semidualizing complex that is neither free nor dualizing. The set $\mathfrak{S}(\widehat{R})$ then has a chain $[D] \lhd [C] \lhd [R]$, so Lemma 3.3 yields power series $P_1(t), P_2(t)$ with positive integer coefficients such that $I_R^R(t) = t^g P_1(t) P_2(t)$. Write $P_1(t) = \sum_{i=0}^{\infty} a_i t^i$ and $P_2(t) = \sum_{i=0}^{\infty} b_i t^i$. It follows that, for each index $i \geqslant g$, we have

$$\mu_R^i(R) = \sum_{j=0}^{i-g} a_j b_{i-g-j}. \tag{4.3.1}$$

(a) Since each $a_j, b_j \geqslant 1$, the equation (4.3.1) implies that

$$\mu_R^i(R) = \sum_{j=0}^{i-g} a_j b_{i-g-j} \geqslant \sum_{j=0}^{i-g} 1 = i - g + 1 > i - g.$$

(b) Assume that R is Cohen–Macaulay. Lemma 3.3 implies that $a_0, b_0 \geqslant p$. Assuming that $i > g$, equation (4.3.1) reads

$$\mu_R^i(R) = \sum_{j=0}^{i-g} a_j b_{i-g-j} \geqslant a_0 + b_0 + \sum_{j=1}^{i-g-1} 1 \geqslant 2p + i - g - 1.$$ □

The next example shows how Proposition 4.3 applies to the ring from 2.9.

Example 4.4. Let k be a field and set $R = k[\![X,Y]\!]/(X^2,XY)$. Then R is a complete local ring of dimension 1 and depth 0. From Example 2.9 we have $\mu_R^2(R) = 2$, so Proposition 4.3 implies that $\mathfrak{S}(R) = \{[R],[D]\}$.

We conclude this section with some questions that arise naturally from this work and from the literature on Bass numbers, followed by some discussion.

Question 4.5. Let R be a local ring and C a non-free semidualizing R-complex.

(a) Must the sequence $\{\beta_i^R(C)\}$ eventually be strictly increasing?

(b) Must the sequence $\{\beta_i^R(C)\}$ be nondecreasing?

(c) Must the sequence $\{\beta_i^R(C)\}$ be unbounded?

(d) Can the sequence $\{\beta_i^R(C)\}$ be bounded above by a polynomial in i?

(e) Must the sequence $\{\beta_i^R(C)\}$ grow exponentially?

(f) If C is not dualizing for R, must the sequence $\{\mu_R^i(R)\}$ be strictly increasing?

Remark 4.6. Question 4.5(a) relates to [14, Question 2] where it is asked whether the Bass numbers of a non-Gorenstein local ring must eventually be strictly increasing. (Note that Example 2.9 shows that they need not be always strictly increasing.) If Question 4.5(a) is answered in the affirmative, then so is [14, Question 2] since the Bass numbers of R are given as the Betti numbers of the dualizing complex for $\widehat{R}$. Part (b) is obviously similar to part (a), and parts (c)–(e) of Question 4.5 relate similarly to Question A.

Question 4.5(f) is a bit different. The idea here is that the existence of a semidualizing R-complex that is not free and not dualizing provides a chain of length 2 in $\mathfrak{S}(\widehat{R})$. Hence, Lemma 3.3 gives a nontrivial factorization $I_R^R(t) = t^g P_1(t)P_2(t)$ where each $P_i(t) = t^{m_i} P_{C^i}^R(t)$ for some non-free semidualizing R-complex C^i. If the coefficients of each $P_i(t)$ are strictly increasing, then the coefficients of the product $I_R^R(t) = t^g P_1(t)P_2(t)$ are also strictly increasing. Note, however, that the positivity of the coefficients of the $P_i(t)$ is not enough to ensure that the coefficients of $I_R^R(t)$ are strictly increasing. For instance, we have

$$(2 + t + t^2 + t^3 + \cdots)(5 + t + t^2 + t^3 + \cdots) = 10 + 7t + 8t^2 + 9t^3 + \cdots .$$

A Homological algebra for complexes

This appendix contains notation and useful facts about chain complexes for use in Sections 2–4. We do not attempt to explain every detail about complexes that we use. For this, we recommend that the interested reader consult a text like [20] or [23]. Instead, we give heuristic explanations of the ideas coupled with explicit connections to the corresponding notions for modules. This way, the reader who is familiar with the homological algebra of modules can get a feeling for the subject and will possibly be motivated to investigate the subject more deeply.

Definition A.1. A *chain complex of R-modules*, or *R-complex* for short, is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

such that $\partial^X_i \partial^X_{i+1} = 0$ for each $i \in \mathbb{Z}$. The ith *homology module* of an R-complex X is the R-module $\mathrm{H}_i(X) = \mathrm{Ker}(\partial^X_i)/\,\mathrm{Im}(\partial^X_{i+1})$. A *morphism* of chain complexes $f\colon X \to Y$ is a sequence of R-module homomorphisms $\{f_i\colon X_i \to Y_i\}_{i\in\mathbb{Z}}$ making the following diagram commute:

$$\begin{array}{ccccccccc}
X & & \cdots & \xrightarrow{\partial^X_{i+1}} & X_i & \xrightarrow{\partial^X_i} & X_{i-1} & \xrightarrow{\partial^X_{i-1}} & \cdots \\
{\scriptstyle f}\big\downarrow & & & & {\scriptstyle f_i}\big\downarrow & & {\scriptstyle f_{i-1}}\big\downarrow & & \\
Y & & \cdots & \xrightarrow{\partial^Y_{i+1}} & Y_i & \xrightarrow{\partial^Y_i} & Y_{i-1} & \xrightarrow{\partial^Y_{i-1}} & \cdots
\end{array}$$

that is, such that $\partial^Y_i f_i = f_{i-1}\partial^X_i$ for all $i \in \mathbb{Z}$. A morphism $f\colon X \to Y$ induces an R-module homomorphism $\mathrm{H}_i(f)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ for each $i \in \mathbb{Z}$. The morphism f is a *quasiisomorphism* if the map $\mathrm{H}_i(f)$ is an isomorphism for each $i \in \mathbb{Z}$.

Notation A.2. The category of R-complexes is denoted $\mathsf{C}(R)$. The category of R-modules is denoted $\mathsf{M}(R)$. Isomorphisms in each of these categories are identified by the symbol $\cong$, and quasiisomorphisms in $\mathsf{C}(R)$ are identified by the symbol $\simeq$.

The derived category of R-complexes is denoted $\mathsf{D}(R)$. Morphisms in $\mathsf{D}(R)$ are equivalence classes of diagrams of morphisms in $\mathsf{C}(R)$. Isomorphisms in $\mathsf{D}(R)$ correspond to quasiisomorphisms in $\mathsf{C}(R)$ and are identified by the symbol $\simeq$.

The connection between $\mathsf{D}(R)$ and $\mathsf{M}(R)$ comes from the following.

Fact A.3. Each R-module M is naturally associated with an R-complex concentrated in degree 0, namely the complex $0 \to M \to 0$. We use the symbol M to designate both the module and the associated complex. With this notation we have

$$\mathrm{H}_i(M) \cong \begin{cases} M & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

This association gives rise to a full embedding of the module category $\mathsf{M}(R)$ into the derived category $\mathsf{D}(R)$. In particular, for R-modules M and N we have $M \cong N$ in $\mathsf{M}(R)$ if and only if $M \simeq N$ in $\mathsf{D}(R)$.

Fact A.4. Let X and Y be R-complexes. If $X \simeq Y$ in $\mathsf{D}(R)$, then we have $\mathrm{H}_i(X) \cong \mathrm{H}_i(Y)$ for all $i \in \mathbb{Z}$. The converse fails in general. However, there is an isomorphism $X \simeq \mathrm{H}_0(X)$ in $\mathsf{D}(R)$ if and only if $\mathrm{H}_i(X) = 0$ for all $i \neq 0$. In particular, we have $X \simeq 0$ in $\mathsf{D}(R)$ if and only if $\mathrm{H}_i(X) = 0$ for all $i \in \mathbb{Z}$.

The next invariants conveniently measure the homological position of a complex.

Definition A.5. The *supremum* and *infimum* of an R-complex X are, respectively

$$\sup(X) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \quad \text{and} \quad \inf(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$$

with the conventions $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$.

Fact A.6. Let X be an R-complex. If $X \not\simeq 0$, then $-\infty \leqslant \inf(X) \leqslant \sup(X) \leqslant \infty$. Also $\inf(X) = \infty$ if and only if $X \simeq 0$ if and only if $\sup(X) = -\infty$. If $M \neq 0$ is an R-module, considered as an R-complex, then $\inf(M) = 0 = \sup(M)$.

The next construction allows us to "shift" a given R-complex, which is useful, for instance, when we want the nonzero homology modules in nonnegative degrees.

Definition A.7. Let X be an R-complex. For each integer i, the ith *suspension* or *shift* of X is the complex $\Sigma^i X$ given by $(\Sigma^i X)_j = X_{j-i}$ and $\partial_j^{\Sigma^i X} = (-1)^i \partial_{j-i}^X$.

Fact A.8. If X is an R-complex, then $\Sigma^i X$ is obtained by shifting X to the left by i degrees and multiplying the differential by $(-1)^i$. In particular, if M is an R-module, then $\Sigma^i M$ is a complex that is concentrated in degree i. It is straightforward to show that $\mathrm{H}_j(\Sigma^i X) \cong \mathrm{H}_{j-i}(X)$, and hence $\inf(\Sigma^i X) = \inf(X) + i$ and $\sup(\Sigma^i X) = \sup(X) + i$.

For most of this investigation, we focus on R-complexes with only finitely many nonzero homology modules, hence the next terminology.

Definition A.9. An R-complex X is *bounded* if $X_i = 0$ for $|i| \gg 0$. It is *homologically bounded below* if $\mathrm{H}_i(X) = 0$ for $i \ll 0$. It is *homologically bounded above* if $\mathrm{H}_i(X) = 0$ for $i \gg 0$. It is *homologically bounded* if $\mathrm{H}_i(X) = 0$ for $|i| \gg 0$. It is *homologically degreewise finite* if each homology module $\mathrm{H}_i(X)$ is finitely generated. It is *homologically finite* if the module $\mathrm{H}(X) = \coprod_{i \in \mathbb{Z}} \mathrm{H}_i(X)$ is finitely generated.

The next fact summarizes elementary translations of these definitions.

Fact A.10. An R-complex X is homologically bounded below if $\inf(X) > -\infty$. It is homologically bounded above if $\sup(X) < \infty$. Hence, it is homologically bounded if $\inf(X) > -\infty$ and $\sup(X) < \infty$, that is, if it is homologically bounded both above and below. The complex X is homologically finite if it is homologically both degreewise finite and bounded.

Each of the properties defined in A.9 is invariant under shift. For instance, an R-complex X is homologically finite if and only if some (equivalently, every) shift $\Sigma^i X$ is homologically finite; see Fact A.8.

For modules, many of these notions are trivial:

Fact A.11. An R-module M is always homologically bounded as an R-complex. It is homologically finite as an R-complex if and only if it is finitely generated.

As with modules, there are various useful types of resolutions of R-complexes.

Definition A.12. Let X be an R-complex. An *injective resolution*[2] of X is an R-complex J such that $X \simeq J$ in $\mathsf{D}(R)$, each J_i is injective, and $J_i = 0$ for $i \gg 0$. The complex X has *finite injective dimension* if it has an injective resolution J such that $J_i = 0$ for $i \ll 0$. More specifically, the injective dimension of X is

$$\mathrm{id}_R(X) = \inf\{\sup\{i \in \mathbb{Z} \mid J_{-i} \neq 0\} \mid J \text{ is an injective resolution of } X\}.$$

Dually, a *free resolution* of X is an R-complex F such that $F \simeq X$ in $\mathsf{D}(R)$, each F_i is free, and $F_i = 0$ for $i \ll 0$. The complex X has *finite projective dimension*[3] if it has a free resolution F such that $F_i = 0$ for $i \gg 0$. More specifically, the projective dimension of X is

$$\mathrm{pd}_R(X) = \inf\{\sup\{i \in \mathbb{Z} \mid F_i \neq 0\} \mid F \text{ is a free resolution of } X\}.$$

A free resolution F of X is *minimal*[4] if for each index i, the module F_i is finitely generated and $\mathrm{Im}(\partial_i^F) \subseteq \mathfrak{m} F_{i-1}$.

For modules, the notions from A.12 are the familiar ones.

Fact A.13. Let M be an R-module. An injective resolution of M as an R-module, in the traditional sense of an exact sequence of the form

$$0 \to M \to J_0 \xrightarrow{\partial_0} J_{-1} \xrightarrow{\partial_{-1}} \cdots$$

where each J_i is injective, gives rise to an injective resolution of M as an R-complex:

$$0 \to J_0 \xrightarrow{\partial_0} J_{-1} \xrightarrow{\partial_{-1}} \cdots .$$

Conversely, every injective resolution of M as an R-complex gives rise to an injective resolution of M as an R-module, though one has to work a little harder. Accordingly,

[2]Note that our injective resolutions are bounded above by definition. There are notions of injective (and projective) resolutions for unbounded complexes, but we do not need them here. The interested reader should consult [3] for information on these more general constructions.

[3]Since the ring R is local, every projective R-module is free. For this reason, we focus on free resolutions instead of projective ones. On the other hand, tradition dictates that the corresponding homological dimension is the "projective dimension" instead of the possibly confusing (though, potentially liberating) "free dimension".

[4]There is also a notion of minimal injective resolutions of complexes, but it is slightly more complicated, and we do not need it here.

the injective dimension of M as an R-module equals the injective dimension of M as an R-complex. Similar comments apply to free resolutions and projective dimension.

The next fact summarizes basic properties about existence of these resolutions.

Fact A.14. Let X be an R-complex. Then X has a free resolution if and only if it is homologically bounded below; when these conditions are met, it has a free resolution F such that $F_i = 0$ for all $i < \inf(X)$; see [5, (2.11.3.4)] or [15, (6.6.i)] or [16, (2.6.P)]. Dually, the complex X has an injective resolution if and only if it is homologically bounded above; when these conditions are satisfied, it has an injective resolution J such that $J_i = 0$ for all $i > \sup(X)$. If X is homologically both degreewise finite and bounded below, then it has a minimal free resolution F, and one has $F_i = 0$ for all $i < \inf(X)$; see [1, Prop. 2] or [5, (2.12.5.2.1)].

These invariants interact with the shift operator as one might expect:

Fact A.15. It is straightforward to show that, if X is an R-complex and i is an integer, then $\mathrm{id}_R(\Sigma^i X) = \mathrm{id}_R(X) - i$ and $\mathrm{pd}_R(\Sigma^i X) = \mathrm{pd}_R(X) + i$.

The next constructions extend Hom and tensor product to the category $\mathsf{C}(R)$.

Definition A.16. Let X and Y be R-complexes. The *tensor product complex* $X \otimes_R Y$ and *homomorphism complex* $\mathrm{Hom}_R(X, Y)$ are defined by the formulas

$$
\begin{aligned}
(X \otimes_R Y)_i &= \coprod_{j \in \mathbb{Z}} X_j \otimes_R Y_{i-j}, \\
\partial_i^{X \otimes_R Y}(\{x_j \otimes y_{i-j}\}_{j \in \mathbb{Z}}) &= \{\partial_j^X(x_j) \otimes y_{i-j} + (-1)^{j-1} x_{j-1} \otimes \partial_{i-j+1}^Y(y_{i-j+1})\}, \\
\mathrm{Hom}_R(X, Y)_i &= \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(X_j, Y_{j+i}), \\
\partial_i^{\mathrm{Hom}_R(X,Y)}(\{\phi_j\}_{j \in \mathbb{Z}}) &= \{\partial_{j+i}^Y \phi_j - (-1)^i \phi_{j-1} \partial_j^X\}.
\end{aligned}
$$

When one of the complexes in this definition is a module, the resulting complexes have the form one should expect:

Fact A.17. Let X be an R-complex and M an R-module. The complexes $X \otimes_R M$, $M \otimes_R X$ and $\mathrm{Hom}_R(M, X)$ are exactly the complexes you would expect, namely

$$
X \otimes_R M = \cdots \xrightarrow{\partial_{i+1}^X \otimes M} X_i \otimes M \xrightarrow{\partial_i^X \otimes M} X_{i-1} \otimes M \xrightarrow{\partial_{i-1}^X \otimes M} \cdots,
$$

$$
M \otimes_R X = \cdots \xrightarrow{M \otimes \partial_{i+1}^X} M \otimes X_i \xrightarrow{M \otimes \partial_i^X} M \otimes X_{i-1} \xrightarrow{M \otimes \partial_{i-1}^X} \cdots,
$$

$$
\mathrm{Hom}_R(M, X) =
$$

$$
\cdots \xrightarrow{\mathrm{Hom}(M, \partial_{i+1}^X)} \mathrm{Hom}(M, X_i) \xrightarrow{\mathrm{Hom}(M, \partial_i^X)} \mathrm{Hom}(M, X_{i-1}) \xrightarrow{\mathrm{Hom}(M, \partial_{i-1}^X)} \cdots.
$$

On the other hand, the complex $\operatorname{Hom}(X, M)$ has the form you would expect, but the differentials differ by a sign:

$$\operatorname{Hom}_R(X, M) =$$
$$\cdots \xrightarrow{(-1)^i \operatorname{Hom}(\partial_i^X, M)} \operatorname{Hom}(X_i, M) \xrightarrow{(-1)^{i+1} \operatorname{Hom}(\partial_{i+1}^X, M)} \operatorname{Hom}(X_{i-1}, M) \cdots .$$

Note that this sign difference does not change the homology since it changes neither the kernels nor the images of the respective maps.

Here are some standard isomorphisms we shall need.

Fact A.18. Let X, Y and Z be R-complexes. The following natural isomorphisms are straightforward to verify, using the counterparts for modules in the first five, and using the definition in the last:

$$\operatorname{Hom}_R(R, X) \cong X, \qquad \text{(cancellation)}$$
$$X \otimes_R Y \cong Y \otimes_R X, \qquad \text{(commutativity)}$$
$$\operatorname{Hom}_R(X \oplus Y, Z) \cong \operatorname{Hom}_R(X, Z) \oplus \operatorname{Hom}_R(Y, Z), \qquad \text{(additivity)}$$
$$\operatorname{Hom}_R(X, Y \oplus Z) \cong \operatorname{Hom}_R(X, Z) \oplus \operatorname{Hom}_R(X, Y), \qquad \text{(additivity)}$$
$$\operatorname{Hom}_R(X \otimes_R Y, Z) \cong \operatorname{Hom}_R(X, \operatorname{Hom}_R(Y, Z)), \qquad \text{(adjointness)}$$
$$\operatorname{Hom}_R(\Sigma^i X, \Sigma^j Y) \cong \Sigma^{j-i} \operatorname{Hom}_R(X, Y). \qquad \text{(shift)}$$

Let S be a flat R-algebra. If each R-module X_i is finitely generated and $X_i = 0$ for $i \ll 0$, then

$$\operatorname{Hom}_S(S \otimes_R X, S \otimes_R Y) \cong S \otimes_R \operatorname{Hom}_R(X, Y). \qquad \text{(base-change)}$$

Bounded complexes yield bounded homomorphism and tensor product complexes. More specifically, the next fact follows straight from the definitions.

Fact A.19. Let X and Y be R-complexes. If $X_i = 0 = Y_j$ for all $i < m$ and all $j < n$, then $(X \otimes_R Y)_i = 0$ for all $i < m + n$. If $X_i = 0 = Y_j$ for all $i < m$ and all $j > n$, then $\operatorname{Hom}_R(X, Y)_i = 0$ for all $i > n - m$.

Here is the notation for derived functors in the derived category $\mathsf{D}(R)$.

Notation A.20. Let X and Y be R-complexes. The left-derived tensor product and right-derived homomorphism complexes in $\mathsf{D}(R)$ are denoted, respectively $X \otimes_R^{\mathbf{L}} Y$ and $\mathbf{R}\operatorname{Hom}_R(X, Y)$.

The complexes $X \otimes_R^{\mathbf{L}} Y$ and $\mathbf{R}\operatorname{Hom}_R(X, Y)$ are computed using the same rules as for computing Tor and Ext of modules:

Fact A.21. Let X and Y be R-complexes. If F is a free resolution of X and G is a free resolution of Y, then

$$X \otimes^{\mathbf{L}}_R Y \simeq F \otimes_R Y \simeq F \otimes_R G \simeq X \otimes_R G.$$

If F is a free resolution of X and I is an injective resolution of Y, then

$$\mathbf{R}\mathrm{Hom}_R(X, Y) \simeq \mathrm{Hom}_R(F, Y) \simeq \mathrm{Hom}_R(F, I) \simeq \mathrm{Hom}_R(X, I).$$

It follows that, if M and N are R-modules, then $\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i(M \otimes^{\mathbf{L}}_R N)$ and $\mathrm{Ext}^i_R(M, N) \cong \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(M, N))$ for every integer i.

The next isomorphisms follow from Fact A.18 using appropriate resolutions.

Fact A.22. If X, Y and Z are R-complexes, then there are isomorphisms in $\mathsf{D}(R)$

$$\mathbf{R}\mathrm{Hom}_R(R, X) \simeq X, \qquad \text{(cancellation)}$$

$$X \otimes^{\mathbf{L}}_R Y \simeq Y \otimes^{\mathbf{L}}_R X, \qquad \text{(commutativity)}$$

$$\mathbf{R}\mathrm{Hom}_R(X \oplus Y, Z) \simeq \mathbf{R}\mathrm{Hom}_R(X, Z) \oplus \mathbf{R}\mathrm{Hom}_R(Y, Z), \qquad \text{(additivity)}$$

$$\mathbf{R}\mathrm{Hom}_R(X, Y \oplus Z) \simeq \mathbf{R}\mathrm{Hom}_R(X, Z) \oplus \mathbf{R}\mathrm{Hom}_R(X, Y), \qquad \text{(additivity)}$$

$$\mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R Y, Z) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(Y, Z)), \qquad \text{(adjointness)}$$

$$\mathbf{R}\mathrm{Hom}_R(\Sigma^i X, \Sigma^j Y) \simeq \Sigma^{j-i}\mathbf{R}\mathrm{Hom}_R(X, Y). \qquad \text{(shift)}$$

Let S be a flat R-algebra. If X is homologically both degreewise finite and bounded below, then

$$\mathbf{R}\mathrm{Hom}_S(S \otimes^{\mathbf{L}}_R X, S \otimes^{\mathbf{L}}_R Y) \simeq S \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, Y). \qquad \text{(base-change)}$$

The following homological bounds are consequences of Fact A.19.

Fact A.23. Let X and Y be homologically bounded below R-complexes. Let F and G be free resolutions X and Y, respectively, such that $F_i = 0$ for $i < \inf(X)$ and $G_i = 0$ for $i < \inf(Y)$. It follows that, for $i < \inf(X) + \inf(Y)$, we have

$$\mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y) \cong \mathrm{H}_i(F \otimes_R G) = 0$$

and hence $\inf(X \otimes^{\mathbf{L}}_R Y) \geqslant \inf(X) + \inf(Y)$. Furthermore, the right exactness of tensor product yields the second isomorphism in the next sequence

$$\mathrm{H}_{\inf(X)+\inf(Y)}(X \otimes^{\mathbf{L}}_R Y) \cong \mathrm{H}_{\inf(X)+\inf(Y)}(F \otimes_R G) \cong \mathrm{H}_{\inf(X)}(X) \otimes_R \mathrm{H}_{\inf(Y)}(Y).$$

This corresponds to the well-known formula $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$ for modules M and N. If $\mathrm{H}_{\inf(X)}(X)$ and $\mathrm{H}_{\inf(Y)}(Y)$ are both finitely generated, e.g., if X and Y are both homologically degreewise finite, then Nakayama's Lemma implies that

$$\mathrm{H}_{\inf(X)+\inf(Y)}(X \otimes^{\mathbf{L}}_R Y) \cong \mathrm{H}_{\inf(X)}(X) \otimes_R \mathrm{H}_{\inf(Y)}(Y) \neq 0$$

and thus $\inf(X \otimes^{\mathbf{L}}_R Y) = \inf(X) + \inf(Y)$. Note that this explicitly uses the assumption that R is local.

A similar argument shows that, when Z is homologically bounded above, then the complex $\mathbf{R}\mathrm{Hom}_R(X, Z)$ is homologically bounded above: there is an inequality $\sup(\mathbf{R}\mathrm{Hom}_R(X, Z)) \leqslant \sup(Z) - \inf(X)$ and an isomorphism

$$\mathrm{H}_{\sup(Z)-\inf(X)}(\mathbf{R}\mathrm{Hom}_R(X, Z)) \cong \mathrm{Hom}_R(\mathrm{H}_{\inf(X)}(X), \mathrm{H}_{\sup(Z)}(Z)).$$

The next fact is a derived category version of the finite generation of Ext and Tor of finitely generated modules. It essentially follows from A.21.

Fact A.24. Let X and Y be R-complexes that are homologically both degreewise finite and bounded below. Let F and G be free resolutions of X and Y, respectively, such that each F_i and G_i is finitely generated. Then $F \otimes_R G$ is a free resolution of $X \otimes^{\mathbf{L}}_R Y$, and each R-module $(F \otimes_R G)_i$ is finitely generated. In particular, the complex $X \otimes^{\mathbf{L}}_R Y$ is homologically both degreewise finite and bounded below. If F and G are minimal, then $F \otimes_R G$ is a minimal free resolution of $X \otimes^{\mathbf{L}}_R Y$.

It takes a little more work to show that, if Z is homologically both degreewise finite and bounded above, then the R-complex $\mathbf{R}\mathrm{Hom}_R(X, Z)$ is homologically both degreewise finite and bounded above.

Here are some homological invariants that are familiar for modules.

Definition A.25. Let X be a homologically finite R-complex. The ith *Bass number* of X is the integer $\mu^i_R(X) = \mathrm{rank}_k(\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, X)))$, and the *Bass series* of X is the formal Laurent series $I^X_R(t) = \sum_{i\in\mathbb{Z}} \mu^i_R(X)t^i$. The ith *Betti number* of X is the integer $\beta^R_i(X) = \mathrm{rank}_k(\mathrm{H}_i(k \otimes^{\mathbf{L}}_R X))$, and the *Poincaré series* of X is the formal Laurent series $P^R_X(t) = \sum_{i\in\mathbb{Z}} \beta^R_i(X)t^i$.

Fact A.26. If M is an R-module, then we have $\mu^i_R(M) = \mathrm{rank}_k(\mathrm{Ext}^i_R(k, M))$ and $\beta^R_i(M) = \mathrm{rank}_k(\mathrm{Tor}^R_i(k, M)) = \mathrm{rank}_k(\mathrm{Ext}^i_R(M, k))$.

We conclude with useful formulas for the Poincaré and Bass series of, respectively, derived tensor products and derived homomorphism complexes.

Fact A.27. Let X and Y be R-complexes that are homologically both degreewise finite and bounded below. If F is a minimal free resolution of X, then $\beta^R_i(X) = \mathrm{rank}_R(F_i)$ for all $i \in \mathbb{Z}$. (Indeed the complex $k \otimes_R F$ has zero differential, and hence

$$\mathrm{H}_i(k \otimes^{\mathbf{L}}_R X) \cong \mathrm{H}_i(k \otimes_R F) \cong (k \otimes_R F)_i \cong k \otimes_R F_i.$$

The k-vector space rank of this module is precisely $\mathrm{rank}_R(F_i)$.) Combining this with Fact A.24, we conclude that

$$P^R_{X\otimes^{\mathbf{L}}_R Y}(t) = P^R_X(t)P^R_Y(t).$$

Furthermore, the equality $\beta_i^R(X) = \operatorname{rank}_R(F_i)$ for all $i \in \mathbb{Z}$ implies that $P_X^R(t) = 0$ if and only if $F = 0$, that is, if and only if $X \simeq 0$. See also Fact A.23.

Given an R-complex Z that is homologically both degreewise finite and bounded above, a different argument yields the next formula

$$I_R^{\mathbf{R}\mathrm{Hom}_R(X,Z)}(t) = P_X^R(t) I_R^Z(t).$$

Furthermore, we have $I_R^Z(t) = 0$ if and only if $Z \simeq 0$. See [4, (1.5.3)].

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Every numerical semigroup is one over d of infinitely many symmetric numerical semigroups

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Abstract. For every numerical semigroup S and every positive integer $d > 1$ there exist infinitely many symmetric numerical semigroups $\overline{S}$ such that $S = \{n \in \mathbb{Z} : dn \in \overline{S}\}$. If $d \geq 3$, there exist infinitely many pseudo-symmetric numerical semigroups $\overline{S}$ such that $S = \{n \in \mathbb{Z} : dn \in \overline{S}\}$.

Keywords. Numerical semigroup, symmetric semigroup.

AMS classification. 20M14.

This note was motivated by the recent results of Rosales and García-Sánchez [5, 6] that for every numerical semigroup S there exist infinitely many symmetric numerical semigroups $\overline{S}$ such that $S = \{n \in \mathbb{Z} : 2n \in \overline{S}\}$. The main result in this note, Theorem 5, is that 2 is not a special integer, bigger positive integers work as well. The Rosales–García-Sánchez construction for $d = 2$ gives all the possible $\overline{S}$, whereas the construction below does not.

Throughout, S stands for a numerical semigroup, $\mathrm{F}(S)$ stands for its Frobenius number, $\mathrm{PF}(S)$ for the set of all pseudo-Frobenius numbers (i.e., all $n \in \mathbb{Z} \setminus S$ such that $n + (S \setminus \{0\}) \subseteq S$), and d for a positive integer strictly bigger than 1. The notation dS stands for the set $\{ds : s \in S\}$ and $\frac{S}{d}$ stands for $\{n \in \mathbb{N} : dn \in S\}$.

The goal is to construct infinitely many symmetric numerical semigroups T such that $S = \frac{T}{d}$. Another goal is to construct, for $d \geq 3$, infinitely many pseudo-symmetric numerical semigroups T such that $S = \frac{T}{d}$.

The ring-theoretic consequence, by a result of Kunz [1], is that for any affine domain of the form $A = k[t^{a_1}, \ldots, t^{a_m}]$, with $a_1, \ldots, a_m$ positive integers generating a numerical semigroup and with k a field, there exist infinitely many (Gorenstein) affine extension domains R of the same form such that any equation $X^d - a$ with $a \in A$ has a solution in A if and only if it has a solution in R. Such rings R are called d-closed.

It is clear that if $S = \frac{T}{a}$ and $T = \frac{U}{b}$, then $S = \frac{U}{ab}$. Thus it suffices to prove that for every S and for every positive prime integer d there exist infinitely many (symmetric) numerical semigroups T for which $S = \frac{T}{d}$. The proofs below, however, will not assume that d is a prime.

Definition 1. Let d be a positive integer. A numerical semigroup S is said to be d-*symmetric* if for all integers $n \in \mathbb{Z}$, whenever d divides n, either n or $\mathrm{F}(S) - n$ is in S.

Observe that a symmetric numerical semigroup is d-symmetric for all d, that a 1-symmetric numerical semigroup S is symmetric, and that a 2-symmetric numerical semigroup S is symmetric if $\mathrm{F}(S)$ is an odd integer.

Proposition 2. *Let $S \subseteq T$ be numerical semigroups such that* $\mathrm{F}(S) = \mathrm{F}(T)$. *If S is d-symmetric, so is T, and $\frac{S}{d} = \frac{T}{d}$.*

Proof. Let $m \in \mathbb{Z}$ be a multiple of d. If $m \notin T$, then $m \notin S$, so by the d-symmetric assumption on S, $\mathrm{F}(T)-m = \mathrm{F}(S)-m \in S \subseteq T$. Thus T is d-symmetric. It remains to prove that $\frac{T}{d} \subseteq \frac{S}{d}$. Let $m \in \frac{T}{d}$. Suppose that $m \notin \frac{S}{d}$. Then $dm \in T \setminus S$. Since S is d-symmetric, $\mathrm{F}(S)-dm \in S \subseteq T$, whence $\mathrm{F}(T) = \mathrm{F}(S) = (\mathrm{F}(S)-dm)+dm \in T$, which is a contradiction. □

In [4, Theorem 1, Proposition 2], Rosales and Branco show that every numerical semigroup S with odd $\mathrm{F}(S)$ can be embedded in a symmetric numerical semigroup T such that $\mathrm{F}(S) = \mathrm{F}(T)$. There are only finitely many choices for such T, but they are in general not unique. For example, let $S = \langle 12, 16, 21, 22, 23\rangle$. Then S is 4-symmetric, $\mathrm{F}(S) = 41$, and the numbers n for which $n, \mathrm{F}(S) - n$ are not in S are $10, 11, 14, 15, 26, 27, 30, 31$. If one adds 10 to S, then $2 \cdot 10 + 21 = 41$ would be in the numerical semigroup, so the Frobenius number would not be preserved. Thus any symmetric numerical semigroup T containing S with $\mathrm{F}(S) = \mathrm{F}(T)$ needs to contain 31. However, there are symmetric (and 4-symmetric) T that contain 11 and there are those that contain 30. All the possible symmetric T containing S with $\mathrm{F}(S) = \mathrm{F}(T)$ are as follows:

$$\langle 11, 12, 16, 21, 22, 23, 26, 31\rangle, \qquad \langle 12, 14, 16, 21, 22, 23, 31\rangle,$$
$$\langle 12, 15, 16, 21, 22, 23, 31\rangle, \qquad \langle 12, 16, 21, 22, 23, 26, 27, 30, 31\rangle.$$

Proposition 3. *Let S be a numerical semigroup and d, t and e positive integers. Let $g_1, \dots, g_t, h_1, \dots, h_e$ be positive integers such that:*

(1) *For all distinct $i, j \in \{1, \dots, e\}$, d does not divide $h_i - h_j$, and does not divide h_i.*

(2) $h_1 = \min\{h_1, \dots, h_e\}$.

(3) *For all $i = 1, \dots, e$, $h_i - d\,\mathrm{F}(S) > \frac{1}{2}h_1$.*

(4) *$g_1, \dots, g_t$ are not contained in S.*

Set $T = dS + \langle h_i - dg_j : i = 1, \dots, e; j = 1, \dots, t\rangle + \langle h_1+1, h_1+2, \dots, 2h_1+1\rangle$. Then T is a numerical semigroup, $\mathrm{F}(T) = h_1$, *and $S = \frac{T}{d}$.*

If $\mathrm{PF}(S) \subseteq \{g_1, \dots, g_t\}$, *then T is d-symmetric.*

Proof. The set $\langle h_1 + 1, \dots, 2h_1 + 1\rangle$ is contained in T, and thus T is a numerical semigroup with $\mathrm{F}(T) \le h_1$. Suppose that $h_1 \in T$. Then

$$h_1 = ds + \sum_{i,j} a_{ij}(h_i - dg_j)$$

for some $s \in S$ and some non-negative integers a_{ij}. Since d does not divide h_1, at least one a_{ij} is non-zero. By condition (3), at most one a_{ij} is non-zero, and so it is necessarily 1. Then $h_1 - h_i = ds - dg_j$, so that by condition (1), $i = 1$ and $s = g_j \in S$, which is a contradiction. This proves that $h_1 \notin T$, whence $h_1 = \mathrm{F}(T)$.

By (3), $h_1 > 2d\,F(S) > d\,F(S)$.

Clearly $dS \subseteq T$, so $S \subseteq \frac{T}{d}$. Let $n \in \mathbb{Z}$ such that $dn \in T$. We want to prove that $n \in S$. If $dn \geq h_1$, by the previous paragraph $n > F(S)$, whence $n \in S$. Now suppose that $dn < h_1$. Since $dn \in T$, write

$$dn = ds + \sum_{i,j} a_{ij}(h_i - dg_j),$$

for some $s \in S$ and some non-negative integers a_{ij}. As before, either $dn = ds + h_i - dg_j$ for some i, j, or $dn = ds$. The former case is impossible as h_i is not a multiple of d, so necessarily $dn = ds$ and so $n = s \in S$. This proves that $S = \frac{T}{d}$.

It remains to prove that T is d-symmetric if $\mathrm{PF}(S) \subseteq \{g_1, \dots, g_t\}$. Let $n \in \mathbb{Z}$ with $n = dm$ for some $m \in \mathbb{Z}$. If $n \notin T$, then $m \notin S$, and by [3, Proposition 12] there exists $g_i \in \mathrm{PF}(S)$ such that $g_i - m \in S$. Then $h_1 - n = h_1 - dm = (h_1 - dg_i) + d(g_i - m) \in T$. Thus T is d-symmetric. □

Corollary 4 (Rosales–García-Sánchez [6]). *Every numerical semigroup is one half of infinitely many symmetric numerical semigroups.*

Proof. Let $\mathrm{PF}(S) = \{g_1, \dots, g_t\}$ and let h_1 be an arbitrary odd integer bigger than $4\,F(S)$. Then by Proposition 3, there exists a 2-symmetric numerical semigroup T such that $\frac{T}{2} = S$ and such that $F(T) = h_1$. We already observed that a 2-symmetric numerical semigroup with an odd Frobenius number is symmetric. Since there are infinitely many choices for h_1, we are done. □

In general, the construction in the proof of Proposition 3 does not necessarily give a symmetric numerical group T. Say $S = \langle 3, 4\rangle$ and $d = 4$. Then $F(S) = 5$, $\mathrm{PF}(S) = \{5\}$. The maximal possible e is $d - 1 = 3$, so if we take $h_1 = 41$, $h_2 = 42$, $h_3 = 43$, the hypotheses of the theorem are satisfied, and the construction gives $T = \langle 12, 16, 21, 22, 23\rangle$. By the theorem, $F(T) = 41$ and $S = \frac{T}{4}$, but T is not symmetric as neither 10 nor 31 are in T. One can find a symmetric numerical semigroup U such that $S = \frac{U}{4}$ by using the Rosales–García-Sánchez result above twice (with $d = 2$), or one can apply the following main theorem of this paper:

Theorem 5. *Let S be a numerical semigroup and let d be an integer greater than or equal to 2. Then there exist infinitely many symmetric numerical semigroups T such that $S = \frac{T}{d}$.*

Proof. By choosing large odd integers h_1 that are not multiples of d, applying Proposition 3 with $e = 1$ and $\{g_1, \dots, g_t\} = \mathrm{PF}(S)$ gives a d-symmetric numerical semigroup T such that $S = \frac{T}{d}$ and $F(T) = h_1$. But then by Theorem 1 in [4] there exists a symmetric numerical semigroup U containing T such that $F(U) = F(T)$. By Proposition 2, $\frac{T}{d} = \frac{U}{d}$. Thus there exists a symmetric numerical semigroup U such that $S = \frac{U}{d}$ and $F(U) = h_1$. Since there are infinitely many choices of h_1, we are done. □

Recall that a numerical semigroup S is *pseudo-symmetric* if $F(S)$ is even and if for all $n \in \mathbb{Z} \setminus \{F(S)/2\}$, either n or $F(S) - n$ is in S. The following is a modification of the main theorem for pseudo-symmetric semigroups:

Theorem 6. *Let S be a numerical semigroup and let d be an integer greater than or equal to* 3*. Then there exist infinitely many pseudo-symmetric numerical semigroups T such that $S = \frac{T}{d}$.*

Proof. By choosing large even integers h_1 that are not multiples of d, applying Proposition 3 with $e = 1$ and $\{g_1, \ldots, g_t\} = \mathrm{PF}(S)$ gives a d-symmetric numerical semigroup T such that $S = \frac{T}{d}$ and $\mathrm{F}(T) = h_1$. Similar to Theorem 1 in [4], there exists a pseudo-symmetric numerical semigroup U containing T such that $\mathrm{F}(U) = \mathrm{F}(T)$, say $U = T \cup \{n \in \mathbb{N} \mid h_1/2 < n < h_1, h_1 - n \notin T\}$. By Proposition 2, $\frac{T}{d} = \frac{U}{d}$. Thus U is a pseudo-symmetric numerical semigroup such that $S = \frac{U}{d}$ and $\mathrm{F}(U) = h_1$. Since there are infinitely many choices of h_1, we are done. □

The integer $d = 2$ has to be excluded from the theorem above: if T is pseudo-symmetric with even Frobenius number $\mathrm{F}(T)$ and $S = \frac{T}{2}$, then necessarily $\mathrm{F}(T) \leq 2\,\mathrm{F}(S)$. But there are only finitely many such T.

A related result is in Rosales [2]: every numerical semigroup is of the form $\frac{T}{4}$ for some pseudo-symmetric numerical semigroup. Also, Rosales [2] proves that a numerical semigroup is irreducible if and only if it is one half of a pseudo-symmetric numerical semigroup.

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